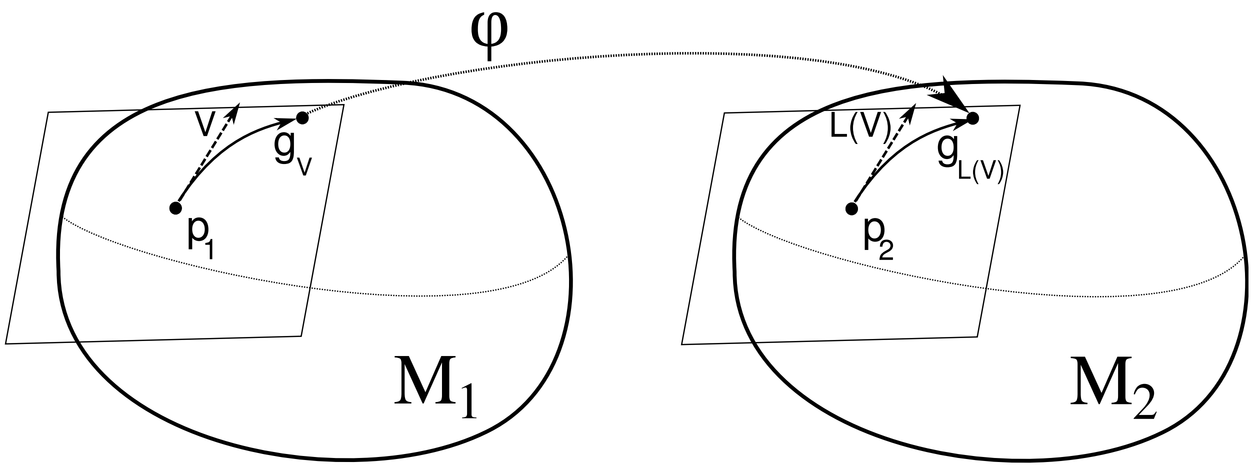
Linking curves, sutured manifolds and the Ambrose conjecture for generic 3-manifolds

Pablo Angulo (ECM7 Berlin, 2016)



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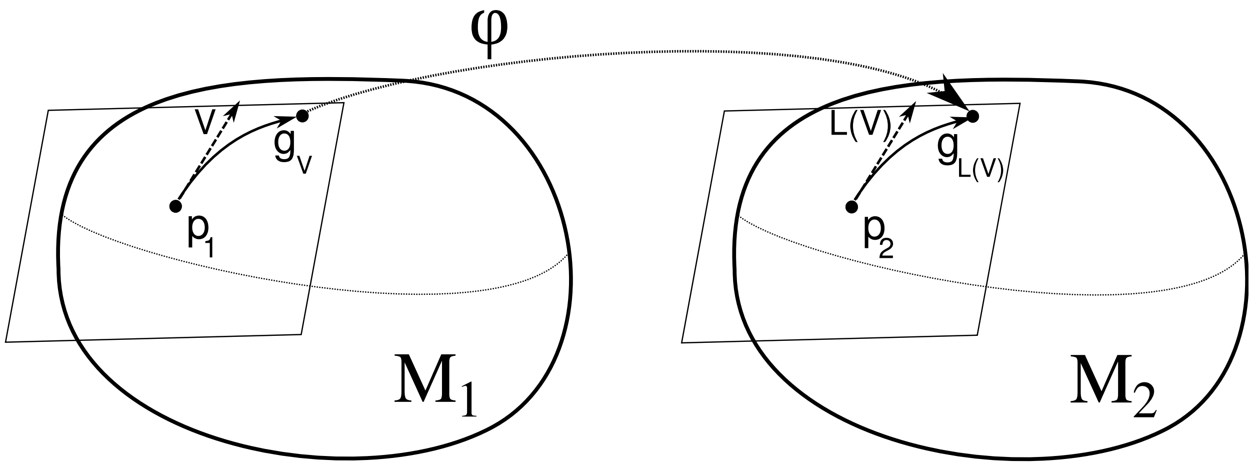
The Ambrose problem

An **isometry** between Riemannian manifolds is determined by its differential at one point p_1:



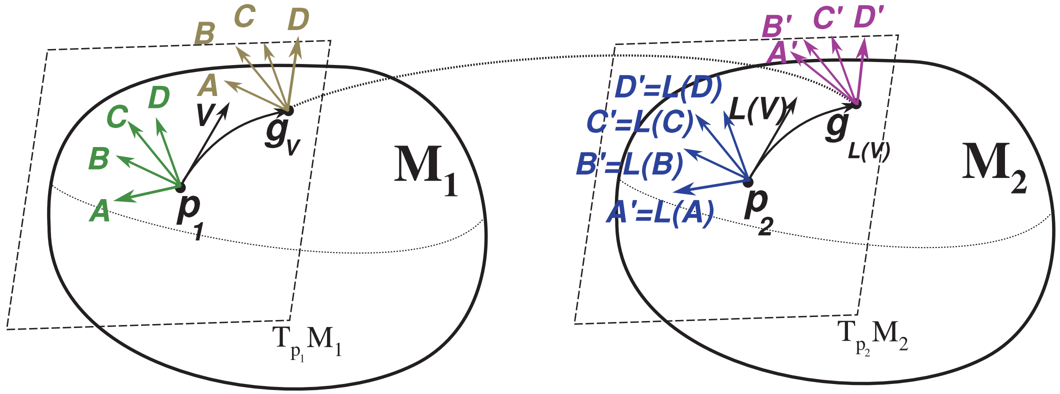
$\varphi\colon M_1 \to M_2$ is an isometry between connected and complete manifolds st
$\varphi(p_1) = p_2,\ L = d_{p_1}\varphi$

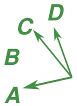
- Fix a *normal neighborhood* U of p_1.

