Review

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New futures for cosmological models

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The discovery of accelerated expansion of the Universe opened up the possibility of new scenarios for the doom of our space–time, besides eternal expansion and a final contraction. In this paper, we review the chances that may await our universe. In particular, there are new possible singular fates (sudden singularities, big rip, etc.), but there also other evolutions that cannot be considered as singular. In addition to this, some of the singular fates are not strong enough in the sense that the space–time can be extended beyond the singularity. For deriving our results, we make use of generalized power and asymptotic expansions of the scale factor of the Universe.

This article is part of the theme issue 'The future of mathematical cosmology, volume 1'.

1. Introduction

The discovery of accelerated expansion of our universe has posed many new issues in gravitational theory. One of them is that the final fate is not constrained to eternal expansion or recollapse into a Big Crunch singularity. One by one, quite a few names have been added to such a list of fates (Big Rip, Little Rip, sudden singularities, etc.). Some of them are singular (strong or weak), some are not. In this paper, we would like to review this issue and provide a thorough classification of them.

The issue of defining what a singularity is in general relativity is definitely not a simple one [1–3]. Most field theories are defined in a fixed background space–time manifold, for instance, Minkowski space–time. A field is then singular when it diverges at a certain event in the background space–time. This is what happens, for instance, with the electric Coulombian field due

to a single charged particle, which is singular at the location of the particle. Many of such singularities are removed on considering quantum field theories.

In general relativity, the problem is a bit more involved and plural. We cannot consider a field such as the metric of the space–time as singular if it diverges at some set of coordinates, since the theory is invariant under changes of coordinates. There are no privileged coordinates and it could happen that the metric does not diverge in another system of coordinates. This is what happens, for instance, for the horizon of the Schwarzschild metric, which appears to be singular in some coordinates, but not in Eddington–Finkelstein or Kruskal coordinates. Furthermore, if the gravitational field is singular, the space–time manifold would be singular too. We cannot say that the gravitational field is singular at one *event* as in other field theories, since a singularity of the gravitational field is a singularity of the space–time itself and hence we cannot talk about events or points of the manifold.

One could relate singularities to divergences of the curvature scalars. These are named scalar polynomial curvature singularities (s.p. curvature singularities). This is a common test for the appearance of singularities, since it is easy to check whether the curvature scalars diverge, pointing out that we are leaving the space–time manifold.

However, such polynomials do not fully characterize the curvature tensor. One can resort then to scalar polynomials in the derivatives of the curvature tensor. On the other hand, curvature tensor components may diverge along a curve, though the scalar polynomials remain finite. A parallelly propagated (p.p.) curvature singularity appears when at least a component of the curvature tensor in a parallelly propagated basis along a curve grows unboundedly. These p.p. curvature singularities include s.p. curvature singularities.

Another way of dealing with this is considering causal geodesics in the manifold, that is, time-like and light-like geodesics. Free-falling observers, subject just to gravitational interaction, follow causal geodesics in General Relativity. And these geodesics can be parametrized using the proper time or affine parameter, which is related to time as experienced by the free-falling observer in the case of time-like geodesics.

This provides us with another way of defining singularities. In principle, proper time should range from minus infinity to plus infinity. If this cannot be done, after a finite proper time our observer would be out of the manifold and this behaviour can be considered as a token of a singularity. A singularity appears in a manifold if a causal geodesic is incomplete in this sense, but we must take care of having a maximal extension for our space–time (the horizon of Schwarschild space–time is not singular in Kruskal's extension, but it appears to be singular in other charts).

This is one of the most common definitions of singularities and most theorems are referred to geodesic incompleteness. Of course, this definition comprises curvature singularities, but there are space–times with incomplete geodesics that are not related to curvature singularities (geodesic imprisonment). This is what happens, for instance, in Taub-NUT space–time.

One reason for considering geodesic completeness is that in Riemannian geometry metric completeness (every Cauchy sequence is convergent) is equivalent to geodesic completeness, although it is not so in Lorentzian geometry.

