

Singularity-free cylindrical cosmological model

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Abstract. A cylindrically-symmetric perfect-fluid spacetime with no curvature singularity is shown. The equation of state for the perfect fluid is that of a stiff fluid. The metric is diagonal and non-separable in comoving coordinates for the fluid. It is proven that the spacetime is geodesically complete and globally hyperbolic.

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1. Introduction

Due to the powerful singularity theorems (cf, for instance, [1]), it was widely believed that cosmological models must have an initial singularity. However, in 1990 Senovilla [2] showed the first cosmological perfect-fluid solution of the Einstein equations with regular scalar curvature invariants. This corresponds to a cylindrically-symmetric spacetime filled with an isotropic radiation perfect fluid. In [3] it was shown that this spacetime is indeed singularity free. This is in full accordance with the singularity theorems, as is to be expected, since although it fulfils the energy, generic and causality conditions, it does not have any of the causally trapped sets required by the theorems (closed trapped surfaces, compact achronous sets without edge, . . .). This special solution was generalized in a subsequent paper [4], where an ansatz of separability of variables for diagonal orthogonally transitive commuting G_2 metrics was extensively explored. A thorough discussion of this ansatz can be found in [5]. This family is shown to be included in a wider class of separable cosmological models, which comprises FLRW universes [6]. Other properties of these solutions, such as their inflationary behaviour, their generalized Hubble law or the feasibility of constructing a realistic non-singular cosmological model, are studied therein. Other non-singular cylindrically-symmetric perfect-fluid spacetimes have been found [7, 8]. Together with some solutions in [9], these are the only non-diagonal ones that are known. Furthermore, in the family of solutions included in [10], there are non-separable non-singular perfect-fluid spacetimes, with and without a symmetry axis [11].

In this paper we present a cylindrically-symmetric stiff perfect-fluid solution which is non-separable in comoving coordinates and has non-singular scalar curvature invariants. Due to these facts, it is not included in the families quoted before. The physical properties of these metrics and their relation to other families of solutions are investigated.

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2. The metric

The line element for a spacetime that admits an Abelian two-dimensional orthogonally transitive group of isometries acting on spacelike surfaces can be written in the form

$$ds^2 = e^{K(t,r)}(-dt^2 + dr^2) + e^{-U(t,r)} dz^2 + e^{U(t,r)} r^2 d\phi^2, \quad (1a)$$

where ϕ and z are coordinates adapted to the commuting generators (in this chart, $\xi = \partial_\phi$, $\zeta = \partial_z$) of the group. The remaining coordinates, $\{t, r\}$, have been chosen so that the line element induced in the subspaces orthogonal to the Killing orbits is isotropic in these coordinates.

Following [11], if we are to have a regular symmetry axis on the locus where $\Delta = g(\xi, \xi) = 0$, then we have to impose that

$$\frac{g(\text{grad } \Delta, \text{grad } \Delta)}{4\Delta} \rightarrow 1, \quad (1b)$$

on approaching the axis.

The metric functions for the spacetime that we want to show are given in this set of coordinates by

$$K(t, r) = \frac{1}{2}\beta^2 r^4 + (\alpha + \beta)r^2 + 2t^2\beta + 4t^2\beta^2 r^2 \quad (1c)$$

$$U(t, r) = \beta(r^2 + 2t^2) \quad (1d)$$

$$-\infty < t, z < \infty, \quad 0 < r < \infty, \quad 0 < \phi < 2\pi \quad (1e)$$

and are easily checked to satisfy the regularity condition (1b) on the set of events defined by $r = 0$. There are no further isometries other than the ones that have been implemented from the beginning, and therefore this spacetime is properly said to admit cylindrical symmetry. The only restriction that we impose on the parameters α and β is that both of them are positive.