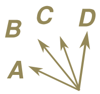
- Any point q of U can be reached from p_1 by a unique geodesic contained in U, with speed V (parametrized by $[0, 1]$).

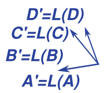
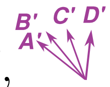
- $\varphi(q)$ is the endpoint of the geodesic with initial conditions p_2 and $L(V)$.

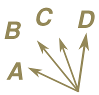
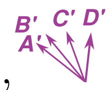
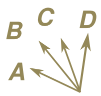
Parallel transport of curvature



- We parallel displace  along g_V up to $q = g_V(1)$ and compute

  R_1(V, )=$<R_{g_V(1)}(A, B)C, D>$

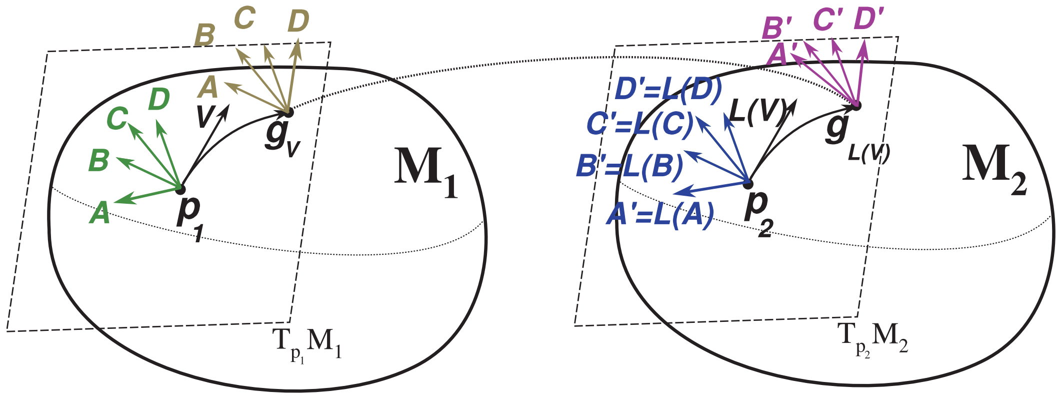
- We apply L to the vectors: , parallel displace them along $g_{L(V)}$ up to $\varphi(q) =$ $g_{L(V)}(1)$ and compute R_2(V, )=$<R_{g_{L(V)(1)}}(A', B')C', D'>$

- If φ is an isometry, R_1(V, )=R_2(V, ) for any V and  .

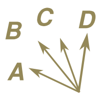
Cartan's lemma

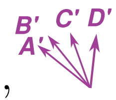
Let's do the opposite thing:

let $L\colon T_{p_1}M_1 \to T_{p_2}M_2$ be a linear isometry	$\Rightarrow$	we get a map φ_L defined in a convex neighborhood of p_1



Cartan's lemma

If R_1(V, 

)$=R_2$(V, 

)

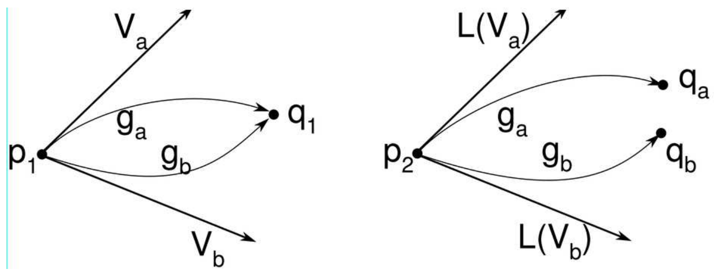
for any A, B, C, D, V, then φ_L is a local isometry from a convex neigborhood of p_1 to a neighborhood of p_2

Global version of Cartan's lemma

(Cartan-)**Ambrose**(-Hicks) theorem
If parallel transport of the curvature of M_1 y M_2
along geodesics *with one elbow* coincide,
and both manifolds are *simply connected*,
φ_L is an isometry from M_1 onto M_2

Ambrose Conjecture (1956)
If parallel transport of the curvature of M_1 y M_2
along *smooth* geodesics coincide,
and both manifolds are *simply connected*,
then φ_L is an isometry from M_1 onto M_2

The proof of the Cartan lemma proves the Ambrose conjecture if there are no conjugate points. The problem is: there may be many geodesics from p_1 to q_1: are the corresponding points in M_2 the same?



History

- Up to 1987, Cartan's lemma is generalized (Hicks59, Hicks66, O'Neill68, Amici-Casciaro86, BlumenthalHebda87,PawelReckziegel02). The global version (involving geodesics with one elbow), is automatic.

- In **1987**, **James Hebda proved the conjecture for surfaces**, assuming that the «distance to the cut locus» is an absolutely continuous function.

- J. Hebda (1994) and J-I Itoh (1996) prove independently that this function is indeed absolutely continuous for any smooth surface.

- J-I Itoh and M. Tanaka (2000) prove that it is indeed Lipschitz for a manifold of any dimension, but they could not prove the conjecture from this, since Hebda's proof for surfaces does not work in higher dimensions.