Finally, one can extend the previous definition to non-geodesic curves (accelerated observers) by introducing a generalized affine parameter for them. A space–time is bundle complete (b-complete), and in this sense singularity-free, if every finite-length curve has an endpoint in the manifold. That is, an observer does not leave the space–time manifold in a finite time.

In the next section, we derive and solve the geodesic equations for flat Friedman–Lemaître–Robinson–Walker (FLRW) cosmological models. In §3, we review the concept of strength of a singularity and apply it to our models. Section 4 is devoted to power and asymptotic expansions of the deceleration parameter in order to find all possible singular behaviours, by relating the expansions to the energy-momentum content of the models. A thorough classification of the models in terms of their final behaviour is provided in §5.

2. Geodesics in cosmological models

For dealing with cosmological models, we consider FLRW homogeneous and isotropic space–times. We introduce coordinates t, r, θ, ϕ with the usual meaning and ranges, such that the metric can be written as

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)), \tag{2.1}$$

in terms of the scale factor of the Universe $a(t)$. We just consider spatially flat models for their match with observations, but the analysis can be done for non-flat models as well.

Geodesics can be parametrized with their proper, afine or internal time τ, such that $d\tau^2 = -ds^2$. The velocity of the parametrization of a geodesic would be then $u = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$, where the dot stands for derivation with respect to proper time.

We may define a conserved quantity δ,

$$\delta := -u \cdot u = \dot{t}^2 - a^2(t)(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2)), \tag{2.2}$$

which is zero for light-like geodesics, one for time-like geodesics and minus one for space-like geodesics.

Since the space–time is homogeneous and isotropic, geodesics are straight lines and without losing generality we may take $\dot{\theta} = 0$, $\dot{\phi} = 0$. Furthermore, there is a conserved momentum associated with ∂_r along such straight lines,

$$\pm P = u \cdot \partial_r = a^2(t)\dot{r},$$

and this allows us to write down the geodesic equations in a simple fashion without resorting to calculation with Levi–Civita symbols,

$$\dot{t} = \sqrt{\delta + \frac{P^2}{a^2(t)}} \tag{2.3a}$$

and

$$\dot{r} = \pm\frac{P}{a^2(t)}, \tag{2.3b}$$

where the sign of $\dot{t}$ is chosen as positive to deal with future-pointing geodesics.

As we see, there are three kinds of causal geodesics, namely, light-like ($\delta = 0$, $P \neq 0$), radial time-like ($\delta = 1$, $P \neq 0$) and time-like co-moving ones, ($\delta = 1$, $P = 0$), which are the worldlines of a fluid of velocity u.

In order to go on with the analysis of geodesic behaviour, we need more information about the scale factor $a(t)$. Without much loss of generality, we may assume that it can be expanded around an event at t_0 [4–6],

$$a(t) = c_0|t - t_0|^{\xi_0} + c_1|t - t_1|^{\xi_1} + \cdots, \tag{2.4}$$

with real (not necessarily integer) and ordered exponents $\xi_0 < \xi_1 < \cdots$.

At first view, we note that the value of ξ_0 already conveys information about the possible singular behaviour of the geodesic,

- - - If $\xi_0 > 0$, $a(t)$ vanishes at t_0 and we find a Big Bang or Big Crunch singularity.
- - - If $\xi_0 = 0$, $a(t)$ is finite at t_0 and we may encounter a singularity depending on whether $a(t)$ is analytical or there is a divergent derivative of the scale factor.
- - - If $\xi_0 < 0$, $a(t)$ diverges at t_0 and we associate this with a Big Rip singularity.

There is another possibility that cannot be overlooked, corresponding to an infinite value of the time coordinate t [7–9]. This needs to be considered if there are observers that may reach $t = \infty$

in finite proper time. In the case of light-like geodesics, we have

$$\dot{t} = \frac{P}{a(t)} \quad \text{and} \quad \int_{t_0}^{t} a(t') \, dt' = P(\tau - \tau_0),$$

and hence $t = \infty$ is reached in finite proper time if the integral

$$\int_{t}^{\infty} a(t') \, dt' \tag{2.5}$$

is finite for large t.

