Cut and conjugate points of the exponential map, with applications

Doctoral dissertation of Pablo Angulo Ardoy

Directed by Luis Guijarro Santamaría

UNIVERSIDAD AUTONOMA DE MADRID

Summary

1. Introduction

2. The Cut Locus and Boundary Value Problems for Hamilton-Jacobi equations

3. Structure of the cut locus

4. Balanced split loci

5. The Ambrose problem

6. Conjectures

Introduction

Riemannian manifold. Smooth n manifold M, with or without boundary, with an *scalar product* g_p at each T_pM

Finsler manifold. Smooth n manifold M, with or without boundary, with a *norm* φ_p at each T_pM: convex, 1-homogeneous, (it may not satisfy $\varphi_p(-v) = \varphi_p(v)$)

Lenght of a curve. $\text{length}(\alpha) = \int_0^T \varphi_p(\alpha'(t))$

Distace on M induced by φ. $d(p, q) = \inf\{\text{lenght}(\alpha) \colon \alpha(0) = p, \alpha(1) = q\}$.

Geodesic. An extremal of the lenght functional $\alpha \to \text{lenght}(\alpha)$ with *fixed endpoints*. In equations: $\varphi_p(\gamma(t), \gamma'(t)) = \frac{d}{dt}\ (\ \varphi_v(\gamma(t), \gamma'(t)))$

Tip: a geodesic is *unique* if its starting point and speed are given.

Minimizing geodesic. $\text{lenght}(\gamma) = d(\gamma(0), \gamma(T))$

Tip: every geodesic is minimizing when restricted to *small enough* interval.

Tip: every minimizing curve is a geodesic.

Exponential map at p. Send $v \in T_pM$ to the tip of the unit time geodesic with initial speed v

Tip: it is a diffeomorphism from an *injectivity set* onto a *normal neighborhood of* p, but may develop singularities later (related to *conjugate points along geodesics*).

The Cut Locus

M: Finsler manifold with boundary. The cut locus $Cut_{\partial M}$ with respect to ∂M can be defined in several *equivalent* ways:

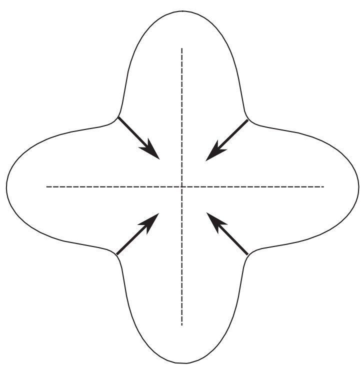
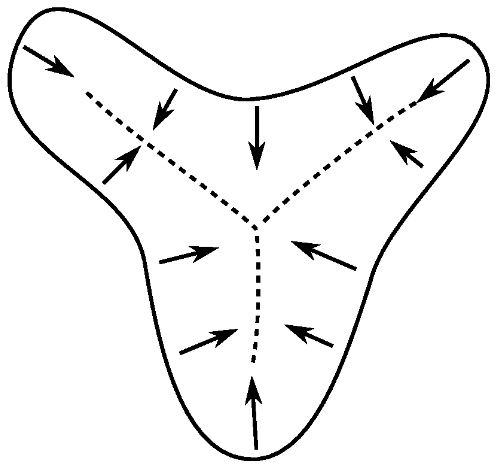
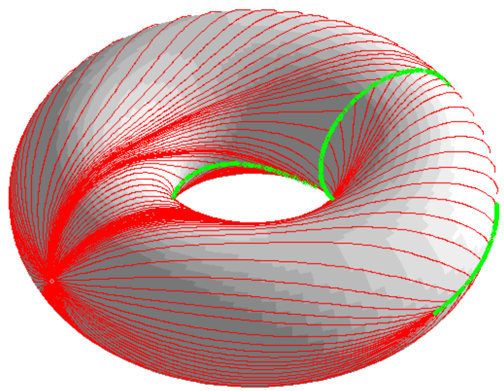
- For $p \in \partial M$, let γ_p be the unit speed geodesic γ with initial point p and initial speed orthogonal to ∂M (and inner-pointing). Define

  $$t_{cut}(p) = \sup \{t : d(\gamma_p(0), \gamma_p(t)) = t\}$$

  Then

  $$Cut = \{\gamma_p(t_{cut}(p)) : p \in \partial M\}$$

- $Cut = \overline{\{p \in M : d_{\partial M} \text{ is singular}\}}$ (the closure of the set of points where $d_{\partial M}$ is *singular*)
- ...there are more definitions.

Boundary Value Problems for Hamilton-Jacobi equations

Find $H\colon M \to \mathbb{R}$ *such that:*

$$\begin{aligned} H(p, du(p)) &= 1 \qquad & p \in M \\ u(p) &= g(p) & p \in \partial M \end{aligned} \tag{1}$$

- $H^{-1}(1) \cap T_p^* M$ is **convex** for every p, and contains 0.

- M is a smooth and **compact manifold with boundary**, H and g are smooth.

- The boundary data satisfies a **compatibility condition** (more about it later)

$$|g(y) - g(z)| < d(y, z) \tag{2}$$

A geometrical interpretation

1. We can assume H is a norm in each vector space T_p^*M (if necessary, replace H with $\tilde{H}(p,w)=t$, for the only $t>0$ such that $H(p,\frac{1}{t}w)=1$)

2. Define *the dual norm* φ_p in T_pM

$$\varphi_p(v) = \sup\{\langle v, \alpha\rangle_p : \alpha \in T_p^*M\,,\, H(p,\alpha)=1\}$$

3. This is a **Finsler** metric, which induces a distance d in M

4. The metric is Riemannian iff H is quadratic in its second argument.

Classical solution by characteristic curves

The HJ equations are first order PDEs, and thus *there is a solution using characteristic curves*, defined *only in a neighborhood of* ∂M.

$$\boxed{\text{If } x=\gamma(t) \text{ for a characteristic } \gamma \text{ with } \gamma(0)=y, \text{ then } u(x)=t+g(y)}$$

The **characteristic curves are geodesics** of φ, whose initial condition at $y\in\partial M$ is the vector V_y satisfying:

$$\varphi_y(V_y)=1 \qquad \widehat{V_y}|_{T_y(\partial M)}=dg \qquad V_y \text{ points inwards}$$

In particular, if g is constant and φ Riemannian, V is perpendicular to ∂M.

For a vector V in a Finsler space, $w=\hat{V}$ is its **dual one-form**, given by:

$$w_j=\frac{\partial\varphi}{\partial V^j}(p,V)$$

This is the usual definition of dual form if φ is a riemannian metric.

Viscosity solution

A *viscosity solution* is a solution in a weak sense, **defined in all $\boldsymbol{M}$**. There is a definition of viscosity solution, and a lot of literature. We only need the Lax-Oleinik formula, which describes the viscosity solution in terms of the Finsler distance and the boundary data:

$$u(p) = \inf_{q \in \partial M} \{d(p,q) + g(q)\}$$

Comments

- The compatibility condition $|g(y) - g(z)| < d(y,z)$ is necessary and sufficient for solutions to exist.
- If $g = 0$, then u is the *distance to the boundary*.
- The solution is not C^1 in all of $M \Rightarrow$ It has a **singular set**

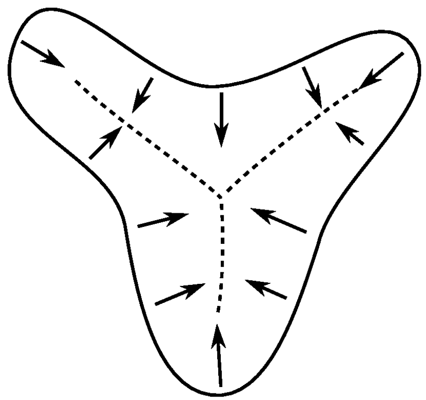
$$\boxed{\text{The solution obtained with characteristic curves coincides with the viscosity solution where both are defined.}}$$

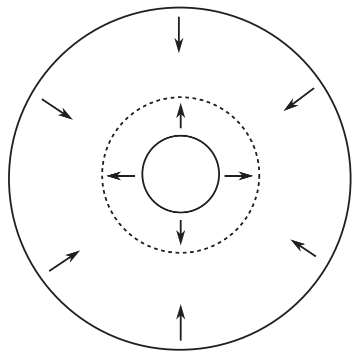
The singular set

Characteristic curves from ∂M intersect each other if continued indefinitely.

The extra information required to *get the viscosity solution from the classical one* is a criterion to decide which characteristic curve is used to compute the value of u at a given point.