- J. Hebda (2010) proves the conjecture for *generic riemannian manifolds*. The proof does not "pass to the limit", so new strategies are needed.

Quick review of 1987 James Hebda's proof

- Cartan's lemma provides an isometric immersion of $M_1 \setminus \mathrm{Cut}_{p_1} = \exp_{p_1}(O_{p_1})$ into M_2

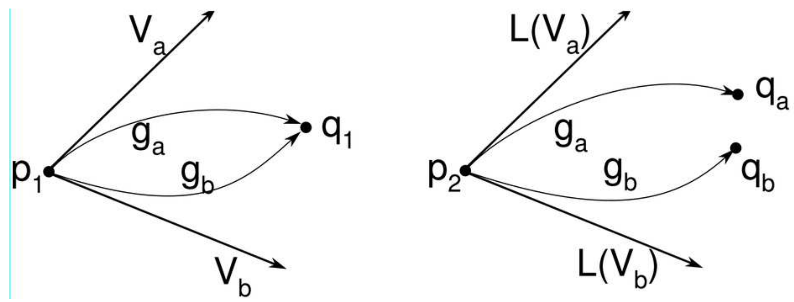
  Injectivity set. $O_{p_1} = \{x \in T_{p_1}M_1 \colon d_{M_1}(\exp_{p_1}(p_1, t\,x)) = t\}$

  Tangent cut locus. $\mathrm{TCut}_{p_1} = \partial O_{p_1}$

  Cut locus. $\mathrm{Cut}_{p_1} = \exp_{p_1}(\mathrm{TCut}_{p_1})$

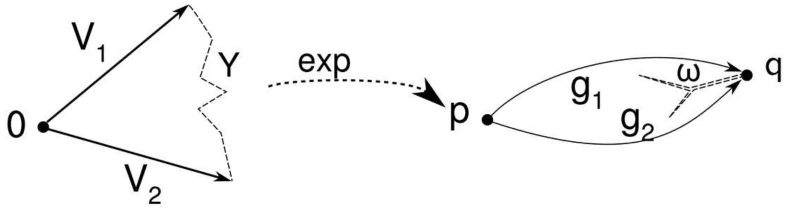
$$\mathrm{Cut}_{p_1} = \left\{ q \in M_1 \colon \begin{array}{c} \text{there are at least two minimizing geodesics from } p_1 \text{ to } q \\ \text{or the unique minimizing geodesic is focal} \end{array} \right\}$$

- If a point $q_1 = \exp_{p_1}(V_a) = \exp_{p_1}(V_b)$ is connected to p_1 by *two minimizing geodesics*, there are two possible images: $\varphi(q_1)$ can be $\exp_{p_2}(L(V_a))$ or $\exp_{p_2}(L(V_b))$. The goal is to prove that it's the same point.



Tree-formed (or tree-like) paths

A central part of the strategy is to find a path Y in TCut_p that joins V_a and V_b (for any point $q = \exp_p(V_a) = \exp_p(V_b) \in \mathrm{Cut}_p$), and maps by $\exp_p$ to a **tree-formed curve** (contained in Cut_p)



Definition 1. *Let $u: [0,1] \to M$ be an absolutely continuous curve. It is **fully tree formed** if:*

- *$\exists T: [0,1] \to \Gamma$, a quotient map with $T(0) = T(1)$*
- *u factors through T ($u = \bar{u} \circ T$ for $\bar{u}: \Gamma \to M$)*
- *for any continuous 1-form $\varphi(x) \in T^*_{\bar{u}(x)} M$:*

$$\int_0^1 \varphi(T(s))(u'(s))\, d s = 0$$

Remark: Tree-formed paths reappeared later in the theory of *Rough Path*.

It is easy to find curves in the **tangent cut locus** whose image by exp is tree-formed in 2D (there are one or two choices, both work)

but *in 3D and higher, this is not possible...*

Look for linking curves outside the cut locus

$$e_1 = \exp_{p_1}; \quad e_2 = L \circ \exp_{p_2}$$

Definition 2. *A **linking curve** is an absolutely continuous curve* $Y : [0, l] \to T_p M$ *such that* $e_1 \circ Y$ *is a fully tree formed curve.*

Definition 3. *Two points* $x, y \in T_p M$ *are **strongly linked** iff* $\exists$ *linking curve* Y, $Y(0) = x$, $Y(1) = y$

$$\boxed{x, y \text{ strongly linked } \Rightarrow e_1(x) = e_1(y),\ e_2(x) = e_2(y)}$$

Definition 4. $O \in T_p M$ *is **unequivocal** iff* $e_1(O)$ *is open, and there is an isometry* $\varphi_O: e_1(O) \to e_2(O)$ *such that* $\varphi_O \circ e_1|_O = e_2|_O$

Definition 5. $x \in T_p M$ *is **unequivocal** if there is a sequence of sets* W_n *such that* $e_1(W_n)$ *is a neighborhood basis of* $e_1(x)$.

$$\boxed{\text{The idea is to link singularities of } \exp_p \text{ to unequivocal points}}$$

Main theorem

Theorem 6. *Let M_1, M_2 be simply connected Riemannian manifolds with L-related curvature, such that* ***every $\boldsymbol{x} \in \boldsymbol{V_1}$ is linked to some unequivocal $\boldsymbol{y} \in \boldsymbol{V_1}$, with $|\boldsymbol{y}| \leqslant |\boldsymbol{x}|$****.*

Then there is a ***strong synthesis*** *of M_1 and M_2 through e_1 and e_2.*

Proof. A manifold M is defined as a **quotient** of a subset of $T_{p_1}M_1$, **identifying linked points**. The maps e_1 and e_2 induce maps π_1 and π_2.

