Class. Quantum Grav. **13** (1996) 753–764. Printed in the UK

Magnetic surfaces in stationary axisymmetric general relativity

L Fernández-Jambrina
Departamento de Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, E-28040-Madrid, Spain

Received 14 July 1995, in final form 12 January 1996

Abstract. A new method is derived for constructing electromagnetic surface sources for stationary axisymmetric electrovac spacetimes endowed with non-smooth or even discontinuous Ernst potentials. This can be viewed as a generalization of some classical potential theory results, since lack of continuity of the potential is related to dipole density and lack of smoothness, to monopole density. In particular, this approach is useful for constructing the dipole source for the magnetic field. This formalism involves solving a linear elliptic differential equation with boundary conditions at infinity. As an example, two different models of surface densities for the Kerr–Newman electrovac spacetime are derived.

PACS numbers: 0420C, 0420J, 0440N

1. Introduction

One of the major challenges of general relativity is the description of a compact rotating material gravitational source and the vacuum surrounding it. Its astrophysical interest is clear, since it would be useful for modelling relativistic rotating stars and galaxies. Nevertheless, there are no known exact solutions for this problem. If we allow the vacuum spacetime to have isometries such as stationarity and axial symmetry, the number of solutions of the Einstein equations that can be obtained by means of generation techniques (cf for instance, [1]) is huge. On the contrary there is only a limited number of exact solutions, sharing the same symmetries, that could be regarded as inner material sources (perfect fluids with a physically reasonable equation of state, for instance) and none of these has been smoothly matched to an asymptotically flat vacuum spacetime (cf [2] for a recent review on the subject).

In section 2 an outline of the mathematical problem of matching spacetimes is provided before dealing with thin layers, that is, two-dimensional sources that could be interpreted as limits of physical configurations in which one of the characteristic lengths of the source can be neglected when compared with the others. In order to shed light on the method for constructing dipole distributions, a reminder of some formulae of potential theory is given in section 3. These expressions are generalized to stationary axisymmetric electrovacuum spacetimes in section 4. The role of the scalar potential is taken on by the complex Ernst potential [3], which was introduced for solving the Einstein–Maxwell system with a two-dimensional group of isometries on the underlying manifold. The lack of continuity of this potential will give rise to a source for the fields consisting of monopole and dipole layers. The results will be applied to the Kerr–Newman metric in section 5 and compared with the ones achieved by Israel in [4]. A brief discussion is provided.

0264-9381/96/040753+12$19.50

2. The matching problem

Our aim is the mathematical description of a relativistic compact object and the gravitational field in the vacuum that surrounds it. Therefore it is convenient to consider two Lorentzian manifolds (M_+, g_+), (M_-, g_-) corresponding to the outer vacuum and the inner material source, respectively, whose matter content satisfies the Einstein equations,

$$\text{Ricci}\,(g_\pm) - \tfrac{1}{2} R g_\pm = 8\pi T_\pm. \tag{1}$$

The stress tensor in (M_+, g_+) can be either electromagnetic or vacuum whereas in (M_-, g_-) it can be the one corresponding to an isentropic (charged) perfect fluid.

As a simplification we shall allow both spacetimes to have isometries and restrict ourselves to stationary axisymmetric spacetimes. The Killing fields that implement these isometries, ξ, η, respectively the generators of the stationary and axial symmetry, must fulfil certain conditions,

$$g(\xi, \xi) < 0 \qquad g(\eta, \eta) \geqslant 0 \qquad [\xi, \eta] = 0. \tag{2}$$

and η must have closed orbits.

The symmetry axis will be then defined by the set of events where the axial Killing field is a null vector,

$$\Delta = g(\eta, \eta) = 0, \tag{3}$$

and in order to avoid conical singularities on it, we have to impose a regularity condition, namely, [5]

$$\frac{g(\text{grad}\,\Delta, \text{grad}\,\Delta)}{4\Delta} \to 1, \tag{4}$$

on approaching the axis. One can also impose that test particles moving along the axis do not experience gravitational or electromagnetic forces that push them away from it. This means [6] that in a chart where the coordinates $\{x^0, x^1, x^2, x^3\}$ are required to satisfy $\xi = \partial_{x^0}$, $\eta = \partial_{x^1}$, $x^2 = \sqrt{g(\eta, \eta)}$ and the $x^2 =$ constant and $x^3 =$ constant hypersurfaces are orthogonal to each other, the connection coefficients $\Gamma^2_{\mu\nu}$ vanish on the axis for $\mu \neq 2 \neq \nu$. Furthermore, the same restriction on charged test particles implies that the partial derivatives of the electromagnetic connection along the x^2 direction also vanish on the axis.

We shall also require that not only the metric but also the electromagnetic curvature, F, and its four-dimensional Hodge dual, ${}^4{*}F$, and the 4-velocity, the density and the pressure of the fluid have zero Lie derivatives along the Killing fields.

