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Grand rip and grand bang/crunch cosmological singularities

L. Fernández-Jambrina[*]

Matemática Aplicada, E.T.S.I. Navales, Universidad Politécnica de Madrid, Arco de la Victoria 4, E-28040 Madrid, Spain

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The present accelerated expansion of the Universe has enriched the list of possible scenarios for its fate, singular or not. In this paper a unifying framework for analyzing such behaviors is proposed, based on generalized power and asymptotic expansions of the barotropic index w, or equivalently of the deceleration parameter q, in terms of the time coordinate. Besides well-known singular and nonsingular future behaviors, other types of strong singularities appear around the phantom divide in flat models, with features similar to those of big rip or big bang/crunch, which we have dubbed "grand rip" and "grand bang/crunch," respectively, since energy density and pressure diverge faster than t^{-2} in coordinate time. In addition to this, the scale factor does not admit convergent generalized power series around these singularities with a finite number of terms with negative powers.

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I. INTRODUCTION

Our Universe is expanding in an accelerating fashion, as has been tested with many sources of observational data [1–5], and this fact has led to several attempted explanations, either by postulating the existence of a new component of the energy of the Universe, dubbed dark energy [6–8], or by suggesting modifications of the theory of gravitation which would be consistent with observations at the cosmological scale [9–12].

This has led to previously unregarded scenarios for the future behavior of the Universe, since dark energy violates some of the conditions that were taken for granted for standard matter, such as the energy conditions [13].

When all energy conditions were taken into account, the future evolution of our Universe was restricted to collapse in a big crunch if the energy content were over a critical value or expansion forever if it were below such value.

Violation of energy conditions has increased the number of possible singular fates from just big crunch to a list of new scenarios. One of the attempts to classify them [14] resorts to the behavior of the scale factor $a(t)$, the Hubble ratio $H(t)$ and the energy and pressure of the content of the Universe at a value of time. This classification has been refined and enlarged in [15,16] and [17]. The latter one also includes other types of future behaviors such as little rip and pseudorip.

(i) Type 0: Big crunch: Vanishing scale factor; blow up of Hubble ratio, energy density and pressure.

(ii) Type I: Big rip [18]: Blow up of scale factor, density and pressure. This was the first nonclassical scenario that was considered. Just the timelike geodesics are incomplete at the big rip [19], but not the lightlike ones.

(iii) Type II: Sudden singularities [20]: Finite scale factor, Hubble ratio, and density; blow up of pressure. They enclose big brake [21] and big boost [22] as a subcase. These singularities do not violate the weak and strong energy conditions. They are weak singularities [23] and in this sense the Universe can be extended after the singular event.

(iv) Type III: Big freeze [24] or finite scale factor singularities: finite scale factor; blow up of Hubble ratio, density and pressure. These can be either weak or strong singularities depending on the criterion [19].

(v) Type IV [25]: Finite scale factor, Hubble ratio, energy density and pressure; blow up of higher derivatives. These are also weak singularities. They are named [17] "generalized sudden singularities" if the barotropic index w remains finite, and big separation if it becomes infinite with vanishing pressure and energy density.

(vi) Type V: w singularities [26]: Finite scale factor, vanishing density and pressure. Just the barotropic index w blows up. These singularities are weak [27].

Some of these singularities can be solved in loop quantum cosmology [28].

To this list we could add another exotic type of singularities, which do not take place at a finite coordinate time:

(i) Type ∞: Directional singularities: These singularities are located at infinite coordinate time, but some observers meet them in finite normal time. Curvature scalars vanish there [29], though they are strong singularities.