This extra information is the singular set of the solution u:



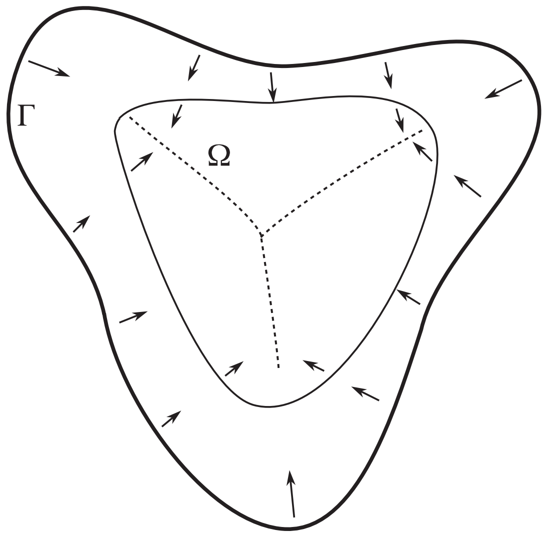


Let Sing be the closure of the singular set of u

What do we know about this singular set?

If $g=0$: u is the distance to the boundary, **Sing** is the **cut locus**.

Theorem. (2.1.6) *A solution with $g\neq 0$ in M is the* restriction *of the solution with $g=0$ in a bigger set $\Gamma\supset M$:*



$$H(p,dv(p))=1 \qquad p\in\Gamma$$
$$v(p)=0 \qquad p\in\partial\Gamma$$

$$u=v|_M$$
$$\mathrm{Sing}(u)=\mathrm{Sing}(v)$$

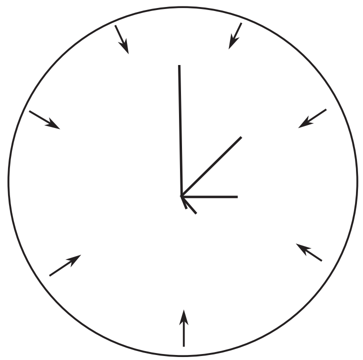
So Sing *is always the cut locus of some Finsler manifold.*

Structure of the singular set

Cut loci (singular sets for HJBVP) are studied by PDE and geometry people

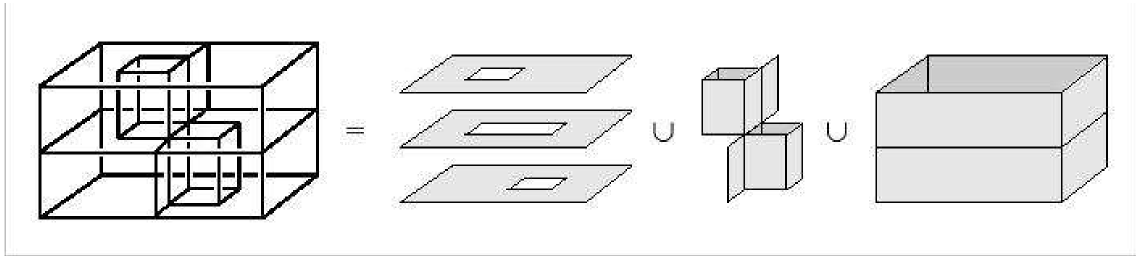
- The singular set is a deformation retract of M (obvious).
- It is the union of a $(n-1)$-dimensional smooth manifold consisting of *"cleave points"* with two minimizing geodesics and a set of Hausdorff dimension at most $n-2$ (Hebda87, Itoh-Tanaka98, Barden-Le97, Mantegazza-Menucci03 for the riemannian case).
- The singular set is stratified by the dimension of the subdifferential ∂u (Alberti-Ambrosio-Cannarsa-Etcetera92-94).
- It has finite Hausdorff measure $\mathcal{H}^{n-1}$ (Itoh-Tanaka00 for riemannian manifolds, Li-Nirenberg05 for Finsler).
- If we add a generic perturbation to H or M, **Sing** becomes a stratified smooth manifold (Buchner78).

However, a cut locus can be pretty bad:



In Gluck-Singer78, the authors show there are plenty of *non-triangulable* cut loci.

Sing has the homotopy of M, but its topology may be non-trivial. The cut locus of a ball in $\mathbb{R}^3$ could be *the house with two rooms*:



This figure was taken from the book *Algebraic Topology* by *Allen Hatcher*

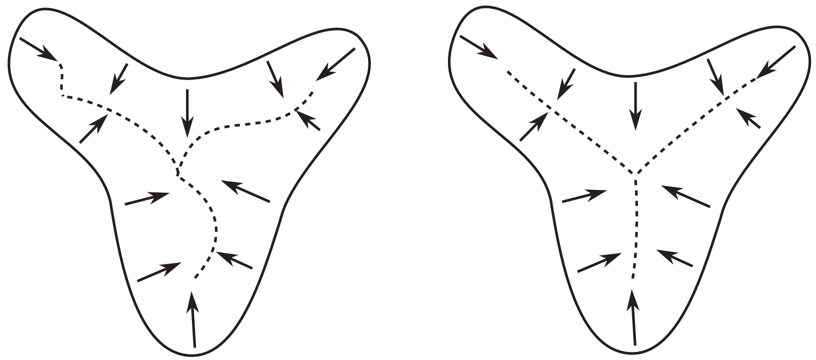
Balanced split locus

Let's start looking at the cut locus from scratch:

Definition. (2.2.1) *We say* $\boldsymbol{S \subset M}$ ***splits*** $\boldsymbol{M}$ *iff every point* $p \in M \setminus S$ *belongs to a unique normal geodesic from* ∂M *contained in* $M \setminus S$.

If S splits M, and $p \in M \setminus S$, $\boldsymbol{R_p}$ is the speed of the normal geodesic from ∂M to p in $M \setminus S$. If $p \in S$, let R_p be the limit set of vectors R_q when $q \to p$.

Definition. (2.2.6) *S is a split locus iff* $S = \overline{\{p \in S : \# R_p \geqslant 2\}}$



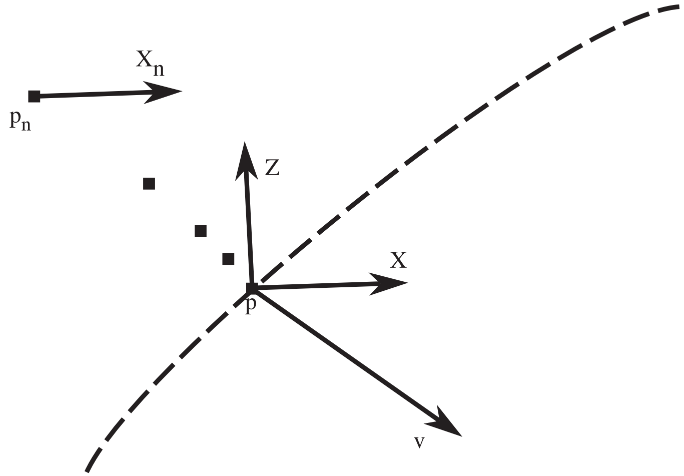
An arbitrary split locus and the cut locus of M

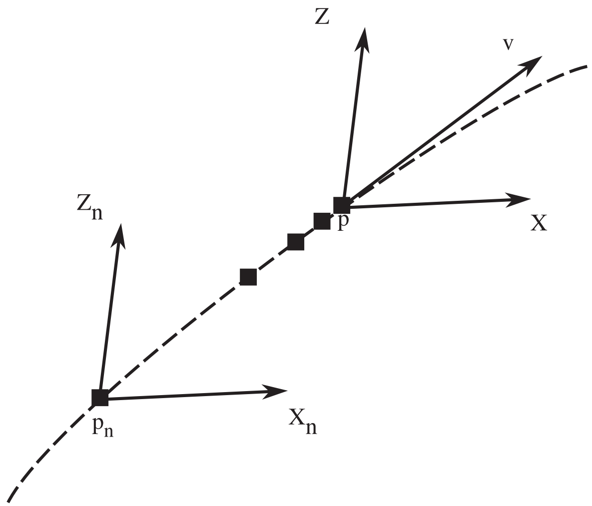
Equivalently, S is a **split locus** iff S is closed, it splits M, and no closed $S' \subsetneq S$ splits M.