A topology is defined ad-hoc, we prove that π_1, π_2 are local homeomorphisms, etc.

The condition $|y| \leqslant |x|$ is important to prove they are covering maps.

Thus π_1 and π_2 are covering maps and local isometries. We say M is a *synthesis* of M_1 and M_2.

M_1, M_2 simply connected $\Rightarrow$ π_1 and π_2 are global isometries. □

In other words, *if we can find linking curves* starting at conjugate points, *the Ambrose conjecture follows.*

But can we find enough linking curves?

Generic 3D riemannian manifolds

For a *generic* set of riemannian metrics in a 3D manifold, T_pM admits the following descomposition (canonical form of $\exp_p$):

- An open set consisting of non-conjugate points (**NC**) ($(x_1, ..., x_n) \to (x_1, ..., x_n)$).
- Strata of dimension **2**, of points of type $\boldsymbol{A_2}$ (*fold* singularities)

  $(x_1, x_2..., x_n) \to (x_1^2, x_2, ..., x_n)$.
- Strata of dimension **1**, of points of type $\boldsymbol{A_3}$ (*cusp* singularities). We further split them into $\boldsymbol{A_3}$**(I)** and $\boldsymbol{A_3}$**(II)** (minima and maxima, roughly)

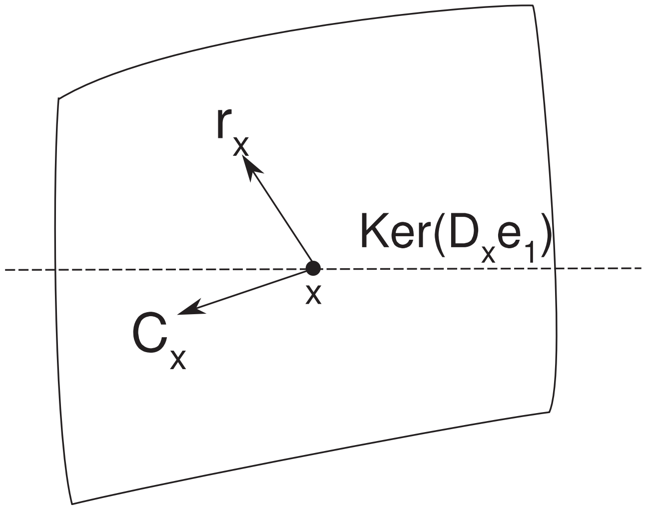
  $(x_1, x_2..., x_n) \to (x_1^3 \pm x_1 x_2, x_2, ..., x_n)$
- Isolated points of type $\boldsymbol{A_4}$ ($(x_1, x_2..., x_n) \to (x_1^4 + x_1^2 x_2 + x_1 x_3, x_2, ..., x_n)$)
- Isolated points of type $\boldsymbol{D_4^+}$ ($(x_1, x_2..., x_n) \to (\frac{1}{2}x_1^2 + x_2 x_3, \frac{1}{2}x_2^2 + x_1 x_3, ..., x_n)$)
- Isolated points of type $\boldsymbol{D_4^-}$ ($(x_1, x_2..., x_n) \to (\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + x_1 x_3, -x_1 x_2 + x_2 x_3, ..., x_n)$)

Conjugate flow at points of type A_2

At points of type A_2 *the kernel of the exponential is transversal to the set of conjugate points* (which is a smooth hypersurface). The exponential e_1 is given in adapted coordinates by:

$$(x_1, ..., x_{n-1}, x_n) \to (x_1, ..., x_{n-1}, x_n^2)$$

e_1 is not a local homeomorphism at an A_2 point, so ¿how can we link the A_2 point x to an unequivocal point y? The conjugate flow tells us how we can start:



Gauss' lemma implies that the radial vector r_x is transversal to the kernel of the exponential. The sum of both spaces is a plane that we intersect with the tangent to the set of conjugate points:

$$(\ker D_x e_1 \oplus <r_x>) \cap T\,\mathrm{Conj} = <C_x>$$

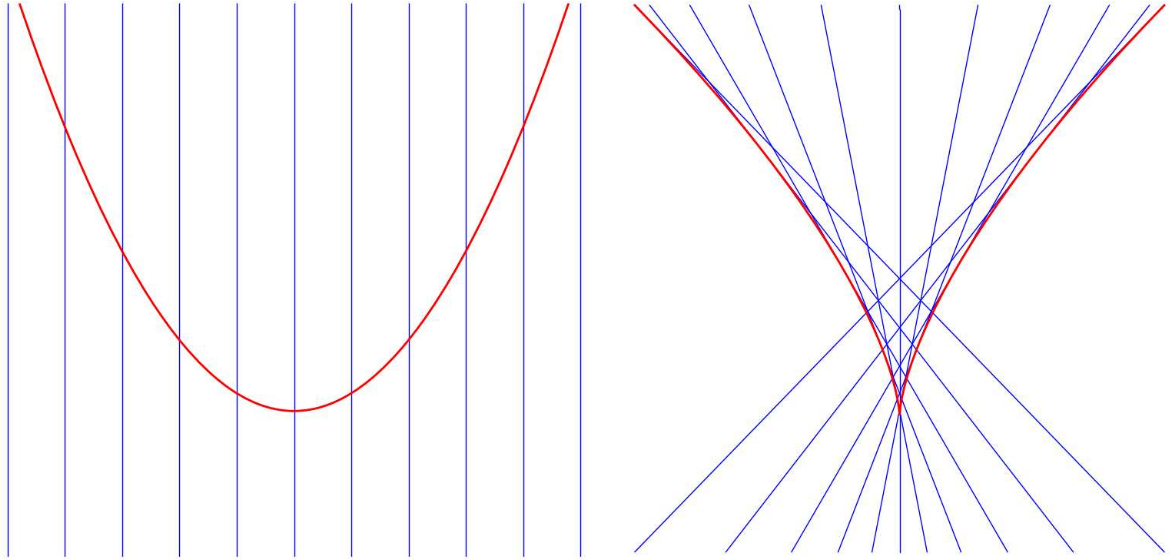
We choose C_x such that $C_x \cdot r_x < 0$

