Necessary, and often sufficient, conditions for the existence of Conformal Factorizations

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$$-\Delta_g u + qu = 0, \quad u_{|\partial\Omega} = f$$
$$\Lambda_{g,q}(f) = (\nabla u \cdot n)_{|\partial\Omega}$$

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Geometric Calderón Problem (one version of it)

Let (M, g) be a Riemannian manifold and $\Omega \subset M$ an open set:

$$-\Delta_g u + qu = 0, \quad u_{|\partial\Omega} = f$$

$$\Lambda_{g,q}(f) = (\nabla u \cdot n)_{|\partial\Omega}$$

If we know $\Lambda_{g,q} : H^{\frac{1}{2}} \to H^{-\frac{1}{2}}$. Can we recover the potential q?

History

- 81 Calderón rediscovered the problem.
- 87 $n = 3$ Sylvester-Uhlmann $g = g_E\ \rho \in C^2$. Introduced **Complex Geometric Optics** (CGO) solutions.
- 2006 $n = 2$ Astala-Päivärinta: Ok for $\rho \in L^\infty$ (Scattering transform method plus quasiconformal maps)
- 2014 Caro-Rogers $n = 3$: ρ Lipschitz (Sylvester-Uhlmann + Haberman Tatar)
- 2006 **Dos Santos Ferreira-Kenig-Salo-Uhlman**. CGO method might work if ($\sim$ and only if $\sim$) (M, g) admits a limiting Carleman weight (LCW).
- 2010 Liimatainen Salo: "generic" metrics do not admit LCWs.

Limiting Carleman weights

Let (M, g) a Riemannian metric. A **limiting Carleman weight** is a function whose gradient is parallel in a conformally equivalent metric. A vector field X is parallel if for every $Y \in \mathfrak{X}(M)$

$$\nabla_X(Y) = 0$$

g admits a limiting Carleman weight φ *if and only if* there exists local coordinates such that $\partial_1 = \nabla\varphi$ and

$$g(x) = e^{2f(x)} \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}$$

In other words, near each point, $g = e^{2f}(e \oplus g_0)$ where g_0 is the metric of an $(n-1)$-manifold, and e is the euclidean metric in $\mathbf{R}$.

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how can I know if a conformal multiple of the metric is a product?

Famous tensors

R=Curvature tensor, Ric=Ricci Tensor, s=Scalar Curvature, W=Weyl Tensor, C=Cotton tensor
Schouten Tensor

$$S = \frac{1}{n-2}\left(Ric - \frac{1}{2(n-1)}sg\right) \tag{1}$$

and

$$R = W + S \owedge g \tag{2}$$

where $\owedge$ is the Kulkarni-Nomizu product of two symmetric 2-tensors which is defined by

$$(\alpha \owedge \beta)_{ijkl} = \alpha_{ik}\beta_{jl} + \beta_{ik}\alpha_{jl} - \alpha_{il}\beta_{jk} - \alpha_{jk}\beta_{il}$$

and R and W are understood as $(0,4)$ tensors. The Cotton tensor:

$$C_{ijk} = (\nabla_i S)_{jk} - (\nabla_j S)_{ik} \tag{3}$$

Conformally invariant tensors

- If $n = 3$, then $W = 0$.
- If $n = 4$, then $(n-3)C = div(W)$.
- The $(1,3)$ version of the Weyl tensor is conformally invariant.
- If $n = 3$, and $\tilde{g}$ is a multiple of g, the Cotton tensor of $\tilde{g}$ is a multiple of the Cotton tensor of g.
- If $n \geq 4$, $W = 0$ implies g is *conformally flat*.
- If $n = 3$, $C = 0$ implies g is *conformally flat*.