Definition. (2.2.8) *A split locus S is balanced iff the following holds:*

Let p_n be a sequence of points and $X_n \in R_{p_n}$ be a sequence of vectors. If $p_n \to p$, $X_n \to X$, and the vector from p_n to p converges to v, *then:*

$$\hat{X}(v) \geqslant \hat{Z}(v) \quad \forall Z \in R_p$$





In riemannian geometry, $\hat{X}(v) = <X, v> = |v||X|\cos(\angle(X, v))$, so the balanced property means that *the angle of the incoming vector with the limit vector X is smaller* than the angle it makes with any other vector of R_p.

Our structure results

It turns out that *most existing structure results for the cut locus also hold for balanced split locus*. Suspicious, uh?

To prove our results we first had to *adapt the existing structure results* to Finsler geometry and/or to balanced split locus: **(3.3.2), (3.3.3), (4.2.5), (4.2.6)**.

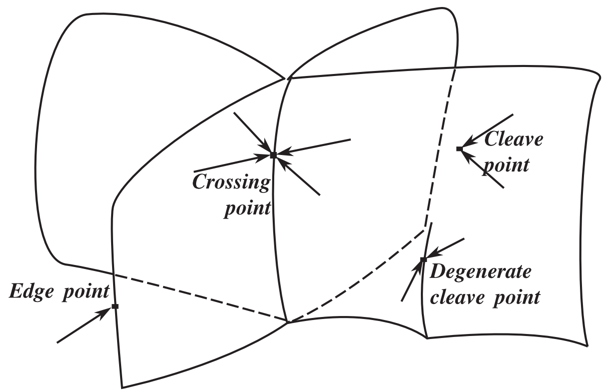
We also proved the following:

Theorem. (3.2.4) *The set of points $p \in M$ such that R_p contains a conjugate geodesic of order $\geqslant 2$ has Hausdorff dimension $\leqslant n-3$.*

Proof. The set of conjugate points of order 2 is the union of two sets: Q_2^1 and Q_2^2. The image of Q_2^2 has Hausdorff dimension $\leqslant n-3$ (uses Morse-Sard-Federer), and vectors in Q_2^1 do not map to vectors in the sets R_p. □

Remark. In more standard terminology, this can be rephrased as *"the set of points that can be joined to ∂M with a minimizing geodesic conjugate of order 2 has Hausdorff dimension $\leqslant n-3$"*.

The restriction to *minimizing* geodesics is essential: the Hausdorff dimension of $F(Q_2^1)$ may well be $n-2$.



Corollary. (3.1.2) *A balanced split locus S consists of:*

- ***Cleave points*** *($R_p = \{X_1, X_2\}$, each X_i is regular)(a smooth non-connected hypersurface)*
- ***Edge points*** *(R_p consists of one conjugate vector of order 1) (Hausdorff dimension $n-2$)*
- ***Degenerate cleave points*** *($R_p = \{X_1, X_2\}$, X_i may be conjugate of order 1) (Hausdorff dimension $n-2$)*
- ***Crossing points*** *($\widehat{R_p} = \{\hat{X} : X \in R_p\}$ is contained in an affine 2D plane, R_p has regular and conjugate vectors of order 1) (*rectifiable *set of dimension $n-2$)*
- ***Remainder*** *(Hausdorff dimension $n-3$)*

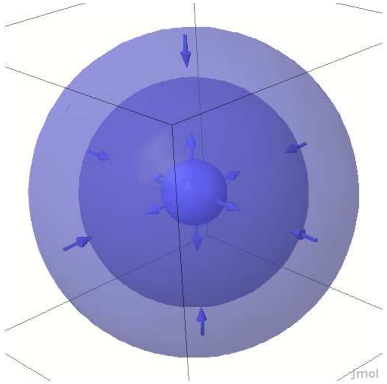
Comment: this is interesting to study *brownian motion on manifolds*.

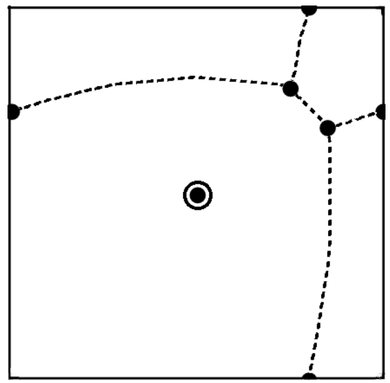
Characterization of the singular set for HJBVP

The singular set of the viscosity solution to a HJBVP is a balanced split locus **(2.3.1, 2.3.2)**

Is it the only balanced split locus?

M is simply connected *and* ∂M connected **(4.2.1)**	$\rightarrow$	The singular set is the unique balanced split locus
M is simply connected, ∂M is **not** connected **(4.2.2)**	$\rightarrow$	We can add a different constant to g at each component of ∂M and get different balanced split loci
General case **(4.2.4)**	$\rightarrow$	Balanced split loci are parametrized by a neighborhood of 0 in $H_{n-1}(M, \mathbb{R})$





Proof of main theorem: a current

Each characteristic curve carries a value for u. A point in $M \setminus S$ gets only one value, but a point in S gets a possible value from each geodesic from ∂M contained in $M \setminus S$.

Let $\mathcal{C}_j$ be the *connected components of* the set of *cleave points*. Each cleave point gets one candidate value for u from either side: u_l and u_r

We define a current T of dimension $n-1$:

$$T(\phi) = \sum_j \left(\int_{\mathcal{C}_{j,l}} \phi u_l + \int_{\mathcal{C}_{j,r}} \phi u_r \right) \tag{3}$$

here $\mathcal{C}_{j,i}$ means $\mathcal{C}_j$ with the orientation induced by a fixed orientation in M, and the vector tangent to the geodesic coming from side $i = l, r$.

If $T = 0$, then u can be defined unambiguously, and it's continuous **(4.3.7)**.

$\boxed{\textit{The main step of the proof is to show } \boldsymbol{\partial T = 0}}$

Once we have this, it is not hard to show that if two currents T_1 and T_2 obtained from two balanced split loci S_1 and S_2 represent the same homology class in $H_{n-1}(M)$, then $T_1 = T_2$.

For example, if M is simply connected and ∂M connected, and T is closed, then $T = dP$, where $P(\phi) = \int \phi f$ for a density $f \in L^n$. But $d\,P|_{M\setminus S} = T|_{M\setminus S} = 0$ implies f is locally constant outside S. Under our hypothesis, f is constant and $T = 0$.

For ϕ with support in a neighborhood of a cleave point:

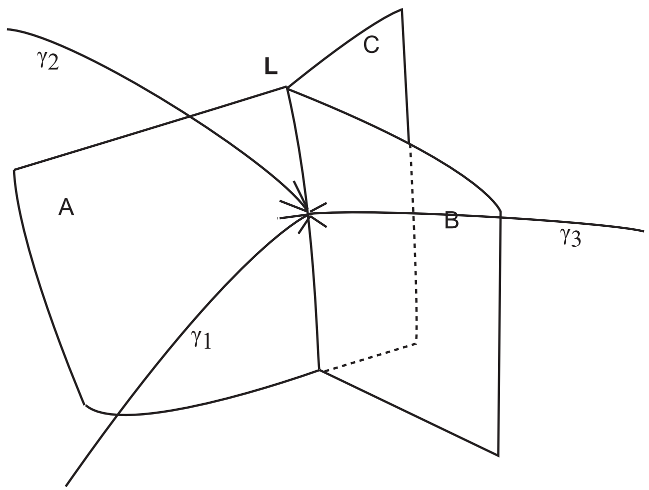
$$\partial T(\phi) = T(d\phi) = \int_{\mathcal{C}_{j,r}} d\phi(u_r - u_l) = \int_{\mathcal{C}_{j,r}} \phi d(u_r - u_l)$$

But $du_i = \widehat{X_i}$ for the incoming vector X_i $(i = l, r)$.
By the balanced condition, $T\mathcal{C}_j \subset \ker(\widehat{X_r} - \widehat{X_l})$, so the integral is 0.

For ϕ with support in a neighborhood of a (generic) edge point:

Near a generic edge point q, S is a smooth *hypersurface with boundary*, with q a boundary point. $u_r - u_l$ is contant, and converges to zero as we approach the boundary.