Besides singularities, there are other types of future behavior, which are not singular, but mimic some of their features, though at an infinite time. They all have in common that for them the barotropic index is close to $w = -1$ and so they can be viewed as deviations from the Λ-cold dark matter model:

[*]leonardo.fernandez@upm.es; http://dcain.etsin.upm.es/ilfj.htm

(i) Little rip: [30] At infinite time the Hubble ratio diverges. It shares the features of big rip, but at an infinite time.
(ii) Pseudorip: [31] Monotonic increase of the Hubble ratio, though finite, even at infinite time.
(iii) Little sibling of the big rip: [32] The same features of little rip (the Hubble ratio blows up at infinite time), but with finite derivative of the Hubble ratio.

It would be interesting to unify all previous future behaviors, singular or not, in one single framework or classification. Our proposal is to look at generalized power and asymptotic expansions in coordinate time of the barotropic index w (or the deceleration parameter q). We shall see that all future behaviors arise naturally in this framework. As a byproduct, new types of strong singularities come up in the vicinity of the phantom divide $w = -1$, sharing features of big crunch or big rip singularities, depending on a sign, and so we have dubbed them grand crunch and grand rip, respectively, since energy density and pressure diverge faster than t^{-2} in coordinate time. They have been overlooked in previous frameworks since for them the scale factor does not admit convergent generalized power expansions around the singularity with a finite number of terms with negative powers, though the barotropic index, the energy density and the pressure do. We shall focus on them.

The paper is organizes as follows. In Sec. II we solve the Friedman equations for a FLRW cosmological model in terms of the barotropic index w. This is shown useful to postulate several kinds of behavior for w, such as power expansions at at finite time event or asymptotic expansions at infinity, which we deal with in Sec. III, and translate them to the scale factor, the energy density and pressure of the Universe. Features of the new types of singularities as well as their geodesic incompleteness and strength are analyzed, respectively, in Secs. IV and V. We end with Sec. VI, Conclusions.

II. SINGULARITIES, BAROTROPIC INDEX *w*, AND DECELERATION PARAMETER *q*

We consider spatially flat homogeneous and isotropic spacetimes with a metric tensor of the form

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)), \qquad (1)$$

where $a(t)$ is the scale factor of the Universe in cosmological time t. Einstein equations for such spacetimes reduce to the usual Friedman equations,

$$\rho = \frac{3\dot{a}^2}{a^2}, \qquad p = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}, \qquad (2)$$

in terms of the energy density $\rho(t)$ and pressure $p(t)$ of the content of the Universe. The dot stands for derivation with respect to the time coordinate. We are using geometrized units for which $c = 1 = 8\pi G$.

Defining the time-dependent barotropic index of the Universe $w(t)$ as the ratio between pressure and energy density allows us to write it in terms of the scale factor and its derivatives,

$$w = \frac{p}{\rho} = -\frac{1}{3} - \frac{2}{3}\frac{a\ddot{a}}{\dot{a}^2}.$$

This formula is valid just for flat models. If curvature is taken into account, additional terms are necessary.

Tha barotropic index w is closely related to the deceleration parameter q,

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{1+3w}{2},$$

again for flat models. Otherwise the relation between both parameters becomes more complicated, involving also the Hubble parameter $H = \dot{a}/a$. This allows direct translation of our results for the barotropic index to the deceleration parameter.

We may see this equation the other way round as the differential equation governing the evolution of the scale factor for a given barotropic index $w(t)$. In fact, we may appease its nonlinearity by introducing the time function $x = \ln a$,

$$\frac{\ddot{x}}{\dot{x}^2} = -\frac{3}{2}(w+1) = -(q+1),$$

which suggests defining

$$h(t) := \frac{3}{2}(w(t)+1) = -q(t) - 1$$

as a correction around the pure cosmological constant case,

$$w(t) = -1 + \frac{2}{3}h(t), \qquad q(t) = -1 - h(t).$$

This change of variables helps us lower the order of the differential equation,

$$h = -\frac{\ddot{x}}{\dot{x}^2} = \left(\frac{1}{\dot{x}}\right)^{\cdot} \Rightarrow \dot{x} = \left(\int h dt + K_1\right)^{-1},$$

which can be solved as a quadrature in terms of two free constants K_1, K_2,

$$a(t) = \exp\left(\int \left(\int h(t)dt + K_1\right)^{-1} dt + K_2\right).$$

The constant K_2 is part of a global constant factor $a(t_0) = \exp(K_2)$,

$$a(t) = a(t_0)\exp\left(\int_{t_0}^{t}\left(\int h(t)dt + K_1\right)^{-1}dt\right), \quad (3)$$

which is fixed by the choice of scale factor equal to one nowadays. Models with such exponential behavior can be found in [15].