The matter content in this spacetime is a stiff perfect fluid, whose density and pressure,

$$\mu = p = \alpha e^{-K(t,r)}, \quad (2)$$

are regular everywhere in this chart. When α equals zero we have a vacuum solution, from which the stiff-fluid metric can be recovered by means of the Wainwright–Ince–Marshman algorithm [12]. However, this solution has not been obtained by this method, although it could be used, as in [8], to generate further models with the same vacuum metric.

The velocity of the fluid takes the form,

$$u = e^{-\frac{1}{2}K(t,r)} \partial_t, \quad (3)$$

and therefore the coordinates are comoving. From the expression for the line element (1a) it is clear that it cannot be rendered separable in comoving coordinates.

The acceleration of the fluid has a projection only on the radial direction,

$$a = r(\beta^2 r^2 + \alpha + \beta + 4\beta^2 t^2) \partial_r, \quad (4)$$

due to the orthogonal transitivity requirement and the fact that the velocity is orthogonal to the orbits of the group of isometries. For the same reason the vorticity is zero.

In an orthonormal coframe $\{\theta^0, \theta^1, \theta^2, \theta^3\}$, where the four independent differential 1-forms take the form,

$$\theta^0 = e^{\frac{1}{2}K(t,r)} dt, \quad \theta^1 = e^{\frac{1}{2}K(t,r)} dr, \quad \theta^2 = e^{-\frac{1}{2}U(t,r)} dz, \quad \theta^3 = e^{\frac{1}{2}U(t,r)} r d\phi, \quad (5)$$

the shear tensor constructed with the derivatives of the velocity u can be written as,

$$\sigma = \frac{4}{3}\beta t e^{-\frac{1}{2}K(t,r)} \left\{ (1 + 2\beta r^2)\theta^1 \otimes \theta^1 - (2 + \beta r^2)\theta^2 \otimes \theta^2 + (1 - \beta r^2)\theta^3 \otimes \theta^3 \right\}. \quad (6)$$

The expression for the expansion, Θ , of the cosmological fluid,

$$\Theta = 2\beta t(1 + 2\beta r^2) e^{-\frac{1}{2}K(t,r)}, \quad (7)$$

allows us to calculate the deceleration parameter, q , for this universe,

$$q = 2 - \frac{3}{2} \frac{1}{\beta t^2(1 + 2\beta r^2)}, \quad (8)$$

from the action of the vector field u on the inverse of Θ , [13]

$$u \left(\frac{1}{\Theta} \right) = \frac{1}{3}(1 + q). \quad (9)$$

Since q is positive in the time span $(-t_{inf}, t_{inf})$,

$$t_{inf} = \sqrt{\frac{3}{4\beta(1 + 2\beta r^2)}}, \quad (10)$$

we are led to conclude that this spacetime has an inflationary epoch, which is longer the closer the observer is to the symmetry axis. Furthermore, the scale factor of the universe can be defined [13] as a solution of the differential equation,

$$u(R) = \frac{1}{3}\Theta R, \quad (11)$$

which in our case can be solved, introducing an arbitrary function C of the spacelike coordinates,

$$R(t, r, z, \phi) = C(r, z, \phi) e^{\frac{1}{3}\beta t^2(1+2\beta r^2)}. \quad (12)$$

3. Regularity of the metric

Since the scalar invariants that can be formed with the metric and the Riemann curvature are polynomials of the density and the pressure of the fluid and the components of the Weyl tensor, we shall calculate the latter ones in order to show that the curvature scalars are not singular. In the null tetrad that can be naturally constructed with the 1-forms of (5), the components of the Weyl tensor can be shown to be,

$$\Psi_0 = \frac{1}{2} (f_1(t, r) + f_2(t, r)) e^{-K(t,r)} \quad (13a)$$

$$\Psi_1 = 0 \quad (13b)$$

$$\Psi_2 = \frac{1}{6} (3\beta^2 r^2 + 3\beta - \alpha - 12\beta^2 t^2) e^{-K(t,r)} \quad (13c)$$

$$\Psi_3 = 0 \quad (13d)$$

$$\Psi_4 = \frac{1}{2} (f_1(t, r) - f_2(t, r)) e^{-K(t,r)}, \quad (13e)$$

$$f_1(t, r) = -3\beta + 2\beta^3 r^4 + 3\beta^2 r^2 + \alpha(1 + 2\beta r^2) + 24\beta^3 r^2 t^2 + 12\beta^2 t^2 \quad (13f)$$

$$f_2(t, r) = 12\beta^3 r^3 t + 12\beta^2 r t + 16\beta^3 t^3 r + 4\alpha\beta t r. \quad (13g)$$