For instance, if we allow an asymptotic behaviour for the scale factor of the form

$$a(t) \simeq c|t|^{\xi},$$

for large t, we see that, according to (2.3a) t diverges at finite proper time τ_0 if $\xi < -1$, the affine time of light-like geodesics ranges from $\tau = -\infty$ ($t = 0$) to $\tau = \tau_0$ ($t = \infty$) and are future-incomplete. Since the curvature scalar polynomials vanish at infinity time coordinate, we find here a simple example of p.p. curvature singularities.

3. Strength of the singularities

The concept of strength of singularities arises in the context of gravitational collapse, not in cosmological frameworks. Big Bang and Bing Crunch singularities are strong in the sense that, as respectively the origin and end of the Universe, no observer should ellude them. This is not so in gravitational collapse and, as we shall see, with other types of singularities.

The concept of strength of a singularity requires considering finite extended bodies instead of point-like observers (causal geodesics). A geodesic may be not continued beyond a singularity, but one has to take into account if tidal forces are capable of disrupting an extended body on approaching the singularity [10]. If this is the case, the singularity is considered strong.

There are several ways of conveying mathematical rigour to this concept. In these definitions, the finite object is depicted as the infinitesimal volume spanned by an orthonormal basis parallelly-transported along the causal geodesic. In Tipler's definition [11], the singularity is strong if such volume tends to zero at the singularity. The main consequence is that the space–time may be extended beyond weak singularities and in this sense a weak singularity cannot be considered the final fate.

However, according to the less restrictive Królak's definition [12], the singularity is strong if the derivative of this volume is just negative.

For checking the fulfilment of these definitions, there are simple necessary and sufficient conditions [13] that require calculation of integrals of the curvature tensor along the incomplete geodesic.

According to Tipler's definition, an incomplete light-like geodesic of velocity u comes up a strong singularity at its affine time τ_0 if and only if the integral

$$\int_{0}^{\tau} d\tau' \int_{0}^{\tau'} d\tau'' R_{ij} u^i u^j \tag{3.1}$$

diverges as τ approaches τ_0. R stands for the Ricci tensor.

On the other hand, according to Królak's definition, an incomplete light-like geodesic of velocity u comes up a strong singularity at its affine time τ_0 if and only if the integral

$$\int_{0}^{\tau} d\tau' R_{ij} u^i u^j \tag{3.2}$$

diverges as τ approaches τ_0.

In our case, the integrand is readily written in terms of the scale factor and its derivatives,

$$R_{ij}u^i u^j \, d\tau = 2P \left(\frac{a'^2}{a^3} - \frac{a''}{a^2} \right) dt. \tag{3.3}$$

For time-like geodesics, the conditions are the same, but they are just sufficient conditions, not necessary conditions.

For co-moving geodesics, the integrand is just

$$R_{ij}u^i u^j \, d\tau = -\frac{3a''}{a} \, dt, \tag{3.4}$$

whereas for radial time-like geodesics is

$$R_{ij}u^i u^j \, d\tau = \frac{-(3a''/a) + 2P^2(a'^2/a^4 - (a''/a^3))}{\sqrt{1 + (P^2/a^2)}} \, dt. \tag{3.5}$$

These results applied to an expression for $a(t)$ in the form (2.4) readily allow to determine whether a singularity is weak or strong.

4. Generalized power expansion of energy-momentum in cosmological models

The energy-momentum content of FLRW space–times can be seen as a perfect fluid of energy density $\rho(t)$ and pressure $p(t)$, which thanks to the Einstein equations are readily written in terms of the scale factor and its derivatives with respect to the time coordinate t (Friedman equations),

$$\rho = \frac{3a'^2}{a^2} \quad \text{and} \quad p = -\frac{2a''}{a} - \frac{a'^2}{a^2}. \tag{4.1}$$

The energy density is much related to the Hubble ratio $H(t)$,

$$H = \frac{a'}{a}, \tag{4.2}$$

a measure of the expansion of the Universe.