Curvature operators

Simmetries of Curvature allow to interpret Curvature as Curvature Operators. i,e elements of $S^2(\Lambda^2(TM))$:

$$R(x \wedge y, z \wedge t) = R(x, y, z, t)$$

- *Curvature Operators:* $\mathcal{R} = ker(b)$. ($b : S^2(\Lambda^2(TM)) \to \Lambda^4(TM)$ is the Bianchi Operator).
- *Weyl Operators:* $\mathcal{W} = \mathcal{R} \cap ker(r)$ ($r : S^2(\Lambda^2(TM)) \to S^2(TM)$ is the Ricci Operator).

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Definition

W satisfies the Eigenflag property with eigenflag direction $v \in TM$ if

$$W(v \wedge v^{\perp}) \subset v \wedge v^{\perp}$$

$$\mathcal{EW} = \{W \in \mathcal{W} : W \text{ satisfy Eigenflag for some } v\}$$

Theorem

Let (M, g) be a Riemannian manifold of dimension $n \geq 4$ which admits a LCW. Then

$$W_g(p) \in \mathcal{EW}$$

The parallel vector field is an eigenflag direction.

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Proof: Show that this is true if g admits a parallel vector field and use the conformal invariance of the Weyl Tensor.

- A bivector $\omega \in \Lambda^2(TM)$ is simple if $\omega = v \wedge w$ for $v, w \in TM$. $e_1 \wedge e_2 + e_3 \wedge e_4$ is not simple.
- (M, g) be a 4 dimensional Riemannian manifold such that some $\tilde{g} \in [g]$ admits a parallel vector field. Then all the eigenvectors of the Weyl operator of g are simple.

Theorem

The complex projective plane (CP^2) with its canonical metric does not have a limiting Carleman weight.

Consider the following basis of non simple eigenvectors.

$$\phi_1 = e_1 \wedge e_2 + e_3 \wedge e_4, \quad \phi_2 = e_1 \wedge e_3 - e_2 \wedge e_4, \quad \phi_3 = e_1 \wedge e_4 + e_2 \wedge e_3, \tag{4}$$

for its self-dual component, $\Lambda^+ = \langle \phi_1, \phi_2, \phi_3 \rangle$ and

$$\psi_1 = e_1 \wedge e_2 - e_3 \wedge e_4, \quad \psi_2 = e_1 \wedge e_3 + e_2 \wedge e_4, \quad \psi_3 = e_1 \wedge e_4 - e_2 \wedge e_3, \tag{5}$$

$\Lambda^- = \langle \psi_1, \psi_2, \psi_3 \rangle$
Then W diagonalize in this basis with eigenvalues $(4, -2, -2, 0, 0, 0)$ i.e the eigenspaces have no simple bivectors !!

In dimension 3 we should try to read the result from the Cotton tensor. By its simetries the Cotton tensor is equivalent to a new (2,0) tensor called the Cotton York tensor (By the Hodge Operator)

$$CY_{ij} = \frac{1}{2} C_{kli} g_{jm} \frac{\epsilon^{klm}}{\sqrt{\det g}} = g_{jm} \left(\nabla_k S \right)_{li} \frac{\epsilon^{klm}}{\sqrt{\det g}} \tag{6}$$

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Theorem

Let $n = 3$. If a metric $\tilde{g} \in [g]$ admits a parallel vector field, then for any $p \in M$, there is a tangent vector $v \in T_pM$ such that

$$CY_p(v, v) = CY_p(w_1, w_2) = 0$$

for any pair of vectors $w_1, w_2 \in v^{\perp}$.
This is equivalent to $det(CY_p) = 0$.

Hint: the parallel vector field is v.

Theorem

Among the eight Thurston geometries, only Nil *and* $\mathrm{SL}(2,\mathbf{R})$ *do not admit limiting Carleman weights while the other six are locally conformal to products of* $\mathbf{R}$ *and a surface.*

Example: $\mathrm{SL}(2,\mathbf{R})$ with its left invariant metric, Iwasawa decomposition

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$CY_{(\theta,0,0)} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

with non-zero determinant. Since the metric is left invariant, the same happens at any other point.