For ϕ in a neighborhood of a (generic) crossing point:



$$
\begin{aligned}
\partial T(\phi) &= T(d\phi) \\
&= \int_{A_1} d\phi u_1 + \int_{A_2} d\phi u_2 + \int_{B_1} d\phi u_1 + \\
&\quad + \int_{B_3} d\phi u_3 + \int_{C_2} d\phi u_2 + \int_{C_3} d\phi u_3 \\
&= \int_{A_1} \phi d(u_1 - u_2) + \int_{B_3} \phi d(u_3 - u_1) \\
&\quad + \int_{C_2} \phi d(u_2 - u_3) \\
&\quad + \int_{L} \phi (u_1 - u_1 + u_2 - u_2 + u_3 - u_3) \\
&= 0
\end{aligned}
$$

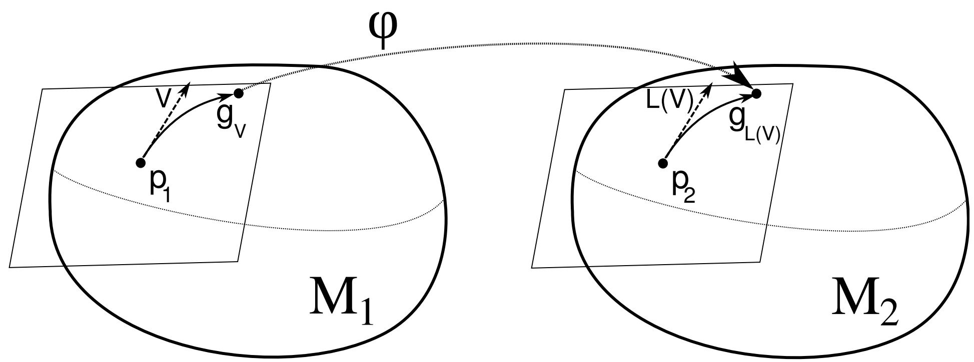
Proof for general points:

Non-generic edge and crossing points can be quite more complicated than that, with *a countable amount of components* $\mathcal{C}_j$ in any neighborhood. Thanks to the structure results, **we only have to deal with non-conjugate geodesics and geodesics of order 1**

However, this required further structure results **(4.6.2-4.6.11)**.

The Ambrose problem

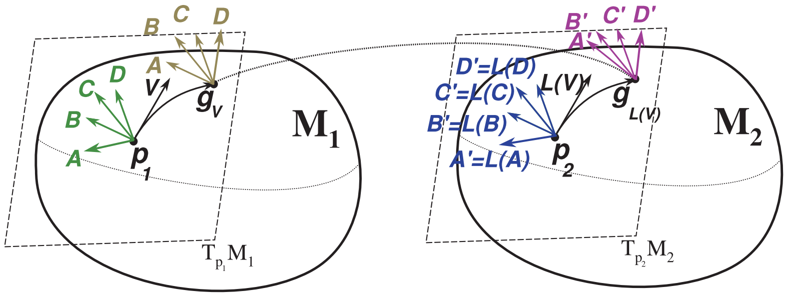
An isometry between Riemannian manifolds is determined by its differential at any point:

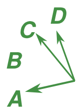
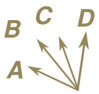


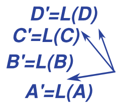
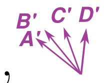
$\varphi\colon M_1 \to M_2$ is an isometry between connected and complete manifolds st
$\varphi(p_1) = p_2,\ L = d_{p_1}\varphi$

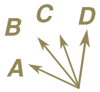
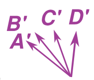
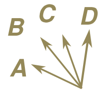
- Fix a *normal neighborhood* U of p_1.
- Any point q of U can be reached from p_1 by a unique geodesic contained in U, with speed V (parametrized by $[0, 1]$).
- $\varphi(q)$ is the endpoint of the geodesic with initial conditions p_2 and $L(V)$.

Parallel transport of curvature



- We parallel displace  along g_V up to $q = g_V(1)$ and compute $R_1(\mathrm{V},$  $) = < R_{g_V(1)}(A, B)C, D >$

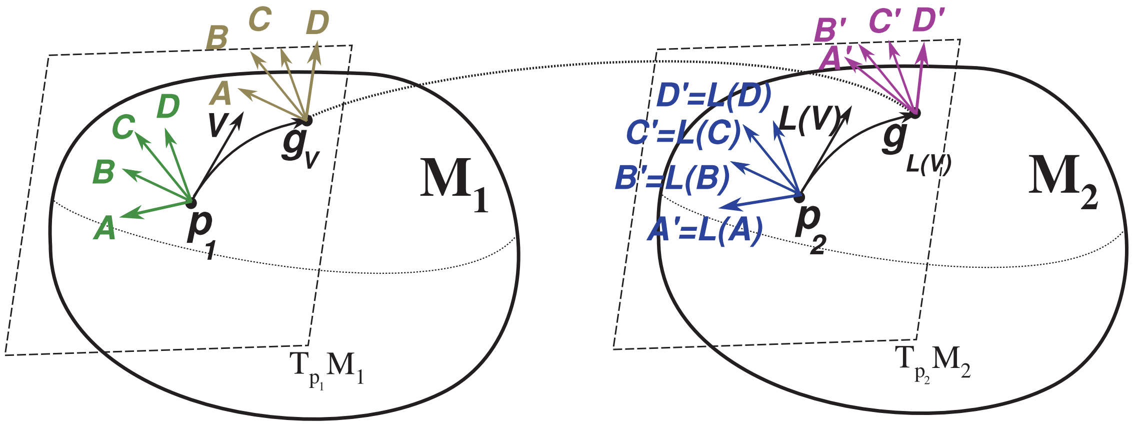
- We apply L to the vectors:  , parallel displace them along $g_{L(V)}$ up to $\varphi(q) = g_{L(V)}(1)$ and compute $R_2(\mathrm{V},$  $) = < R_{g_{L(V)(1)}}(A', B')C', D' >$

- If φ is an isommetry, $R_1(\mathrm{V},$  $) = R_2(\mathrm{V},$  $)$ for any V and  .

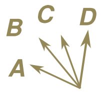
Cartan's lemma

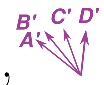
Let's do the opposite thing:

let $L\colon T_{p_1}M_1 \to T_{p_2}M_2$ be a linear isometry	$\Rightarrow$	we get a map φ defined in a convex neighborhood of p_1



Cartan's lemma

If R_1(V, 

)$=R_2$(V, 

)

for any A, B, C, D, V, then

φ is a local isometry

from a convex neigborhood of p_1 to one of p_2

Global version of Cartan's lemma

(Cartan-)**Ambrose**(-Hicks)' theorem
If parallel transport of the curvature of M_1 y M_2
along geodesics *with one elbow* coincide,
and both manifolds are *simply connected*,
φ is an isometry from M_1 onto M_2

Ambrose Conjecture (1956)
If parallel transport of the curvature of M_1 y M_2
along *smooth* geodesics coincide,
and both manifolds are *simply connected*,
then φ is an isometry from M_1 onto M_2

History

- Up to 1987, Cartan's lemma is generalized (Hicks59, Hicks66, O'Neill68, AmiciCasciaro86, BlumenthalHebda87,PawelReckziegel02). The global version (involving geodesics with one elbow), is automatic.

- In 1987, James Hebda proved the conjecture for *surfaces*, assuming that the «distance to the cut locus» is an absolutely continuous function.

- J. Hebda (1994) and J-I Itoh (1996) prove independently that this function is indeed absolutely continuous for any smooth surface.

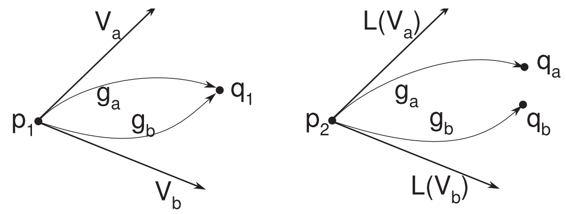
- J-I Itoh and M. Tanaka (2000) prove that it is indeed Lipschitz for a manifold of any dimension, but Ambrose's conjecture does not follow.

- J. Hebda (2010) proves the conjecture for generic riemannian manifolds.

Quick review of 1987 James Hebda's proof

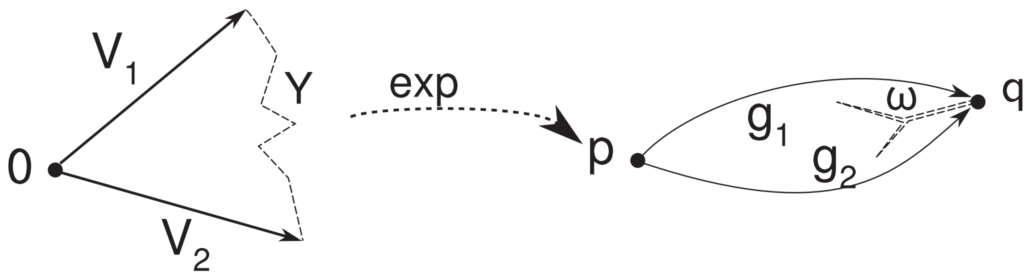
- Notice that Cartan's lemma actually provides an isometric immersion of $M_1 \setminus C$ into M_2 (C is the cut locus of M_1).