The second derivative is usually expressed in terms of the deceleration parameter $q(t)$,

$$q = -\frac{aa''}{a'^2}, \tag{4.3}$$

which is also closely related to the barotropic index of the cosmological model $w(t)$, the quotient between pressure and density,

$$w = \frac{p}{\rho} = -\frac{1}{3} - \frac{2}{3}\frac{aa''}{a'^2}. \tag{4.4}$$

We shall see that it is useful to perform calculations using the deceleration parameter instead of the scale factor.

It is also common to define a time coordinate $x = \ln a$,

$$\frac{x''}{x'^2} = -(q+1), \tag{4.5}$$

which invites us to define the deviation $h(t)$ of the deceleration parameter from the pure cosmological constant case [8],

$$q(t) = -1 + h(t). \tag{4.6}$$

Besides being physically meaningful, this definition allows us to reduce the order of Friedman equations,

$$h = -\frac{x''}{x'^2} = \left(\frac{1}{x'} \right)^{\cdot} \Rightarrow x' = \left(\int h \, dt + K_1 \right)^{-1},$$

and to integrate the scale factor of the model,

$$a(t) = \exp\left(\int \left(\int h(t)\, dt + K_1\right)^{-1} dt + K_2\right). \tag{4.7}$$

We may calculate explicitly the energy density and pressure of the model by fixing the integration limits,

$$\rho(t) = 3x'(t)^2 = 3\left(\int_{t_0}^{t} h(t')\, dt' + K_1\right)^{-2}, \tag{4.8}$$

$$p(t) = -2x''(t) - 3x'(t)^2 = \frac{2h(t) - 3}{\left(\int_{t_0}^{t} h(t')\, dt' + K_1\right)^2}, \tag{4.9}$$

and interpreting the integration constants.

One constant can be absorbed as a global constant factor $a(t_0) = \exp(K_2)$, which is just the value of the scale factor nowadays,

$$a(t) = a(t_0) \exp\left(\int_{t_0}^{t} \left(\int_{t_0}^{t''} h(t')\, dt' + K_1\right)^{-1} dt''\right). \tag{4.10}$$

The other constant may be interpreted in terms of the energy density, $K_1 = \sqrt{3}\rho(t_0)^{-1/2}$, except in the case of infinite $\rho(t_0)$, for which $K_1 = 0$, which is relevant for the appearance of singularities. For the sake of simplicity, we take $t_0 = 0$, $a(t_0) = 1$.

Without much loss of generality, we assume for our analysis of singularities that the relevant function $h(t)$ can be expanded in real powers of t around the event at $t = 0$,

$$h(t) = h_0 t^{\eta_0} + h_1 t^{\eta_1} + \cdots, \quad \eta_0 < \eta_1 < \cdots, \tag{4.11}$$

where we have assumed positive t, but the same analysis can be performed just exchanging t for $-t$.

The scale factor, the energy density and the pressure behave at lowest order in t as

$$x(t) = \begin{cases} -\dfrac{\eta_0 + 1}{\eta_0 h_0} t^{-\eta_0} + \cdots & \text{if } -1 \neq \eta_0 \neq 0 \\[2ex] \dfrac{1}{h_0} \displaystyle\int \dfrac{dt}{\ln|t|} + \cdots & \text{if } \eta_0 = -1 \\[2ex] \dfrac{\ln|t|}{h_0} + \cdots & \text{if } \eta_0 = 0. \end{cases} \tag{4.12}$$

$$\rho(t) = \begin{cases} 3\left(\dfrac{\eta_0 + 1}{h_0}\right)^2 t^{-2(\eta_0 + 1)} + \cdots & \text{if } -1 \neq \eta_0 \neq 0 \\[2ex] \dfrac{3}{h_0^2} \dfrac{1}{\ln^2|t|} + \cdots & \text{if } \eta_0 = -1 \\[2ex] \dfrac{3t^{-2}}{h_0^2} + \cdots & \text{if } \eta_0 = 0, \end{cases} \tag{4.13}$$