In order to model compact objects such as stars, it is expected that the gravitational field will decay to zero at infinity and therefore it is reasonable to require that the (M_+, g_+) be asymptotically flat in a convenient set of coordinates. We shall expand on this subject in the following sections.

Since we are dealing with rotating objects, nonstaticity has to be achieved. Otherwise we would have the Schwarzchild solution. Therefore *every* timelike Killing field ζ must be non-surface forming,

$$\zeta^\flat \wedge \mathrm{d}\zeta^\flat \neq 0, \tag{5}$$

where $\flat$ denotes the isomorphism between the tangent and the cotangent bundle induced by the metric structure.

If both spacetimes are to be matched, then there must be in each of them a closed three-dimensional timelike surface $\Sigma_\pm$ that can be imbedded in both (M_+, g_+) and (M_-, g_-) as a submanifold $(\Sigma_\pm, i^*_\pm g_\pm)$

$$i_\pm : \Sigma_\pm \hookrightarrow M_\pm. \tag{6}$$

According to Darmois' junction conditions [7] both spacetimes are suitable for matching if the induced metric and the extrinsic curvature on the hypersurfaces are continuous,

$$[i^* g] = 0 \tag{7}$$

$$[K] = 0, \tag{8}$$

after identifying events on Σ_+ and Σ_- by means of an isometry. The difference between the values of a quantity on Σ_+ and Σ_- has been denoted by a square bracket.

If the electromagnetic field F is non-zero, for the sake of continuity it must fulfil the condition [4],

$$[i^* F(n)] = 0, \tag{9}$$

on the matching hypersurface, whose outer unit normal is denoted by n.

As Israel has shown [8], whenever the equation (8) is not satisfied, the jump in the extrinsic curvature K reveals the presence of an energy–momentum surface density, S, on Σ,

$$S = \frac{1}{8\pi}\left([K] - \mathrm{Tr}[K] i^* g\right), \tag{10}$$

and in this case the inner source is surrounded by a 'crust' of matter.

In the same fashion, there is a non-zero electromagnetic surface current j on Σ, given by the expression,

$$[i^* F(n)] = -4\pi j, \tag{11}$$

if the continuity condition (9) on F is not satisfied.

Thin layers are a special case of non-smooth matching. If we try to match two vacuum spacetimes across a common hypersurface so that the induced metric on both sides fulfils (7), then the only possibility of having matter content in the global manifold is given by (10), (11). In this case the 'crust' is all we have.

We would be interested then in determining physical information about the material source that it is located on the hypersurface, such as the densities of the physical quantities. In particular dipole densities will be the main concern in the following sections.

3. Classical dipole densities

Before describing our approach to calculating dipole surface densities in the relativistic situation, it will be useful to recall some results of classical potential theory in order to establish a comparison when dealing with general relativity.

Let us consider a classical field which can be derived from a scalar potential V that satisfies the Laplace equation. Assume that the field is generated by a source located on a closed surface S (a sheet of monopoles and dipoles). Then the density of the monopole layer σ_Q and the density σ_M of the projection of the moment in the z direction (its unit vector will be represented by u_z),

$$\sigma_Q = -\frac{1}{4\pi}\left[\frac{\mathrm{d}V}{\mathrm{d}n}\right] \tag{12a}$$

$$\sigma_M = \frac{1}{4\pi}\left\{n \cdot u_z [V] - z\left[\frac{\mathrm{d}V}{\mathrm{d}n}\right]\right\}, \tag{12b}$$

can be calculated in terms of the discontinuities (denoted by square brackets) of the potential and its derivative along the outer unit normal n to the surface.

The latter expression can be easily obtained by applying the Green identity to the flat space Ω, excluding the source,

$$0 = \int_{\Omega} \mathrm{d}^3x \, (V \Delta z - z \Delta V)$$
$$= \int_{\partial\Omega} \mathrm{d}S \left(V \frac{\mathrm{d}z}{\mathrm{d}n} - z \frac{\mathrm{d}V}{\mathrm{d}n} \right)$$
$$= \int_{S^2(\infty)} \mathrm{d}S \left(V \frac{\mathrm{d}z}{\mathrm{d}n} - z \frac{\mathrm{d}V}{\mathrm{d}n} \right) - \int_S \mathrm{d}S \left([V] \frac{\mathrm{d}z}{\mathrm{d}n} - z \left[\frac{\mathrm{d}V}{\mathrm{d}n} \right] \right), \qquad (13)$$

taking into account that both the Cartesian coordinate z and the potential V are solutions of the Laplace equation in Ω and that the boundary $\partial\Omega$ consists of the sphere at infinity and the surface S.

The integral at infinity can be performed if V has the usual multipolar expression,

$$V = \frac{Q}{r} + \frac{M \cos\theta}{r^2} + \mathrm{O}\,(r^{-3}), \qquad (14)$$

and it yields the value $4\pi M$. Hence the integrand on S can be interpreted as the surface density for the dipole moment M of the source.