Conjugate flow

A **conjugate descending curve (CDC)** is *an integral curve of the vector field* C_x. The curve stays within the set of conjugate points, and can be continued until it hits a point that is not A_2.

The most simple and most important case is that the conjugate flow hits an $\boldsymbol{A_3}$ point. For A_3 points, e_1 is given in adapted coordinates by:

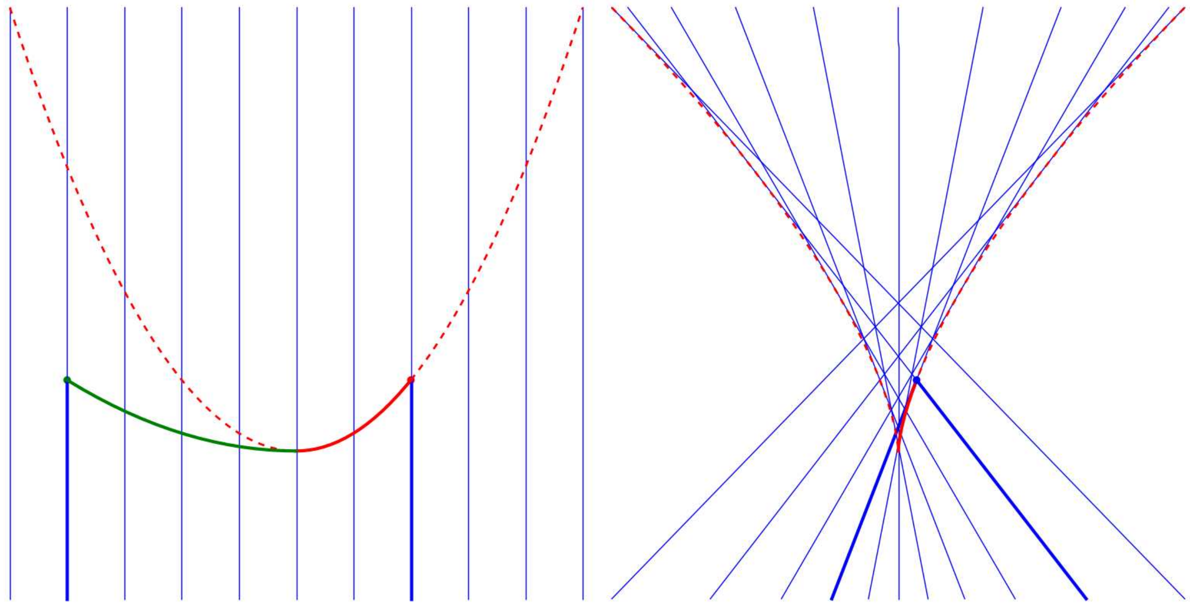
$$(x_1, ..., x_{n-1}, x_n) \to (x_1, ..., x_{n-1}, x_n^3 + x_1 x_n)$$



Replying a CDC

Once the CDC reaches an A_3 point, we can find a **reply** to the CDC: it is a curve of NC points whose image by e_1 is the same as the CDC, but run in the opposite direction.

The concatenation of both segments is a curve **whose image is tree-formed**.



A CDC α is **unbeatable**: suppose β replies to α:

$$|\alpha(0)| - |\alpha(t_0)| = \text{length}(\exp \circ \alpha) = \text{length}(\exp \circ \beta) > |\beta(t_0)| - |\beta(0)|$$

Building linking curves

Remember: *our goal is to show our A_2 point x is linked to an unequivocal point y*

Starting at x, suppose we follow the CDC up to an A_3 point. We can keep replying as long as the reply stays within V_1, but **we may hit a singularity**. If that happens, we descend along the conjugate descending flow again, etcetera:

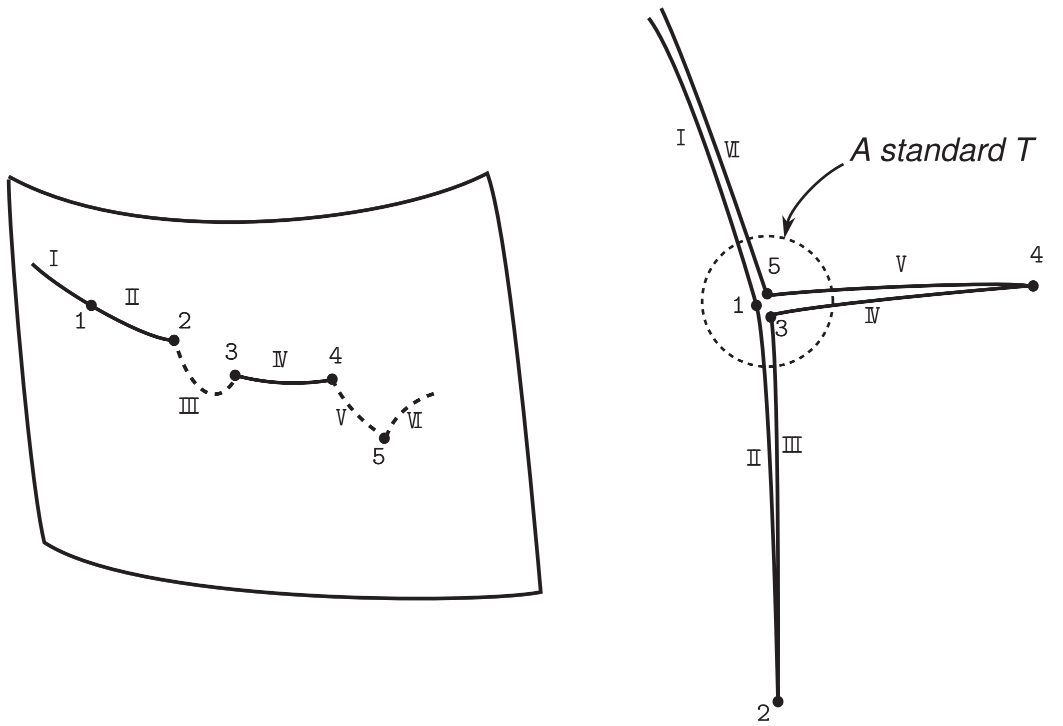
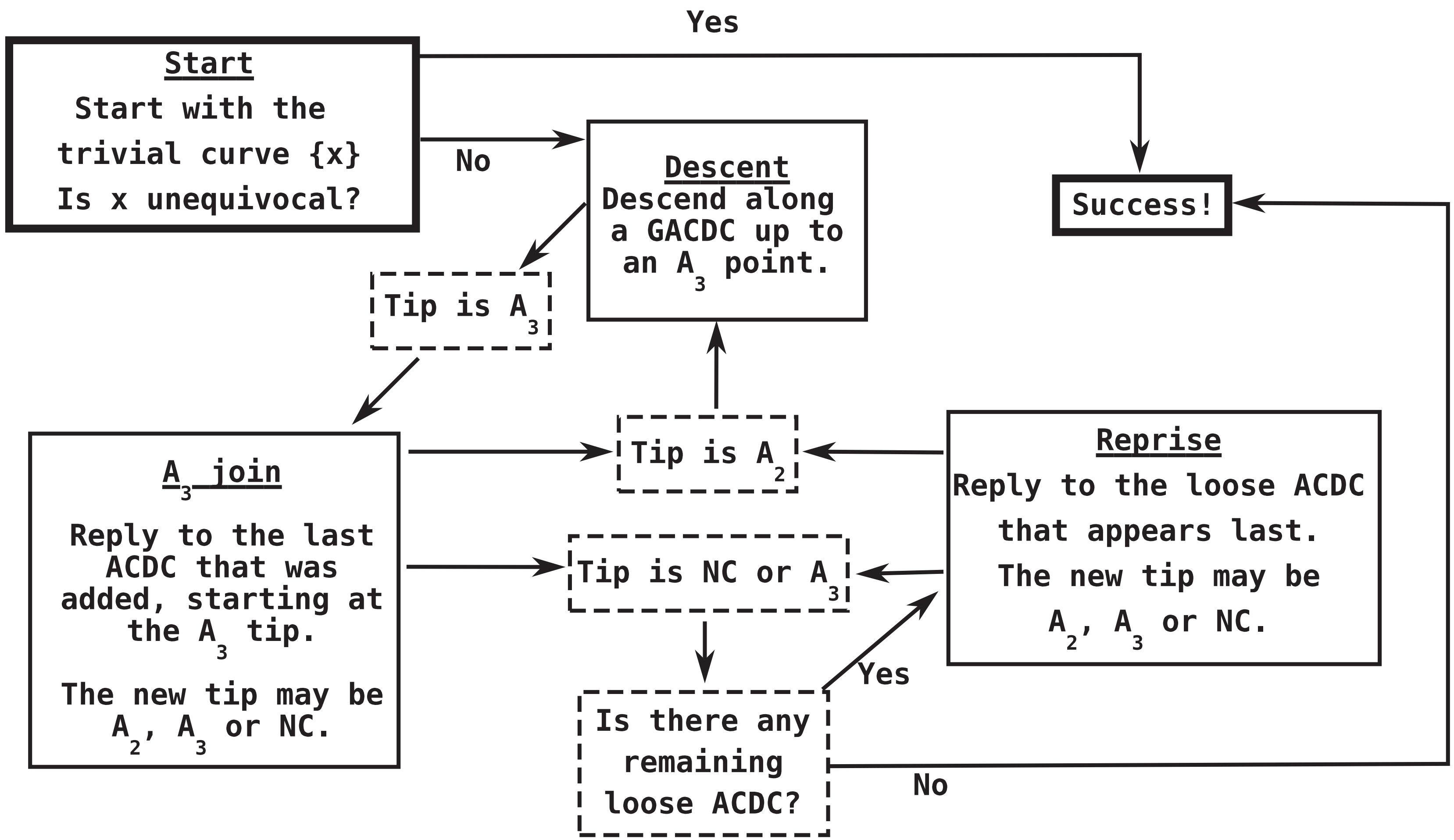


Figure. Splitter (1), A_3-join (2,4), hit(3), reprise(5)

The algorithm

We summarize the procedure in this flow diagram:



Does it work?