Table 1. Singularities in terms of the power expansion of $q(t)$

η_0	a_s	ρ_s	p_s	w_s	sing.
$(-\infty, -2)$	finite	0	0	∞	IV or V
-2	finite	0	finite	∞	IV
$(-2, -1]$	finite	0	∞	∞	II
$(-1, 0), K_1 \neq 0$	finite	finite	∞	∞	II
$(-1, 0), K_1 = 0$	finite	∞	∞	∞	III
0	$0/\infty$	∞	∞	finite	big crunch/rip
$(0, \infty)$	$0/\infty$	∞	∞	-1	grand crunch/rip

$$
p(t) = \begin{cases}
\dfrac{2(\eta_0 + 1)^2}{h_0} t^{-\eta_0 - 2} + \cdots & \text{if } -1 \neq \eta_0 < 0 \\[2ex]
\dfrac{2}{h_0} \dfrac{1}{t \ln^2 |t|} + \cdots & \text{if } \eta_0 = -1 \\[2ex]
\dfrac{2h_0 - 3}{h_0^2} t^{-2} + \cdots & \text{if } \eta_0 = 0 \\[2ex]
-3 \left(\dfrac{\eta_0 + 1}{h_0} \right)^2 t^{-2(\eta_0 + 1)} + \cdots & \text{if } \eta_0 > 0.
\end{cases}
\tag{4.14}
$$

Calculations are lengthy and one has to be careful in order to prevent overlooking subcases. From these expressions, one can determine if the scale factor a_s, the energy density ρ_s, the pressure p_s or even the barotropic index w_s diverge or not at $t = 0$, depending on the expansion (4.11), which is what most classifications of singularities [14–17] take into consideration. The strength or weakness of the singularities can be determined with the results of the previous section.

We have summarized the possible cases in table 1, in terms of our power expansion. The last column enlarges classifications [14–17] and shall be explained later.

After considering the case of finite t_0, we are not to forget the infinite case. For this, we need asymptotic expressions for our magnitudes,

$$
a(t) = \exp \left(\int_t^\infty \left(\int_{t''}^\infty h(t') \, dt' + K_1 \right)^{-1} dt'' \right),
\tag{4.15}
$$

$$
\rho(t) = 3 \left(\int_t^\infty h(t') \, dt' + K_1 \right)^{-2}
\tag{4.16}
$$

and

$$
p(t) = \frac{2h(t) - 3}{\left(\int_t^\infty h(t') \, dt' + K_1 \right)^2},
\tag{4.17}
$$

which require $K_1 = 0$ for ρ and p to diverge at infinity.

Again, calculations are involved and the results are summarized in table 2, where the asymptotic behaviour of $h(t)$ for large t is related to the asymptotic values of the scale factor, the energy density, the pressure and the barotropic index and to the type of singularity or future behaviour according to our classifications. Again, the last column refers to the thorough classification of the singularities.

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Table 2. Possible behaviour in terms of the asymptotic behaviour of $q(t)$

h	signum (h)	K_1	a_∞	ρ_∞	p_∞	w_∞	behaviour
finite $\int_t^\infty h(t')\,dt'$	$+$	0	0	∞	∞	-1	∞
	$-$	0	∞	∞	∞	-1	little rip / sibling
	$\pm$	positive	0	finite	finite	-1	non-singular
	$\pm$	negative	∞	finite	finite	-1	pseudo-rip
$t^{-1} \lesssim |h(t)| \to 0$	$+$	any	∞	0	0	-1	little rip with 0 ρ and p
	$-$	any	0	0	0	-1	∞
K	$+$	any	∞	0	0	$-1 + 2K/3$	non-singular
$K \in (-1, 0)$	$-$	any	0	0	0	$-1 + 2K/3$	∞
$K \in (-\infty, -1]$	$-$	any	0	0	0	$-1+2K/3$	non-singular
$|h(t)| \to \infty$, infinite $\int_t^\infty\,dt''/\int_{t''}^\infty h(t')\,dt'$	$+$	any	∞	0	0	∞	non-singular
$|h(t)| \to \infty$, infinite $\int_t^\infty\,dt''/\int_{t''}^\infty h(t')\,dt'$	$-$	any	0	0	0	∞	non-singular
$|h(t)| \to \infty$, finite $\int_t^\infty\,dt''/\int_{t''}^\infty h(t')\,dt'$	$\pm$	any	finite	0	0	∞	non-singular