This interpretation arises immediately from potential theory (cf for instance, [9]), which provides the expressions for the dipole density of a double distribution,

$$\sigma_{dip} = \frac{1}{4\pi} n \cdot u_z [V], \qquad (15)$$

in terms of the discontinuity of the potential, and the density of a monopole layer (12*a*) related to the jump of the normal derivative of the potential on the surface.

Hence the first term in (12*b*) is the contribution of a dipole layer to the total dipole density and the second term is just the moment density in the z direction of the distribution of monopoles.

4. Relativistic layers

For the stationary axisymmetric Einstein–Maxwell system we shall follow a similar approach, which can be viewed as a generalization of the formalism introduced in [10, 11] for static electrovacs. In that paper magnetic surface-sources were constructed for magnetostatic solutions of the Einstein–Maxwell equations. The magnetic moment density was calculated from the discontinuities of the Ernst magnetic potential [3] involved in the generation of the solutions. However that formalism could not deal with asymptotically monopolar electric fields and with nonstatic metrics and, therefore, many important exact solutions fell out of its scope.

In order to work with the Ernst formalism for the stationary axisymmetric Einstein–Maxwell system, we shall work in charts adapted to the Killing fields throughout the paper so that these can be expressed in the form $\xi = \partial_t$, $\eta = \partial_\phi$ and restrict ourselves to metrics which can be rendered into a reducible matrix in a holonomic frame, that is, there is a set of coordinates in which the metric can be expressed as $g = g_1 \oplus g_2$, where g_1 is the metric in the subspace spanned, at each event, by the Killing fields and g_2 is the metric in the subspace orthogonal to the former one. Having this restriction in mind, it is useful [12] to introduce an orthonormal frame $\{\theta^0, \theta^1, \theta^2, \theta^3\}$ such that the Killing fields have non-zero projection on θ^0 and θ^1 only.

In this orthonormal frame, we shall consider electric and magnetic forms, E and B, with zero projection on the orbits of the Killing fields,

$$F = E \wedge \theta^0 + *B \wedge \theta^1 \qquad {}^4 * F = -B \wedge \theta^0 + *E \wedge \theta^1, \tag{16}$$

so that the electromagnetic Faraday, F, and Maxwell, ${}^4 * F$, forms have a simple expression after introducing the Hodge duality ($*\theta^2 = \theta^3$) in the space orthogonal to the Killing fields [10].

Taking into account Cartan's structure equations, expressing the torsion-free connection coefficients as 1-forms, a, w, s, b and ν, (cf [10, 12] for the physical meaning of these forms)

$$\mathrm{d}\theta^0 = a \wedge \theta^0 - w \wedge \theta^1 \tag{17a}$$

$$\mathrm{d}\theta^1 = (b - a) \wedge \theta^1 - s \wedge \theta^0 \tag{17b}$$

$$\mathrm{d}\theta^2 = -\nu \wedge \theta^3 \tag{17c}$$

$$\mathrm{d}\theta^3 = \nu \wedge \theta^2, \tag{17d}$$

the Maxwell vacuum equations, $\mathrm{d}F = 0$ and $\mathrm{d}\, {}^4 * F = 0$, can be written in a compact fashion after defining a complex 1-form $f = E + \mathrm{i}B$,

$$\mathrm{d}f = -a \wedge f - \mathrm{i}s \wedge *f \tag{18a}$$

$$\mathrm{d} * f = (a - b) \wedge *f + \mathrm{i}w \wedge f. \tag{18b}$$

The orthonormal frame can always be chosen without loss of generality so that the 1-form s is zero. Bianchi's compatibility conditions for (17),

$$\mathrm{d}b = 0 \tag{19a}$$

$$\mathrm{d}a = w \wedge s \tag{19b}$$

$$\mathrm{d}w = -(b - 2a) \wedge w \tag{19c}$$

$$\mathrm{d}s = (b - 2a) \wedge s, \tag{19d}$$

can be then formally integrated after introducing some new functions U, A,

$$a = \mathrm{d}U \qquad b = \mathrm{d}\ln\rho \tag{20}$$

$$w = \rho^{-1}\mathrm{e}^{2U}\mathrm{d}A \qquad s = 0, \tag{21}$$

which allow the solution of Cartan's first structure exterior system of equations (17) and yield the line element in Weyl coordinates,

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}(\mathrm{d}t - A\,\mathrm{d}\phi)^2 + \mathrm{e}^{-2U}\{\rho^2\,\mathrm{d}\phi^2 + \mathrm{e}^{2k}(\mathrm{d}z^2 + \mathrm{d}\rho^2)\}, \tag{22}$$

in terms of the functions U, A, k of ρ and z.