- *Do we always reach an A_3 point?* Yes, we can «dodge» other singularities.

- *How long can we keep replying?* We can always reply to an A_2 point, since the «linking curve under construction» always stays within V_1.

- *What if the reply hits a worse singularity?* We can also dodge that.

- *How do you know the procedure will ever stop?* Because every point in a generic manifold has a «**transient neighborhood**»: if the algorithm starts at a point on that neighborhood, after a finite number of elementary steps of the algorithm, the tip will never again be there (the *unbeatable* property is key).

We show that it works for generic metrics, in 3 dimensions. The algorithm can also start at $\boldsymbol{A_3}$**(II)**, $\boldsymbol{A_4}$, $\boldsymbol{D_4^+}$ and $\boldsymbol{D_4^-}$ points.

Points of type $\boldsymbol{A_3}$**(I)** are unequivocal.

Summary

- For a generic manifold, $V_1 \subset \mathrm{NC} \cup A_2 \cup A_3(I) \cup A_3(\mathrm{II}) \cup A_4 \cup D_4^{\pm}$
- $\mathcal{I} = V_1 \cap (\mathrm{NC} \cup A_3(I))$ $\quad$ $\mathcal{J} = V_1 \cap (A_2 \cup A_3(\mathrm{II}) \cup A_4 \cup D_4^{\pm})$

Theorem 7. *A point in $\mathcal{I}$ is unequivocal; a point in $\mathcal{J}$ is linked to a point of $\mathcal{I}$ of smaller radius.*

Corollary 8. *Ambrose conjecture holds for a generic 3-manifold.*

Thanks for your attention!

- Pablo Angulo. Linking curves, sutured manifolds and the Ambrose conjecture for generic 3-manifolds. arxiv.org/abs/1509.02125

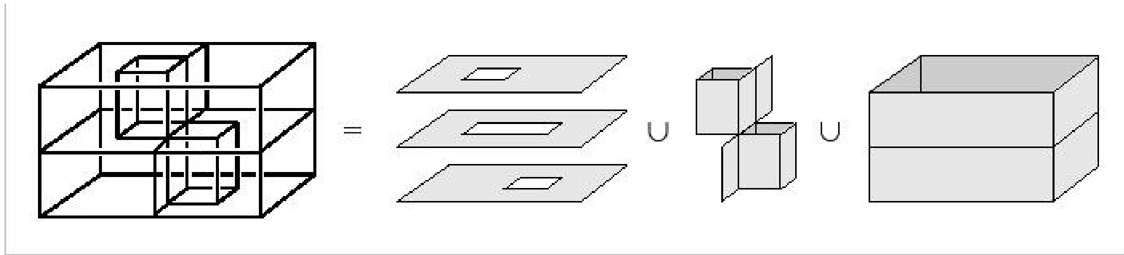
#

FAQ #1 Why can't you extend the proof of James Hebda to three dimensions?

Answer: You mean: Can we find the curve Y in dimension $\geqslant$3?

In **dimension $\geqslant$3** there are many possible choices for Y, none of them is *canonical*.

But it's worse than that: the house with two rooms is the cut locus of a certain manifold, and it doesn't have edge points!