5. Singularities in cosmological models

So far, we have been able to detect all possible singular scenarios in spatially flat cosmological models by use of power expansions in the time coordinate. The resulting possible singularities that have been obtained may be organized in the following fashion with the names and references where they were discovered for the first time:

— Type -1: 'Grand bang/rip': [8] The scale factor vanishes or blows up at $w = -1$. The Hubble ratio, the energy density and the pressure blow up. These are strong singularities.We may see as the counterparts for Big Bang and Big Rip singularities $w \neq -1$ at the phantom divide, but pressure and energy density blow up as a power of t different from -2.The sign of the coefficient h_0 states the kind of singularity: if $h_0 > 0$, we have a sort of exponential Big Bang singularity (Grand bang or Grand crunch if we exchange t for $-t$) and the phantom divide is approached from below. If $h_0 < 0$, the scale factor blows up and we would have an exponential Big Rip at $t = 0$ (Grand Rip). The phantom divide is approached from above.
— Type 0: 'Big bang/crunch': These are well-known *classical* strong singularities. The Hubble ratio, the energy density and the pressure diverge.
— Type I: 'Big rip' [18]: These were the *new* singularities to appear in cosmological models. The scale factor blows up instead of vanishing. In this sense, the final fate of the Universe would be a progressive disruption of all structures instead of a collapse. These are strong singularities. A peculiar feature is that light-like geodesics are complete close to the singularity.
— Type II: 'Sudden singularities' [19–32] or even 'quiescent singularities' [33]: This is the second type of singularity that was considered on introducing new cosmological models, although they had been already introduced in [34]. The main feature of these singularities is that the scale factor, the Hubble ratio and the energy density do not blow up and the models just violate the dominant energy condition. The appearance of fractional exponents in the power expansion of the scale factor produces divergent derivatives starting from second onwards. These singularities are weak and in this sense they cannot be considered the end of the Universe [35], since the space–time can be extended beyond the singularity. In certain contexts, they have been named [36] or big boost [37].
— Type III: 'Big freeze' [38] or 'finite scale factor singularities': Similar to the previous one, but the first divergent derivative of the scale factor is the first and hence the Hubble factor, the energy density and the pressure blow up. That is, the first derivative of the scale factor is singular. Depending on the definition used [11,12], they can be either strong or weak [5].
— Type IV [39]: A generalization of sudden singularities for models with a divergent derivative of the scale factor of order higher than two and, therefore, pressure and density do not blow up. They are also weak singularities.
— Type V: 'w-singularities' [40,41]: They could be considered a subcase of the previous ones, since every derivative of the scale factor is finite and just the barotropic index w blows up. They are weak singularities [42].
— Type ∞: 'Directional singularities' [7,9]: These may come up at infinite coordinate time, though finite proper time. They do not affect all geodesics, since co-moving observers need infinite proper time to reach the singularity. In this sense, they are directional and are p.p. curvature singularities, but they are strong.

In addition to these singularities, there are other asymptotic behaviours that resemble singular ones, although they correspond to regular models. For all of them, the barotropic index is near the phantom divide and could be deemed deviations from the Λ-Cold Dark Matter model:

— Little Rip [43,44]: The Hubble ratio diverges at infinite coordinate and proper time.
— Little Sibling of the Big Rip [45]: A subcase of the previous behaviour, but with a finite derivative of the Hubble ratio.
— Pseudo-rip: [46] Asymptotic monotonic, finite growth of the Hubble ratio.

6. Conclusion

We have provided a thorough classification of possible final fates for our universe in terms of power and asymptotic expansions of the deceleration parameter of the models. This provides a unified framework for both singular and non-singular behaviours. Besides, we have analysed if the singularities are strong or weak. In the latter case, they cannot be considered the final stage for the Universe and the space–time can be continued beyond the singularity.

Among the main interesting features, we may point out that the popular sudden and generalized sudden singularities are not a final fate in the sense they are weak singularities and the space–time can be extended beyond the singularity. And that intriguing directional singularities may come up. That is, singularities which may be experienced by some observers, but not by all of them, although they are strong. Besides, Big Rip singularities are avoided by photons.

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