Also equation (18*a*) can be solved in terms of a complex scalar potential, Φ, the Ernst electromagnetic potential [3],

$$f = E + \mathrm{i}B = -\mathrm{e}^{-U}\,\mathrm{d}\Phi, \tag{23}$$

that satisfies one of the Ernst equations [3],

$$\mathrm{d} * \mathrm{d}\Phi + (b - 2a) \wedge *\mathrm{d}\Phi = \mathrm{i}\frac{\mathrm{e}^{-2U}}{\rho}\,\mathrm{d}A \wedge \mathrm{d}\Phi, \tag{24}$$

which can be easily obtained from (18*b*) and (23). However, it will be convenient to cast it in a different form,

$$\mathrm{d}(\mathrm{e}^{-2U}\rho * \mathrm{d}\Phi - \mathrm{i}A\,\mathrm{d}\Phi) = 0, \tag{25}$$

for future purposes. In a coordinate patch, it bears a resemblance to a complex Laplace equation on the hypersurfaces of constant time,

$$L\Phi \equiv \frac{1}{\sqrt{g}}\partial_\mu\left\{N\sqrt{g}\left(\mathrm{e}^{-2U}g^{\mu\nu} - \frac{\mathrm{i}}{\rho}A\epsilon^{\mu\nu}\right)\partial_\nu\Phi\right\} = 0, \tag{26}$$

but including a correction depending on A due to non-staticity, that prevents the decoupling of the equations for the real and imaginary parts of Φ. In this equation g is the metric induced by (22) on the hypersurfaces t = constant, $N = (-{}^4g^{tt})^{-\frac{1}{2}}$ is the lapse function and ϵ is the Levi–Cività tensor on the surfaces of constant time t and azimuthal angle ϕ. For simplicity the whole equation has been written as the action of a differential operator L on the potential.

As a matter of fact, this is a consequence only of the Maxwell equations in the curved spacetime whose metric is given by (22), regardless of whether the electromagnetic field is the source of the gravitational field, since we have not made use of the Einstein equations. Therefore what follows is valid for any potential that satisfies the equation $L\Phi = 0$ on the previously described geometry.

As was stated in the first section, we are interested in compact sources and therefore we shall only consider metrics which can be rendered asymptotically flat in some coordinates (t, r, θ, ϕ),

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\left(\mathrm{d}t + \frac{2J\sin^2\theta}{r}\,\mathrm{d}\phi\right)^2 + \left(1 + \frac{2m}{r}\right)\{\mathrm{d}r^2 + (r^2 + c_1 r)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\} + \mathrm{O}\,(1/r^2), \tag{27}$$

that allows us to read the total mass m and the total angular momentum of the source J from the Lense–Thirring expansion. In this set of coordinates, the electromagnetic potential of the compact source will be required to have an asymptotic expansion in terms of the first multipole moments, the total charge e and the complex electromagnetic dipole M, whose real and imaginary parts are, respectively, the electric and magnetic dipole moment,

$$\Phi = \frac{e}{r} + \frac{M\cos\theta}{r^2} + \frac{c_2}{r^2} + \mathrm{O}\,(r^{-3}). \tag{28}$$

The constants c_1 and c_2 may arise in some choices of coordinates.

In the case of a non-smooth or discontinuous electromagnetic Ernst potential Φ the equation $L\Phi = 0$ may not hold in the whole manifold, and it may have a distributional source located in the region where the discontinuity takes place. Since the aim of this paper is devoted to thin layers in stationary axisymmetric spacetimes, we shall assume that this region is a closed surface S of outer unit normal n within the constant time hypersurfaces. We shall denote these hypersurfaces as (V_3, g) for our calculations.

In order to mimic the construction done in section 3 it will be necessary to obtain a Green identity for the L operator. A most natural candidate can be checked to be,

$$\int_\Omega \sqrt{g}(ZL\Phi - \Phi L^+ Z)\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 = \int_{\partial\Omega}\mathrm{d}S\,N\left\{\mathrm{e}^{-2U}\left(Z\frac{\mathrm{d}\Phi}{\mathrm{d}n} - \Phi\frac{\mathrm{d}Z}{\mathrm{d}n}\right) + \mathrm{i}\frac{A}{\rho}\big(Z * \mathrm{d}\Phi(n) + \Phi * \mathrm{d}Z(n)\big)\right\}, \tag{29}$$

which is a consequence of the divergence theorem. L^+ is just the complex conjugate of L. We shall apply this identity to the region $\Omega = V_3^+ \cup V_3^-$, the disjoint union of the outer and inner regions of V_3 referred to the surface S. The oriented boundary of V_3^- is just the surface S whereas that of V_3^+ is formed by the sphere at infinity and S.