- A **cleave point** $q = \exp_{p_1}(V_a) = \exp_{p_1}(V_b)$ is connected to p_1 by *exactly two minimizing geodesics*, and both are *non conjugate*. There are two possible images: $\varphi(q)$ can be $\exp_{p_2}(L(V_a))$ or $\exp_{p_2}(L(V_b))$. The goal is to prove that it's the same point.



- *If φ is defined at $M_1 \setminus C$ and the cleave points, it extends to all of M_1.*

- A central part of the strategry is to find a path Y in TCut_p that joins V_a and V_b, and maps by $\exp_p$ to a **tree-formed curve** (contained in Cut_p):



Tree-formed (or tree-like) paths

Definition. *($\sim$J. Hebda) An absolutely continuous path $u\colon [0,1]\to M$ is* ***tree-formed*** *(with respect to T) iff u factors through a* "tree" Γ*: $u = \bar{u}\circ T$, for $T\colon [0,1]\to\Gamma$, $\bar{u}\colon\Gamma\to M$.*

In order to make the above idea precise, Hebda admits any identification map $T\colon [0,1]\to\Gamma$, where Γ has the final topology given by T, but also asks that for any *continuous* 1*-form along* Γ ($\varphi(r)\in T^*_{\bar{u}(r)}M$), we have $\int_{t_0}^{t_1}\varphi(T(s))(u'(s))d\,s = 0$, if $T(t_0)=T(t_1)$. If $T(0)=T(1)$, we say u is *fully tree-formed*.

Remark: Tree-formed paths reappeared later in the theory of *Rough Path*, where they play a central role:

It is easy to find curves in TCut whose image by exp is tree-formed in 2D (there is basically one choice)

but in 3D and higher, this is not possible...

Our approach: synthesis of two manifolds

We allow that M_1 and M_2 be non-simply-connected.

$$
\begin{tikzcd}
& T_pM \arrow[dl, "L_1"'] \arrow[dr, "L_2"] \arrow[dd, "e"] & \\
T_{p_1}M_1 \arrow[rr, "L"] \arrow[dd, "e_1"] & & T_{p_2}M_2 \arrow[dd, "e_2"] \\
& M \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
M_1 & & M_2
\end{tikzcd}
$$

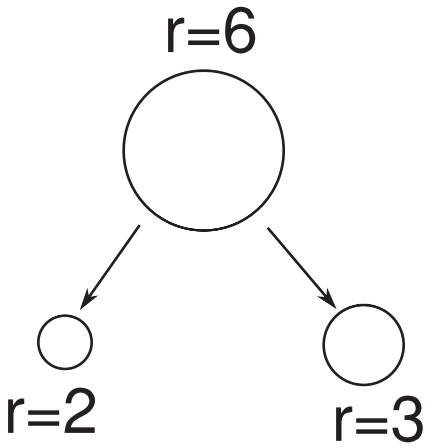
If the manifolds M_1 y M_2 have a common covering space M whose covering maps π_1 y π_2 are local isometries, then M is a **synthesis** of M_1 and M_2.

If π_1 and π_2 are local isometries and local homeomorphisms, then M is a **weak synthesis**.

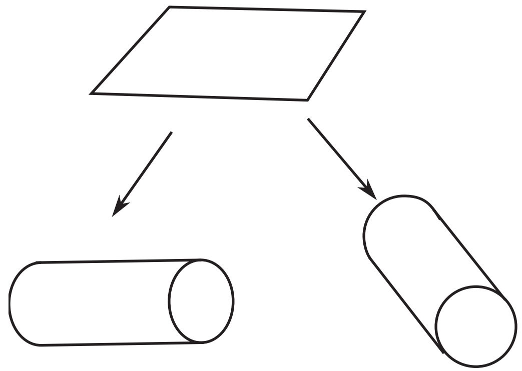
If M_1 and M_2 are simply connected, and they have a synthesis M, then M is isometric to both M_1 and M_2 (by π_1 and π_2).

This idea is found in «Riemannian coverings» by B. O'Neill68.

Examples of synthesis



The exponential of a circle of radius 6 is a synthesis of the exponential maps of two circles of radii 2 and 3.



The synthesis of two cylinders, whose maps e_1 y e_2 are rotated 90°, is a plane, with $\pi_1 = e_1$ and $\pi_2 = e_2$.

An equivalence relation

Notation: $e_1 = \exp_{p_1}$, $e_2 = \exp_{p_2} \circ L$, $V_1 = \{x \in T_{p_1} M_1 \colon |x| \leqslant \lambda_1(\frac{x}{|x|})\}$

Definition. *We say* $\boldsymbol{x} \leftrightsquigarrow \boldsymbol{y}$ *(x is linked to y) iff* $x = y$*, or:*

$$e_1(x) = e_1(y) \text{ and } e_2(x) = e_2(y)$$

and there are neighborhoods U^x *and* V^y *such that*

$$\forall z \in U, w \in V: \quad e_1(z) = e_1(w) \Rightarrow e_2(z) = e_2(w)$$

Definition. *We say that an open set* $O \subset T_{p_1} M_1$ *is* ***unequivocal*** *iff:*

- $e_1(O \cap V_1)$ *is open*
- $e_2(O \cap V_1)$ *is open*
- $\exists$ *an isometry* $\varphi \colon e_1(O \cap V_1) \to e_2(O \cap V_1)$ *such that* $\varphi \circ e_1|_{O \cap V_1} = e_2|_{O \cap V_1}$

$x \in V_1$ *is* ***unequivocal*** *if it has a* neighbourhood base of unequivocal sets

Theorem. (5.4.6) *If* $M_1 = \mathcal{I} \cup \mathcal{J}$*, so that:*

- *all points in* $\mathcal{I}$ *are unequivocal*
- *every point in* $\mathcal{J}$ *is linked to some point in* $\mathcal{I}$

Then $M = V_1 / \leftrightsquigarrow$ *is a* ***weak synthesis*** *of* M_1 *and* M_2*.*

The conjecture for generic riemannian manifolds

For a *generic* set of riemannian metrics in a manifold, T_pM admits the following descomposition:

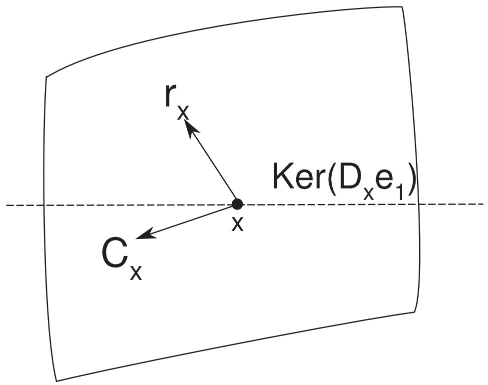
- An open set consisting of non-conjugate points (**NC**).
- Strata of dimension $\boldsymbol{n-1}$, of points of type $\boldsymbol{A_2}$ (*fold* singularities)
- Strata of dimension $\boldsymbol{n-2}$, of points of type $\boldsymbol{A_3}$ (*cusp* singularities). We further split them into $\boldsymbol{A_3(\mathrm{I})}$ and $\boldsymbol{A_3(\mathrm{II})}$ (minima and maxima, roughly)
- Strata of dimension $\boldsymbol{n-3}$, of points of type $\boldsymbol{A_4}$**,** $\boldsymbol{D_4^+}$ and $\boldsymbol{D_4^-}$
- Strata of smaller dimension

Conjugate flow at points of type A_2

At points of type A_2 *the kernel of the exponential is transversal to the set of conjugate points* (which is a smooth hypersurface). The exponential e_1 is given in adapted coordinates by:

$$(x_1, ..., x_{n-1}, x_n) \rightarrow (x_1, ..., x_{n-1}, x_n^2)$$

e_1 is not a local homeomorphism at an A_2 point, so ¿how can we link the A_2 point x to an unequivocal point y? The conjugate flow tells us how we can start:



Gauss' lemma implies that the radial vector r_x is transversal to the kernel of the exponential. The sum of both spaces is a plane that we intersect with the tangent to the set of conjugate points:

$$(\ker D_x e_1 \oplus <r_x>) \cap T\,\mathrm{Conj} = <C_x>$$

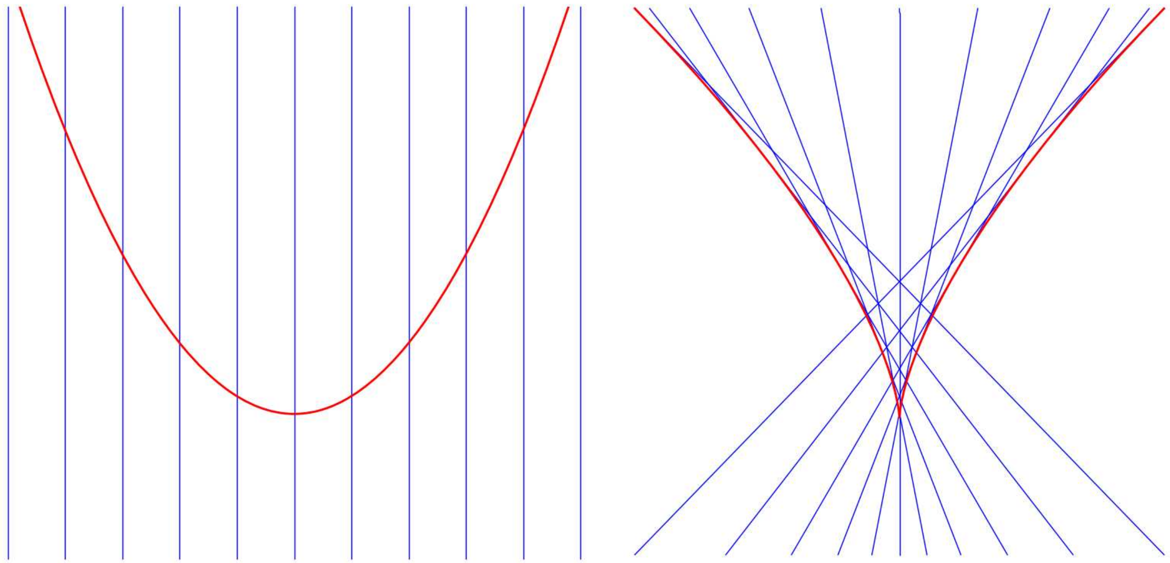
We choose C_x such that $C_x \cdot r_x < 0$

Conjugate flow

We call **conjugate flow curve (CDC)** to a integral curve of the above vector field C_x. The curve stays within the set of conjugate points, and can be continued until it hits a point that is not A_2.

The most simple and most important case is that the conjugate flow hits an $\boldsymbol{A_3}$ point. For A_3 points, e_1 is given in adapted coordinates by:

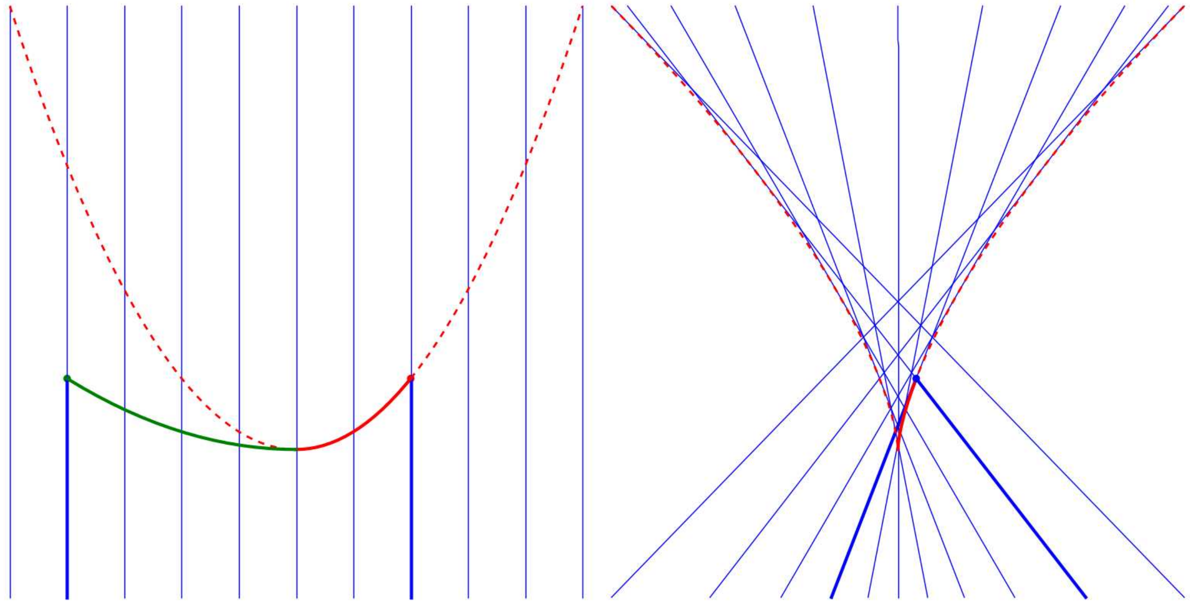
$$(x_1, \ldots, x_{n-1}, x_n) \to (x_1, \ldots, x_{n-1}, x_n^3 + x_1 x_n)$$



Replying a CDC

Once we reach an A_3 point, we can find a **reply** to the CDC: it is a curve of NC points whose image by e_1 is the same as the CDC, but run in the opposite direction.

The concatenation of both segments is a curve **whose image is tree-formed**.



A CDC α is **unbeatable**: $|\alpha(0)| - |\alpha(t_0)| > |\beta(t_0)| - |\beta(0)|$ *for any reply* β. **(5.4.10)**

Building linking curves

Remember: *our goal is to show our A_2 point x is linked to an unequivocal point y*

Starting at x, suppose we follow the CDC up to an A_3 point. We can keep replying as long as the reply stays within V_1, but **we may hit a singularity**. If that happens, we descend along the conjugate flow again, etcetera:

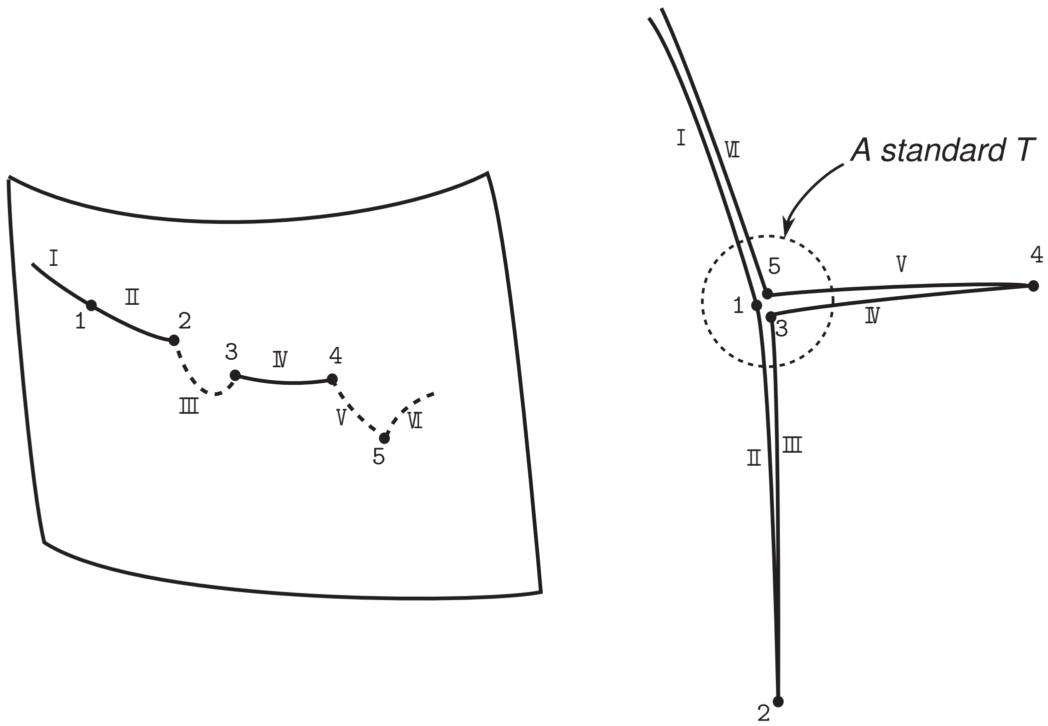
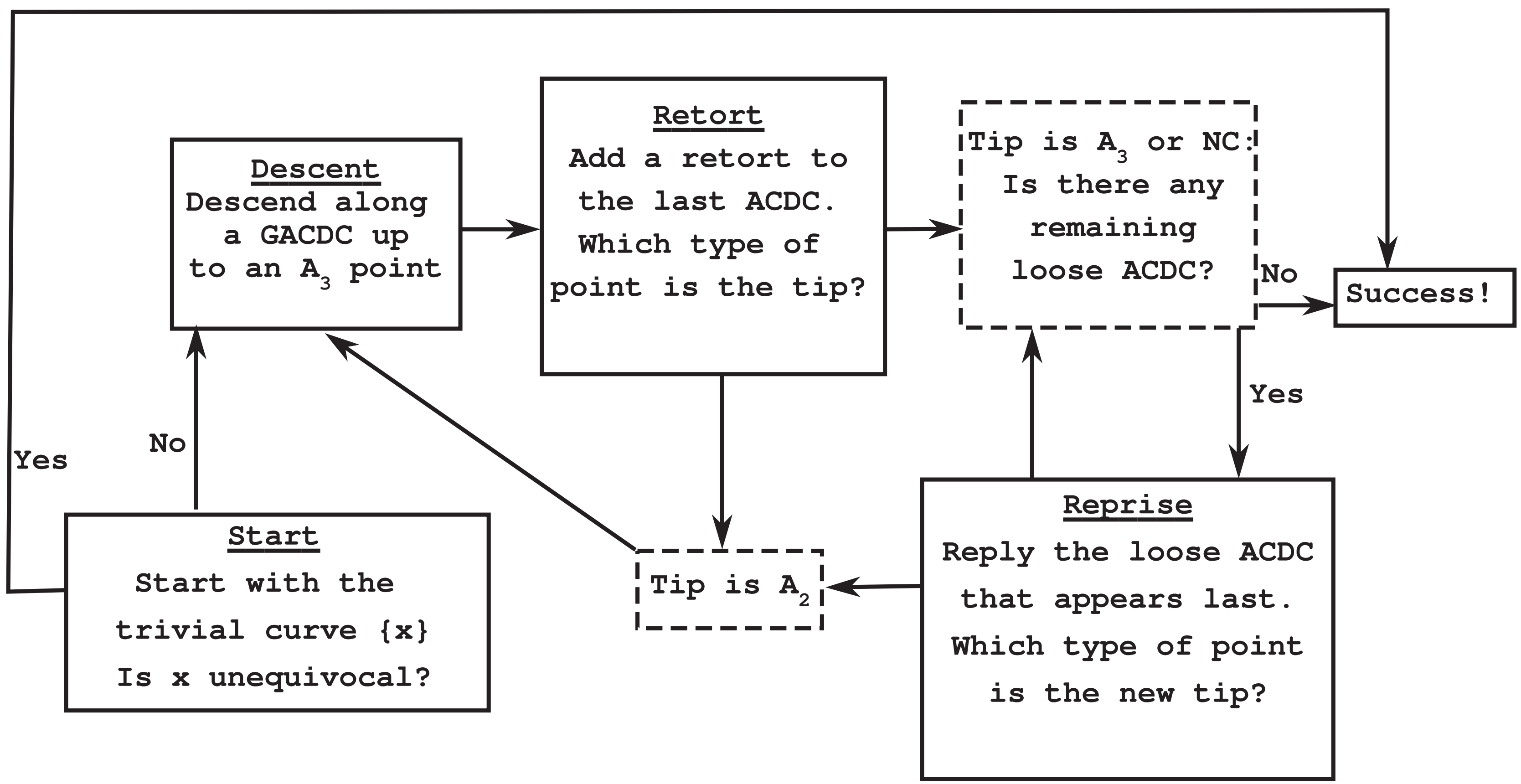


Figure. Splitter (1), A_3-join (2,4), hit(3), reprise(5)

The algorithm

We summarize the procedure in this flow diagram:



Does it work?

- *Do we always reach an A_3 point?* Yes, we can «dodge» other singularities **(5.4.29)**.

- *How long can we keep replying?* As long as the reply stays within V_1. If the reply hits an A_2 point z, we plug in a linking curve from z to a NC point w, and take on the job of replying to whatever is left.

- *What if the reply hits a worse singularity?* We can also dodge that **(5.4.22)**.

- *How do you know the procedure will ever stop?* Because every point in a generic manifold has a «**transient neighborhood**»: if the algorithm starts at a point on that neighborhood, after a finite number of elementary steps of the algorithm, the tip will never again be there (the *unbeatable* property is key). **(5.4.25)**

We show that it works for generic metrics, in 3 dimensions. The algorithm can also start at $\boldsymbol{A_3}$**(II)**, $\boldsymbol{A_4}$, $\boldsymbol{D_4^+}$ and $\boldsymbol{D_4^-}$ points.

Points of type $\boldsymbol{A_3}$**(I)** are unequivocal **(5.4.3)**.

Summary

- For a generic manifold, $V_1 \subset \mathrm{NC} \cup A_2 \cup A_3(I) \cup A_3(\mathrm{II}) \cup A_4 \cup D_4^{\pm}$

- $\mathcal{I} = V_1 \cap (\mathrm{NC} \cup A_3(I)) \qquad \mathcal{J} = V_1 \cap (A_2 \cup A_3(\mathrm{II}) \cup A_4 \cup D_4^{\pm})$

- Points in $\mathcal{I}$ are unequivocal, points in $\mathcal{J}$ are linked to points of $\mathcal{I}$, of smaller radius.

- Thus we have a weak synthesis M. We prove it is actually a synthesis **(5.4.11)**.

Beyond

- The set of points in a Finsler manifold M that can be joined to ∂M with a minimizing geodesic conjugate of order k has Hausdorff dimension $\leqslant n-k-1$.
- Characterize the singular set for more first order PDEs
  - HJ-equations with dependence on u
  - Non-convex H?
  - Sub-riemannian geometry?
- Ambrose in higher dimension, generic metric (easy)
- Ambrose for arbitrary metric (hard). We prove that it follows from the $\mathcal{I}\mathcal{J}\mathcal{K}$ conjecture, where the points in T_pM are classified into $\mathcal{I},\mathcal{J},\mathcal{K}$ depending on their behaviour with respect to the conjugate flow. It would follow if we could bound the lenghts of the linking curves in terms of numbers that depend continuously on the metric.

Theorem. (6.5.2) *If $M_1=\mathcal{I}\cup\mathcal{J}\cup\mathcal{K}$, so that all points in $\mathcal{I}$ are unequivocal, all points in $\mathcal{J}$ are linked to a point in $\mathcal{I}$, and $\mathcal{H}^{n-2}(e_1(\mathcal{K}))=0$.*

*Then $M_0=(V_1\setminus\mathcal{K})/\sim$ is a **weak synthesis** of M_1 and M_2 that extends to a weak synthesis M of M_1 and M_2.*

Thanks for your attention!

- Pablo Angulo and Luis Guijarro. Cut and singular loci up to codimension 3 (Annales de l'Institut Fourier 61, 2011, n4, p 1655-1681)

  http://arxiv.org/abs/0806.2229 (2009)

- Pablo Angulo and Luis Guijarro. Balanced split sets and Hamilton-Jacobi equations (Calculus of Variations and Partial Differential Equations, vol 40, 2011, n1-2, p 223-252)

  http://arxiv.org/abs/0807.2046 (2009)

- Pablo Angulo. Linking curves, synthesis of manifolds and the Ambrose conjecture for generic 3-manifolds. Publication pending.

https://arxiv...

#

FAQ #1 What is FAQ #1?

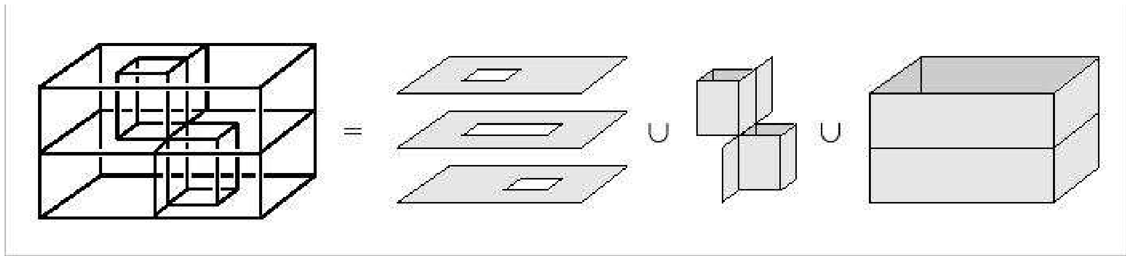
Answer: “What is FAQ #1?”