From the expressions for the components of the Weyl tensor and the density and the pressure of the fluid, it follows that they vanish when either t or r tend to infinity. Hence the spacetime is flat for large values of the time and radial coordinate. It is also easy to check

that the Weyl components are regular everywhere and so are the pressure and the density. Therefore all the curvature invariants are regular. The spacetime has a low matter content for large negative values of the time coordinate and undergoes a contracting epoch until t reaches the zero value. For positive values of t this universe is expanding. The change from contraction to expansion without developing a curvature singularity also appears in other non-singular cosmological models [2, 6, 7].

The gradient of the comoving time coordinate t is always negative. It is therefore a cosmic time [1] and the spacetime is causally stable. In particular, this implies weaker causality conditions such as the chronology condition.

The strong and dominant energy conditions are satisfied, since the density of the fluid is positive everywhere and the equation of state corresponds to a stiff fluid. Moreover, the energy–momentum tensor does not vanish anywhere. This means that for every non-spacelike vector W the contraction with the Ricci tensor $R(W, W)$ is greater than zero, and it implies that the generic condition is also satisfied [14].

There is a static limit for the metric, which amounts to taking the parameter β equal to zero. The resulting metric,

$$ds^2 = e^{\alpha r^2} (-dt^2 + dr^2) + dz^2 + r^2 d\phi^2, \quad (14)$$

can be seen to be also the static limit of [7].

4. Geodesic completeness

In order to determine whether the spacetime is geodesically complete, we have to study the equations for the causal geodesics,

$$\nabla_v v = 0, \quad v = \dot{t} \partial_t + \dot{r} \partial_r + \dot{z} \partial_z + \dot{\phi} \partial_\phi, \quad (15)$$

where v is the tangent vector along the geodesic and the dot stands for the derivative with respect to the affine parameter τ . We shall introduce a constant of motion δ , which takes the zero value for null geodesics and one for timelike geodesics, that is,

$$g(v, v) = -\delta. \quad (16)$$

The existence of isometries simplifies the problem, since two new independent constants of motion arise,

$$L = g(v, \partial_\phi) = r^2 e^{U(t,r)} \dot{\phi}, \quad Z = g(v, \partial_z) = e^{-U(t,r)} \dot{z}, \quad (17)$$

as a consequence of the geodesic and the Killing equation.

Since (16) can be cast in the form,

$$\dot{t}^2 - \dot{r}^2 = e^{-K(t,r)} \left\{ \delta + Z^2 e^{U(t,r)} + L^2 e^{-U(t,r)} r^{-2} \right\}, \quad (18)$$

it is convenient to parametrize \dot{t} and \dot{r} with hyperbolic functions of a new variable, ξ , as is done in [3], in order to lower the order of the geodesic equations. The final system of equations for future-directed geodesics (past-directed ones can be handled in a similar way) is first order,

$$\dot{\xi} = -e^{-\frac{1}{2}K(t,r)} \left\{ \frac{r \cosh \xi (Z^2 A(t,r) + L^2 B(t,r) + \delta C(t,r))}{\sqrt{G(t,r)}} + \frac{t \sinh \xi (Z^2 D(t,r) + L^2 E(t,r) + \delta F(t,r))}{\sqrt{G(t,r)}} \right\} \quad (19a)$$