So far no condition has been imposed on the function Z. In the classical case, it was just the Cartesian coordinate z and the Laplace equation held for it. Since we no longer have a Laplace operator but L, it seems natural to choose Z so that it satisfies $L^+ Z = 0$ in Ω. Hence the left-hand side in (29) will be zero and we shall have to deal with surface integrals only. In addition to the differential equation the function Z will be required to behave near infinity as the Cartesian coordinate. The set of conditions on Z will then be,

$$L^+ Z = 0 \qquad Z = (r + c_3)\cos\theta + \mathrm{O}\,(r^{-1}), \tag{30}$$

which requires solving an elliptic partial differential equation with boundary conditions at infinity. Again c_3 is a constant.

The integral at infinity in (29) can be calculated from the information given by the asymptotic behaviour. We are left with just an integral over the surface S, where the source is located,

$$0 = -4\pi M + \int_S \mathrm{d}S\, N \left\{ \mathrm{e}^{-2U} \left[\Phi \frac{\mathrm{d}Z}{\mathrm{d}n} - Z \frac{\mathrm{d}\Phi}{\mathrm{d}n} \right] - \mathrm{i}\frac{A}{\rho} \left[Z * \mathrm{d}\Phi(n) + \Phi * \mathrm{d}Z(n) \right] \right\}, \tag{31}$$

which allows us to express the total electromagnetic moment as an integral over the source,

$$M = \frac{1}{4\pi} \int_S \mathrm{d}S\, N \left\{ \mathrm{e}^{-2U} \left[\Phi \frac{\mathrm{d}Z}{\mathrm{d}n} - Z \frac{\mathrm{d}\Phi}{\mathrm{d}n} \right] - \mathrm{i}\frac{A}{\rho} \left[Z * \mathrm{d}\Phi(n) + \Phi * \mathrm{d}Z(n) \right] \right\}. \tag{32}$$

As was done in classical potential theory, we can interpret the discontinuity of the integrand as the electromagnetic dipole moment surface density of the source for the field,

$$\sigma_M = \frac{1}{4\pi} N \left\{ \mathrm{e}^{-2U} \left[\Phi \frac{\mathrm{d}Z}{\mathrm{d}n} - Z \frac{\mathrm{d}\Phi}{\mathrm{d}n} \right] - \mathrm{i}\frac{A}{\rho} \left[Z * \mathrm{d}\Phi(n) + \Phi * \mathrm{d}Z(n) \right] \right\}, \tag{33}$$

the real part of σ_M being the electric dipole density and its imaginary part, the magnetic moment density.

When A is zero, that is in the static case, this formula is pretty similar to the classical one (12*b*), corrected by metric factors. Its first term also coincides in the static case with the formula introduced in [10], where the contribution of the moment density arising from the layer of monopoles (second term) was not taken into account.

Also the electric charge density on S can be calculated using the divergence theorem, since equation (26) has the form of a total derivative and therefore its integral on Ω can be reduced to a surface integral on its boundary,

$$0 = \int_\Omega L\Phi = \int_{\partial\Omega} \mathrm{d}S\, N \left\{ \mathrm{e}^{-2U} \frac{\mathrm{d}\Phi}{\mathrm{d}n} + \mathrm{i}\frac{A}{\rho} * \mathrm{d}\Phi(n) \right\}. \tag{34}$$

The asymptotic expansion of the fields provides the necessary information to perform the integral at infinity,

$$0 = -4\pi Q - \left\{ \int_S \mathrm{d}S\, N \left(\mathrm{e}^{-2U} \left[\frac{\mathrm{d}\Phi}{\mathrm{d}n} \right] + \mathrm{i}\frac{A}{\rho} \left[* \mathrm{d}\Phi(n) \right] \right) \right\}, \tag{35}$$

from which we can read the expression for the charge density,

$$Q = \int_S \mathrm{d}S\, \sigma_Q \qquad \sigma_Q = -\frac{1}{4\pi} \,\mathrm{Re} \left\{ N \left(\mathrm{e}^{-2U} \left[\frac{\mathrm{d}\Phi}{\mathrm{d}n} \right] + \mathrm{i}\frac{A}{\rho} \left[* \mathrm{d}\Phi(n) \right] \right) \right\}, \tag{36}$$

in terms of the discontinuities of the derivatives of the electromagnetic Ernst potential Φ. This formula recovers Israel's expression for the electric charge density [4], but written in terms of the electromagnetic Ernst potential.

The function Z can only be considered as a coordinate in the static case. Otherwise the equation (30) is complex and so is the solution Z. This is very similar to what happens with the Ernst potential [3], $\varepsilon = \mathrm{e}^{2U} + \mathrm{i}\chi$. Whereas its first term is a metric function, the second term is just an auxiliary potential, the twist potential χ, that is due to nonstaticity. Similarly, just the real part of Z can be viewed as the coordinate function which determines the projection of the dipole moment that is being calculated. The imaginary part is again an auxiliary potential which states the influence of the electric (magnetic) field on the magnetic (electric) dipole density.