Remark: The answer to FAQ #1 is actually FAQ #1

FAQ #2: Why a handshake picture at the beginning?

If you grab a ball of clay, insert both hands in an attempt to shake them, you are doing the retraction frmo the ball onto a space toplogically equivalent to the house with two rooms.

The house with two rooms lies at the heart of many of the difficulties in this thesis.



FAQ #3 Why can't you extend the proof of James Hebda to three dimensions?

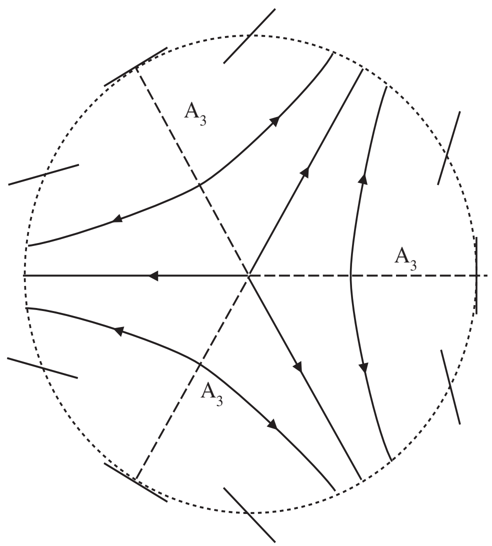
Answer: You mean: Can we find the curve Y in dimension $\geqslant 3$?

In **dimension $\geqslant$3** there are many possible choices for Y, none of them is *canonical.*

But it's worse than that: the house with two rooms is the cut locus of a certain manifold, and it doesn't have edge points!

Weinstein showed that every manifold except S^2 admits a metric such that the cut locus wrta point doesn't have edge points.

FAQ #4 Are there many easy manifolds?



In 3 dimensions, there are *many non-simply connected easy manifolds*, but for example there are *no easy metrics close to the standard metric* on the 3-sphere:

If a 3d manifold admits a metric with positive curvature and no conjugate points of order 2, then the set of first conjugate points is diffeomorphic to the sphere, but has a non-zero vector field.

If the metric is generic, there must be some D4+ points, because it's impossible to «comb» the sphere of first conjugate points with the singularities that arise from D4- singularities.

FAQ #5 Where did the idea to study split loci came from?

The idea was supposed to help in *proving Ambrose conjecture.*

The **cut locus** is a *convenient* tool in the proof of James Hebda for surfaces, but *any split locus could play the same role*.

So I thought: is it possible to find a split locus, in an arbitrary riemannian manifold, that is triangulable, and collapses simplicially to one point?

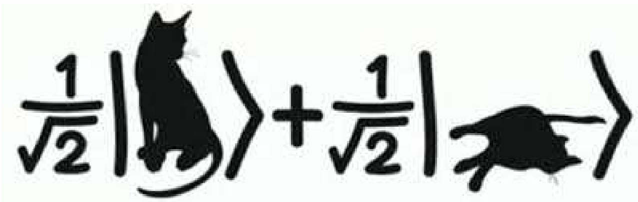
The answer is: if you can do that, you'd have a new, more topological proof of the Poincaré conjecture.

FAQ #6 Why didn't you choose a simpler problem?

If you work in a hard problem, you may or may not succeed. If you don't, you're out of the game. If you do, you can choose where you want to work (and live).

If you work in not-so-hard problem, your chances of failure are smaller, but it's more likely that you have to spend considerable time in remote places with great uncertainty about your future.

In short: I chose a **Schrödinger cat** over a **sick cat**.



$>$

FAQ #7 Is the synthesis the «minimal common covering space»?

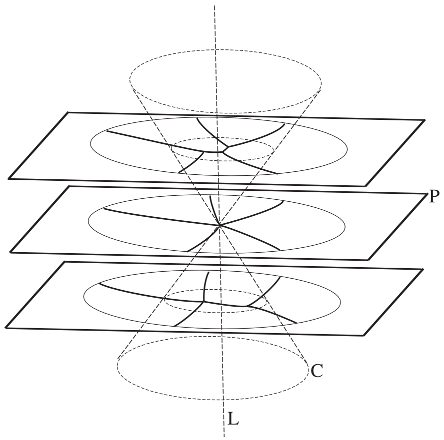
Yes, it satisfies a universal property **(5.4.7)**: Let X be the synthesis of X_1 and X_2. For any Riemannian manifold X', continuous surjective map $e': A \to X'$ and local isometries $\pi_1': X' \to X_1$ and $\pi_2': X' \to X_2$, such that $e_i = \pi_i' \circ e'$, for $i = 1, 2$, there is a local isometry $\pi: X' \to X$ such that:

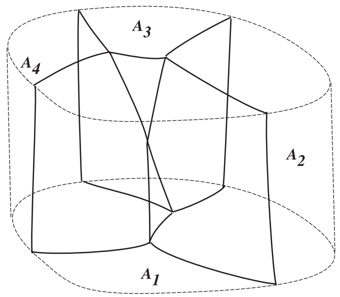
$$
\begin{tikzcd}
& A \arrow[d, "e'"] \arrow[ddl, "e_1"'] \arrow[ddr, "e_2"] & \\
& X' \arrow[dl, "\pi_1'"] \arrow[dr, "\pi_2'"'] & \\
X_1 & & X_2
\end{tikzcd}
\qquad
\begin{tikzcd}
& X' \arrow[d, "\pi"] \arrow[ddl, "\pi_1'"'] \arrow[ddr, "\pi_2'"] & \\
& X \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
X_1 & & X_2
\end{tikzcd}
$$

FAQ #8 How bad do crossing points get?

Lemma. (4.6.5-4.6.7) *Let $p \in S$ be a general crossing point. There is a finite amount of univocal open sets O_i (see lemma 4.6.1) such that any $X \in R_p$ is of the form $X = d_{x_i} F(\frac{\partial}{\partial t})$ for some $x_i \in O_i$.*

- *All $A_i \cap A_j$ are Lipschitz hypersurfaces*
- *Let $\Sigma = \cup (A_i \cap A_j \cap A_k)$. In certain coordinates, the intersections of Σ with coordinate planes $\{x_1 = a_1, \ldots, x_{n-2} = a_{n-2}\}$ are Lipschitz trees*
- *At general crossing points, we also have $\partial T = 0$.*





FAQ #9 What is the motivation for studying Ambrose Problem?

Ambrose's motivation (roughly): to characterize a Riemannian manifold by the parallel transport of its curvature: $L\colon \mathbb{R}^n \times G_2(\mathbb{R}^n) \to \mathbb{R}$

$L(V, P)$: parallel transport the plane P along the geodesic g_V and compute the sectional curvature of the plane.

My motivation: The Ambrose problem is similar to some *inverse problems* in Riemannian geometry. Some of these problems come directly from applications like tomography. Some of those problems are easy to solve if there are no singularities. I planned to build some muscle and try other such inverse problems later.

FAQ #10 Can you bound the lenght of the linking curves and approximate an arbitrary metric by generic ones?

Not without new ideas. I've tried two things:

1. As the *slack* goes to 0, and an A_2 point x becomes a worse singularity, the *gain* of the CDC α through x decreases. It's true that the lenght of its composition with the exponential also decreases, but overall, nor the lenght of α neither that of $e_1 \circ \alpha$ is not bounded.

2. Try to put numbers to the algorithm: Let B_R be the maximum lenght of a linking curve through a point x of radius R. The algorithm starts with a CDC α of lenght l that leaves a transient neighborhood U of x. Then a linking curve starting at the tip of α follows. Its lenght is bounded by B_{R-l}. And after that, we have to reply to α. If we can reply to α at once:

   $$B_R < l + B_{R-l}$$

   but the reply might hit an A_2 point, and then we have to reply to part of α, then plug in another linking curve at the tip of the reply, then reply the rest... if there are k interruptions:

   $$B_R < 2\,l + k\,B_{R-\varepsilon}$$

   where ε is the *gain* of the transient neighborhood. This is exponential growth.

References

Cut and singular loci up to codimension 3 http://arxiv.org/abs/0806.2229 (Annales de l'Institut Fourier, Vol. 61 no. 4, 2011)

Balanced split sets and Hamilton-Jacobi equations http://arxiv.org/abs/0807.2046 (Calculus of Variations and Partial Differential Equations, vol 40, 2011)

Linking curves, synthesis of manifolds and the Ambrose conjecture for generic 3-manifolds (publication pending)