$$\dot{t} = e^{K(t,r)/2} G(t,r) \cosh \xi \quad (19b)$$

$$\dot{r} = e^{K(t,r)/2} G(t, r) \sinh \xi \quad (19c)$$

$$\dot{z} = e^{U(t,r)} Z \quad (19d)$$

$$\dot{\phi} = e^{-U(t,r)} r^{-2} L, \quad (19e)$$

and the last two equations are just quadratures that can be integrated after solving the first three equations. We have introduced seven functions of t and r ,

$$A(t, r) = (\beta^2 r^2 + 4\beta^2 t^2 + \alpha + 2\beta) e^{U(t,r)} \quad (20a)$$

$$B(t, r) = \left(\beta^2 - \frac{1}{r^4} + \frac{\alpha + 4\beta^2 t^2}{r^2} \right) e^{-U(t,r)} \quad (20b)$$

$$C(t, r) = \beta^2 r^2 + \alpha + \beta + 4\beta^2 t^2 \quad (20c)$$

$$D(t, r) = 4\beta(1 + \beta r^2) e^{U(t,r)} \quad (20d)$$

$$E(t, r) = 4\beta^2 e^{-U(t,r)} \quad (20e)$$

$$F(t, r) = 2\beta(1 + 2\beta r^2) \quad (20f)$$

$$G(t, r) = \delta + Z^2 e^{U(t,r)} + \frac{L^2 e^{-U(t,r)}}{r^2}. \quad (20g)$$

The only geodesics that fall out of this scheme are radial null geodesics, $\delta = L = Z = 0$. For them the analysis is fairly simple, since $\dot{t} = |\dot{r}|$ and the only equation that is left for integration is,

$$\ddot{r} + 2r(\beta^2 r^2 + \alpha + \beta + 4\beta^2 t^2) \dot{r}^2 + 4\beta t(1 + 2\beta r^2) \dot{r} = 0, \quad (21)$$

which can be seen to have a first integral, h ,

$$h = \dot{r} e^{2\beta t^2(1+2\beta r^2) + (\alpha+\beta)r^2 + \frac{1}{2}\beta^2 r^4}. \quad (22)$$

The completeness of these geodesics is obvious, since \dot{r} is confined finite in the interval $[-|h|, |h|]$.

Going back to the generic equations (19), we shall explore the appearance of divergences that may occur when the radius r is either too small or too large or when the time coordinate t is too large.

If the radius, r , grows too large, so that $r^4 > \beta^{-2}$, $\dot{r} > 0$ ($\sinh \xi > 0$), then $\dot{\xi}$ is negative when the time coordinate, t , is positive. Therefore ξ decreases and r does not diverge for finite affine parameter since \dot{r} cannot grow arbitrarily large. If t is negative, since the time coordinate grows at least as fast as r , t becomes positive before r diverges.

The terms with negative powers of r do not yield singularities when the geodesics approach the axis ($\dot{r} < 0$, $\sinh \xi < 0$). The reason is that ξ becomes positive when the geodesic is close to $r = 0$ and prevents the radial coordinate from decreasing too quickly. This happens because the only negative terms in the expression of $\dot{\xi}$ are either bounded or overcome by the term in r^{-4} in $B(t, r)$ for decreasing radial coordinate.

Also if t grew too fast, this would mean that r would grow or decrease quickly and the previous reasonings would prevent divergences from appearing.

Hence we are led to conclude that every causal geodesic is complete and therefore the spacetime is *geodesically complete*.

From this analysis it follows that every null geodesic intersects once and only once every timelike hypersurface $t = \text{constant}$. Therefore these hypersurfaces are Cauchy surfaces and the spacetime is globally hyperbolic [15].

5. Discussion

It has been shown that the spacetime is singularity free and globally hyperbolic. Since energy, generic and causal conditions are fulfilled, it is clear that this spacetime does not have causally trapped sets [1, 14]. The gradient of the transitivity surface element is spacelike as in all known non-singular models.

The question of whether there is an open set of non-singular cosmological models in a reasonable topology remains open. However, the fact that both this spacetime and the one shown in [7] have the same static limit suggest that more solutions could be found by adding suitable new parameters to the static solution.

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