As it has already been mentioned, these results are valid for any potential that satisfies (26) in the geometry defined by (22). Therefore the same considerations can be applied to the gravitational Ernst potential ε [3], since it fulfils $L\varepsilon = 0$. It will be convenient, however, to introduce another potential η,

$$\eta = \varepsilon - 1, \tag{37}$$

since the Ernst potential is defined up to a constant, which is usually fixed so that the potential tends to one at infinity and this asymptotic finite value of ε is not very convenient when performing integrals on Ω. If the potential has the following asymptotic behaviour,

$$\eta = -\frac{2m}{r} - \mathrm{i}\frac{2J\cos\theta}{r^2} + \frac{c_4}{r^2} + \mathrm{O}\,(r^{-3}), \tag{38}$$

where c_4 is again a constant, then all the previous calculations can be repeated just substituting Φ for η to yield expressions for the mass and angular momentum densities,

$$\sigma_m = \frac{1}{8\pi} N \left\{ \mathrm{e}^{-2U} \left[\frac{\mathrm{d}\eta}{\mathrm{d}n} \right] + \mathrm{i}\frac{A}{\rho} \left[*\,\mathrm{d}\eta(n) \right] \right\} \tag{39}$$

$$\sigma_J = \frac{1}{8\pi} N \left\{ \mathrm{e}^{-2U} \left[Z\frac{\mathrm{d}\eta}{\mathrm{d}n} - \eta\frac{\mathrm{d}Z}{\mathrm{d}n} \right] + \mathrm{i}\frac{A}{\rho} \left[Z * \mathrm{d}\eta(n) + \eta * \mathrm{d}Z(n) \right] \right\}, \tag{40}$$

on the surface S.

5. An example. The Kerr–Newman spacetime

As an application of this formalism, we shall calculate the sources for the physical quantities of the Kerr–Newman spacetime [13]. The energy–momentum tensor of a source located on the $r = 0$ surface of this spacetime has already been obtained by Israel [4] using the formalism of thin layers [8] and later by López [14] using distributions. We shall focus our attention on the magnetic moment density since it is not calculated in those references.

Instead of allowing the radial coordinate r to become negative, we shall restrict its range to positive values, as it is done in [4]. This amounts to identifying points on the hypersurface $r = 0$ (both sets of coordinates $(t, \phi, 0, \theta)$ and $(t, \phi, 0, \pi - \theta)$ represent the same event), as if we were working with oblate spheroidal coordinates. This identification causes certain functions of the collatitude angle θ to be discontinuous, such as the cosine, and hence the Ernst potentials,

$$\varepsilon = 1 - \frac{2m}{r - \mathrm{i}a\cos\theta} \qquad \Phi = \frac{e}{r - \mathrm{i}a\cos\theta}, \tag{41}$$

and their derivatives will encounter discontinuities on $r = 0$, revealing the presence of the source.

The Kerr–Newman metric, in Boyer–Lindquist coordinates,

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2mr - e^2}{r^2 + a^2\cos^2\theta}\right)\left(dt + \frac{(2mr - e^2)a\sin^2\theta}{r^2 - 2mr + e^2 + a^2\cos^2\theta}\,d\phi\right)^2 \\ &+ \left(1 - \frac{2mr - e^2}{r^2 + a^2\cos^2\theta}\right)^{-1}\Bigg\{(r^2 - 2mr + a^2 + e^2)\sin^2\theta\,d\phi^2 \\ &+ (r^2 - 2mr + a^2\cos^2\theta + e^2)\left(\frac{dr^2}{r^2 - 2mr + a^2 + e^2} + d\theta^2\right)\Bigg\} \end{aligned} \quad (42)$$

if the parameters m, a, e satisfy $m^2 < a^2 + e^2$, induces a line element on the surface $r = 0$, $t = \text{constant}$

$$ds_2^2 = a^2\cos^2\theta\,d\theta^2 + \sin^2\theta(a^2 - e^2\tan^2\theta)\,d\phi^2, \quad (43)$$

and unit normal,

$$n = \frac{\sqrt{e^2 + a^2}}{a\cos\theta}\partial_r, \quad (44)$$

where m is the total mass, ma the total angular momentum and e the total electric charge.

It is clear that the Kerr–Newman metric and Ernst potentials fulfil the previously described asymptotic requirements and therefore the surface densities can be calculated within this formalism. We need a solution for (30),

$$Z = (r - 2m)\cos\theta + \frac{e^2\cos\theta + iam\cos^2\theta}{r + ia\cos\theta}, \quad (45)$$

before calculating the densities on the surface $r = 0$ by introducing these expressions in (33), (36), (39), (40).