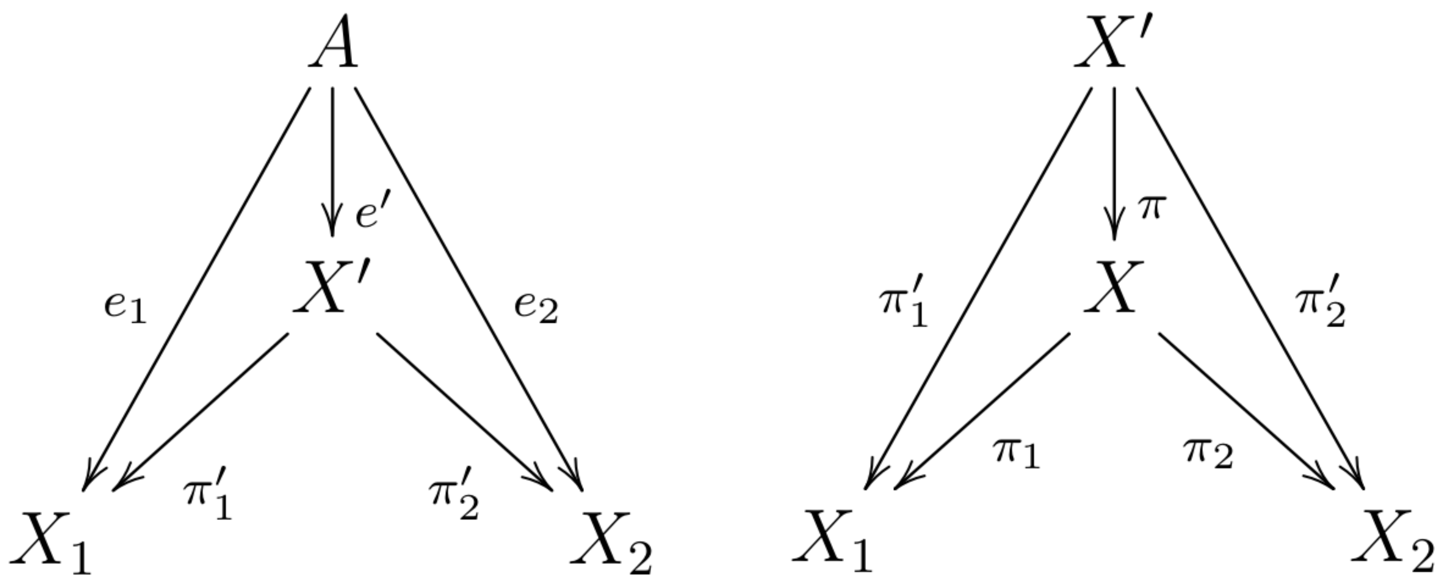


Weinstein showed that every manifold except S^2 admits a metric such that the cut locus wrta point doesn't have edge points.

There cannot be non-trivial linking curves contained in the cut locus of such manifolds.

FAQ #2 Is the synthesis the «minimal common covering space»?

Yes, it satisfies a universal property **(5.4.7)**: Let X be the synthesis of X_1 and X_2. For any Riemannian manifold X', continuous surjective map $e'\colon A \to X'$ and local isometries $\pi_1'\colon X' \to X_1$ and $\pi_2'\colon X' \to X_2$, such that $e_i = \pi_i' \circ e'$, for $i = 1, 2$, there is a local isometry $\pi\colon X' \to X$ such that:



FAQ #3 What is the motivation for studying Ambrose Problem?

Ambrose's motivation (roughly): to characterize a Riemannian manifold by the parallel transport of its curvature: $L\colon \mathbb{R}^n \times G_2(\mathbb{R}^n) \to \mathbb{R}$

$L(V,P)$: parallel transport the plane P along the geodesic g_V and compute the sectional curvature of the plane.

My motivation: The Ambrose problem is similar to some *inverse problems* in Riemannian geometry. Some of these problems come directly from applications like tomography. Some of those problems are easy to solve if there are no singularities. I planned to build some muscle and try other such inverse problems later.

FAQ #4 Can you bound the lenght of the linking curves and approximate an arbitrary metric by generic ones?

Not without new ideas. I've tried two things:

1. As the *slack* of an A_2 point (the angle between the kernel of the exponential and the tangent hyperplane to the conjugate points) goes to 0, and x becomes a worse singularity, the *gain* of the CDC α through x decreases. It's true that the lenght of its composition with the exponential also decreases, but overall, nor the lenght of α neither that of $e_1 \circ \alpha$ is bounded.

2. Try to put numbers to the algorithm: Let B_R be the maximum lenght of a linking curve through a point x of radius R. The algorithm starts with a CDC α of lenght l that leaves a transient neighborhood U of x. Then a linking curve starting at the tip of α follows. Its lenght is bounded by B_{R-l}. And after that, we have to reply to α. If we can reply to α at once:

   $$B_R < l + B_{R-l}$$

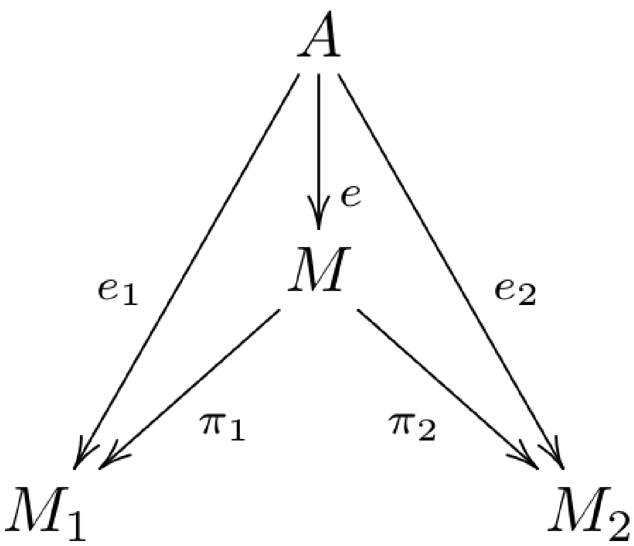
   but the reply might hit an A_2 point, and then we have to reply to part of α, then plug in another linking curve at the tip of the reply, then reply the rest... if there are k interruptions:

   $$B_R < 2\,l + k\,B_{R-\varepsilon}$$

   where ε is the *gain* of the transient neighborhood. This is exponential growth.

FAQ #5 Why "***strong*** synthesis"? Is there a "***weak synthesis***"?

Yes, if π_1 and π_2 are local isometries, but not covering maps:



The corresponding theorem is:

Theorem 9. *Let M_1, M_2 be Riemannian manifolds with L-related curvature, such that* ***every $x \in V_1$ is linked to some unequivocal $y \in V_1$****.*

Then there is a ***strong synthesis*** *of M_1 and M_2 through e_1 and e_2.*

In short: if we do not assume we can find y with $|y| \leqslant |x|$, then π_1 and π_2 may not be covering maps.

FAQ #6 Why "***strongly*** linked"? Are there "***weakly linked" points***?

Yes, but that is rather technical. Let me just say that there are other hypothesis that imply:

$$e_1(x) = e_1(y),\ e_2(x) = e_2(y)$$