Size of Eigenflag Weyl Operators

Theorem

The set $\mathcal{EW}$ of Weyl tensors that satisfy the eigenflag condition is a semialgebraic subset of the space of Weyl tensors with codimension

$$\frac{1}{3}n^3 - n^2 - \frac{4}{3}n + 2.$$

In particular, the codimension is 2 for $n = 4$ and 12 for $n = 5$.

Metrics whose Weyl operator has the eigenflag property

Theorem

The set of metrics which are not locally conformal to a product at any point in M contains and open and dense set.

Proof: the set of all Weyl tensors with the eigenflag property is a *stratified bundle*. The set of metrics for which the map $p \to W_p$ is transverse to this bundle is open and dense.

Is it a sufficient Condition?

More careful analysis of dimension 4.

Lema

$n = 4$ $W \in \mathcal{EW}$. Then there are at most three eigenvalues of multiplicity at least 2 which add up to 0

- If the three eigenvalues are distinct then there are exactly four orthogonal eigenflag directions.
- If two eigenvalues coincide we have two orthogonal planes of eigenflag directions.
- otherwise $W = 0$

If (M, g) is conformal to a product of surfaces, the Weyl tensor has two planes of eigenflag directions.

Criteria for being conformal to a product, *given the candidate distributions*

Theorem

Let (M, g) be a Riemannian metric, and D_1 and D_2 be two orthogonal integral distributions. The following are equivalent:

- *g is locally conformal to the product of a metric on an integral leave of D_1 and an integral leave of D_2*
- *D_1, D_2 are umbilical with mean curvature normals η_1, η_2 and*
$$d(\eta_1 + \eta_2)^\flat = 0$$

Product of Surfaces

Theorem

$(S1, g_1) \times (S_2, g_2)$ is conformally flat iff S_1 has contant curvature c and S_2 has constant curvature $-c$.
If $(S1, g_1) \times (S_2, g_2)$ is not flat, it admits a LCW if and only if (S_1, g_1) or (S_2, g_2) is locally isometric to a surface of revolution.

Ex: The product of two scalene ellipsoids with the metric induced by $\mathbb{R}^3$ does no admit LCWs but has the eigenflag property.

A curious example

On $U \subset \{(t, x, y, z) \in \mathbf{R}^4 : x > 0\}$, define:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x^2 \end{pmatrix} \tag{7}$$

- The Weyl tensor of g has the eigenflag property at every point, with three different eigenvalues. Hence, the metric is not locally conformal to a product of surfaces.
- The eigenflag directions of the Weyl tensor are spanned by the coordinate vector fields.
- The functions t, y, z are the only LCWs.

Another curious example

On $U \subset \{(t, x, y, z) \in \mathbf{R}^4 : x > 0\}$, define:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x^3 & 0 \\ 0 & 0 & 0 & 1/x \end{pmatrix} \tag{8}$$

- The Weyl tensor of g has the eigenflag property at every point, with only two different eigenvalues. However, the metric is not locally conformal to a product of surfaces.
- The functions t, y, z are the only LCWs.

Proof: The Weyl tensor identifies the distributions D_1 and D_2. Apply the criteria for being conformal to a product to find out M is not a product along D_1 and D_2. Then a more careful analysis is required to find out the LCWs.

Thanks!

- Pablo Angulo, Daniel Faraco, Luis Guijarro: Sufficient conditions for the existence of limiting Carleman weights. http://arxiv.org/abs/1603.04201
- Pablo Angulo: On the set of metrics without local limiting Carleman weights. http://arxiv.org/abs/1509.02127
- Pablo Angulo, Daniel Faraco, Luis Guijarro and Alberto Ruiz: Obstructions to the existence of limiting Carleman weights. http://arxiv.org/abs/1411.4887 (Analysis and PDE 9-3 (2016), 575–59)
- This can be done using **SageManifolds** or some other Computer Algebra Software.