From the expressions for the electromagnetic moment and the charge surface density,

$$\sigma_M = \frac{\left(e^2\cos^2\theta + e^2 + a^2\cos^2\theta\right)ie}{2\pi a^2\cos^3\theta|a^2 - e^2\tan^2\theta|^{\frac{1}{2}}} \qquad \theta \in [0, \theta_0) \cup (\theta_0, \pi/2), \quad (46a)$$

$$\sigma_Q = -\frac{e}{2\pi a\cos^3\theta|a^2 - e^2\tan^2\theta|^{\frac{1}{2}}} \qquad \theta \in [0, \theta_0) \cup (\theta_0, \pi/2), \quad (46b)$$

where $\theta_0 = \tan^{-1}(|a/e|)$. The corresponding angular momentum and mass densities are obtained by multiplying them, respectively, by the inverse of the gyromagnetic ratio e/m, since the Ernst potentials are linearly dependent,

$$\sigma_J = \frac{\left(e^2\cos^2\theta + e^2 + a^2\cos^2\theta\right)im}{2\pi a^2\cos^3\theta|a^2 - e^2\tan^2\theta|^{\frac{1}{2}}} \qquad \theta \in [0, \theta_0) \cup (\theta_0, \pi/2), \quad (46c)$$

$$\sigma_m = -\frac{m}{2\pi a\cos^3\theta|a^2 - e^2\tan^2\theta|^{\frac{1}{2}}} \qquad \theta \in [0, \theta_0) \cup (\theta_0, \pi/2). \quad (46d)$$

The electromagnetic dipole density σ_M is imaginary, and hence there is no electric dipole density, just the magnetic moment density.

From the expressions for the differential elements of magnetic moment and electric charge,

$$dM_{mag} = \mathcal{I}\{\sigma_M\}\,dS = \frac{\left(e^2\cos^2\theta + e^2 + a^2\cos^2\theta\right)e\sin\theta}{2\pi a\cos^2\theta}\,d\theta\,d\phi, \quad (47)$$

$$dQ = \sigma_Q\,dS = -\frac{e\sin\theta}{2\pi\cos^2\theta}\,d\theta\,d\phi, \quad (48)$$

we could get by integration on the surface $r = 0$ the total magnetic moment and electric charge of the source. However, instead of obtaining the values $M_{mag} = ea$ and $Q = e$, the integration yields divergent results due to the singular ring at $r = 0$, $\theta = \pi/2$. More precisely the integrals

$$M = \lim_{\varepsilon\to 0}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\cos^{-1}\varepsilon/a} \mathrm{d}\theta\, \sigma_M = \mathrm{i}ea\left(1 + \lim_{\varepsilon\to 0}\frac{e^2}{a\varepsilon}\right) \tag{49a}$$

$$Q = \lim_{\varepsilon\to 0}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\cos^{-1}\varepsilon/a} \mathrm{d}\theta\, \sigma_Q = e\left(1 - \lim_{\varepsilon\to 0}\frac{a}{\varepsilon}\right), \tag{49b}$$

provide the correct results plus a term that blows up on approaching the singular ring. There is an infinite contribution to the physical quantities located on the ring.

It is remarkable that the angular momentum density is integrable in the uncharged case, as it was shown in [15]. From the expression of the induced metric (43) we learn that there is a change of signature at θ_0. For values of θ greater than θ_0, the metric on the surface is semi-Riemannian.

The elements of mass and electric charge are the same as were obtained in [4] using the thin layer formalism. The linear relation between the magnetic surface density and the angular momentum surface density confirms the value of the gyromagnetic ratio e/m for the Kerr–Newman spacetime.

There is another choice of location for the source of the Kerr–Newman spacetime, as it is shown in [16, 17]. Instead of considering the surface $r = 0$, it is possible to match the Kerr–Newman manifold to Minkowski spacetime at the pseudosphere $r = r_0 = e^2/2m$. The electromagnetic Ernst potential in the interior of the pseudosphere is taken to be zero. An advantage of this choice is that the source is valid for all values of the parameters m, a, e, since this surface is always located in the Boyer–Lindquist chart (the horizon, whenever there is one, lies within the surface). Another advantage of this approach lies in the fact that the metric at $r = r_0$,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (r_0^2 + a^2\cos^2\theta)\left(\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{r_0^2 + a^2}\right) + (r_0^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2, \tag{50}$$

is just flat spacetime in oblate spheroidal coordinates, as it happened for the uncharged case at $r = 0$. Therefore the metric tensor is continuous at the matching surface and also the induced metric,

$$\mathrm{d}s_2^2 = (r_0^2 + a^2\cos^2\theta)\,\mathrm{d}\theta^2 + (r_0^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2, \tag{51}$$

on the pseudosphere experiences no change of signature.

The expressions that we obtain for the electromagnetic dipole and monopole densities on the surface $r = r_0$,

$$\sigma_M = \frac{e\cos\theta\left\{2r_0^2(r_0 - m) + \mathrm{i}a\cos\theta(a^2\cos^2\theta + 3r_0^2)\right\}\sqrt{r_0^2 + a^2}}{4\pi\left(r_0^2 + a^2\cos^2\theta\right)^{5/2}} \qquad \theta\in[0,\pi] \tag{52a}$$

$$\sigma_Q = \frac{e\left(r_0^2 - a^2\cos^2\theta\right)\sqrt{r_0^2 + a^2}}{4\pi\left(r_0^2 + a^2\cos^2\theta\right)^{5/2}} \qquad \theta\in[0,\pi], \tag{52b}$$

lead to the correct results, respectively, for the total magnetic moment and electric charge after integration on the pseudosphere. From the fact that σ_M is not imaginary, we learn that there is a non-zero density of electric moment on the matching surface, given by the real part of (52*a*),

$$\sigma_{M_{elec}} = \frac{2e\cos\theta r_0^2 (r_0 - m)\sqrt{r_0^2 + a^2}}{4\pi\left(r_0^2 + a^2\cos^2\theta\right)^{5/2}}, \tag{53}$$

although the total electric moment amounts to zero. The magnetic moment density is then given by the expression

$$\sigma_{M_{mag}} = \frac{ea\cos^2\theta(a^2\cos^2\theta + 3r_0^2)\sqrt{r_0^2 + a^2}}{4\pi\left(r_0^2 + a^2\cos^2\theta\right)^{5/2}}. \tag{54}$$

6. Discussion

In this paper a new method for constructing dipole surface sources for stationary axisymmetric electrovac spacetimes has been introduced. The expressions that are obtained show the contributions of dipole and monopole layers to the total dipole density. For this purpose it is necessary to calculate a function Z as the solution of a linear elliptic differential equation with boundary conditions at infinity. Its real part can be interpreted as the function which determines the projection of the dipole moment that is being calculated whereas the imaginary part is an auxiliary potential related to nonstaticity.

The method has been applied to the Kerr–Newman spacetimes to produce the magnetic moment density of two different choices of source, both of them on hypersurfaces of constant time. To our knowledge, this quantity had not been calculated before. One of them has been located, for parameters m, a, e satisfying $m^2 < a^2 + e^2$, on the surface $r = 0$, that is surrounded by a singular ring, whereas the other lies on the pseudosphere $r_0 = e^2/2m$. The resulting densities are integrable in the pseudosphere model but they are not so in the first model. Besides the pseudosphere model is valid for all ranges of the parameters m, a, e, since the horizons lie within the surface. Therefore the second choice seems to be a more suitable source for the Kerr–Newman fields.

For the future it would be interesting to devise new methods for interpreting the lack of continuity or smoothness of the potentials in terms of distributions in order to cope with other possible sources, such as struts and rings.

Acknowledgments

The present work has been supported in part by DGICYT project PB92-0183 and by a FPI Predoctoral Scholarship from Ministerio de Educación y Ciencia (Spain). The author wishes to thank F J Chinea and L M González-Romero for valuable discussions and the referees for their suggestions.

References

[1] Hoenselaers C and Dietz W (ed) 1984 *Solutions of Einstein's equations: Techniques and Results* (Berlin: Springer)
[2] Chinea F J and González-Romero L M 1993 Rotating objects and other topics *Proc. El Escorial Summer School on Gravitation and General Relativity (1992)* (Berlin: Springer)
[3] Ernst F J 1968 *Phys. Rev.* **168** 1415
[4] Israel W 1970 *Phys. Rev.* D **2** 641

[5] Kramer D, Stephani H, MacCallum M and Herlt E 1980 *Exact Solution's of Einstein's Field Equations* (Cambridge: Cambridge University Press)
[6] Van den Bergh N and Wils P 1984 *Class. Quantum Grav.* **2** 229
[7] Darmois G 1927 *Mémorial des Sciences Mathématiques* **XXV** ch 5 (Paris: Gauthier-Villars)
[8] Israel W 1966 *Nuovo Cimento* B **44** 1
[9] Kellogg O D 1954 *Foundations of Potential Theory* (New York: Dover)
[10] Fernández-Jambrina L and Chinea F J 1994 *Class. Quantum Grav.* **11** 1489
[11] Fernández-Jambrina L 1994 *PhD Thesis* Universidad Complutense de Madrid (unpublished)
[12] Chinea F J and González-Romero L M 1992 *Class. Quantum Grav.* **9** 1271
[13] Newman E T, Couch E, Chinnapared K, Exton A, Prakash A and Torrence R 1965 *J. Math. Phys.* **6** 918
[14] López C A 1983 *Nuovo Cimento* B **76** 9
[15] Fernández-Jambrina L and Chinea F J 1993 *Phys. Rev. Lett.* **71** 2521
[16] López C A 1984 *Phys. Rev.* D **30** 313
[17] Grøn Ø1985 *Phys. Rev.* D **32** 1588