For fixing K_1 we are to resort to one of the Friedman equations (2), since we have made use of just the ratio between pressure and energy density,

$$\rho(t) = 3\dot{x}(t)^2 = 3\left(\int_{t_0}^{t} h(t)dt + K_1\right)^{-2},$$

$$p(t) = -2\ddot{x}(t) - 3\dot{x}(t)^2 = \frac{3(h(t)-1)}{(\int_{t_0}^{t} h(t)dt + K_1)^2},$$

allowing us to determine $K_1 = \sqrt{3}\rho(t_0)^{-1/2}$, unless ρ becomes infinite at $t = t_0$, in which case $K_1 = 0$.

We focus on the latter case since our interest is the possibility of formation of singularities. In order to simplify the notation, a time translation is performed to locate the singular event at $t = 0$. The global factor due to K_2 is also omitted.

From the expression for the scale factor,

$$a(t) = \exp\left(\int \frac{dt}{\int h(t)dt}\right),$$

we learn that there are several qualitative behaviors depending on the rate of growth of $h(t)$. If we assume that this function can be expanded in powers of time around $t = 0$,

$$h(t) = h_0 t^{\eta_0} + h_1 t^{\eta_1} + \cdots, \qquad \eta_0 < \eta_1 < \cdots,$$

we get expressions for the scale factor, the energy density, and the pressure at lowest order in t,

$$x(t) = \begin{cases} -\frac{\eta_0+1}{\eta_0 h_0} t^{-\eta_0} + \cdots & \text{if } -1 \neq \eta_0 \neq 0 \\ \frac{1}{h_0}\int \frac{dt}{\ln|t|} + \cdots & \text{if } \eta_0 = -1 \\ \frac{\ln|t|}{h_0} + \cdots & \text{if } \eta_0 = 0. \end{cases}$$

For simplicity, we have considered $t > 0$. Since our equations are symmetric under time reversal, the same expressions are valid exchanging t by $-t$ in order to consider times before $t = 0$.

Once we know the scale factor, we can derive expressions at lowest order for the energy density,

$$\rho(t) = \begin{cases} 3(\frac{\eta_0+1}{h_0})^2 t^{-2(\eta_0+1)} + \cdots & \text{if } -1 \neq \eta_0 \neq 0 \\ \frac{3}{h_0^2}\frac{1}{\ln^2|t|} + \cdots & \text{if } \eta_0 = -1 \\ \frac{3t^{-2}}{h_0^2} + \cdots & \text{if } \eta_0 = 0, \end{cases}$$

and the pressure,

$$p(t) = \begin{cases} \frac{3(\eta_0+1)^2}{h_0} t^{-\eta_0-2} + \cdots & \text{if } -1 \neq \eta_0 < 0 \\ \frac{3}{h_0}\frac{1}{t\ln^2|t|} + \cdots & \text{if } \eta_0 = -1 \\ \frac{3(h_0-1)}{h_0^2} t^{-2} + \cdots & \text{if } \eta_0 = 0 \\ -3(\frac{\eta_0+1}{h_0})^2 t^{-2(\eta_0+1)} + \cdots & \text{if } \eta_0 > 0, \end{cases}$$

and we come across several possibilities:

(i) For $\eta_0 < -2$, both ρ and p vanish at $t = 0$ whereas w diverges. These are generalized sudden or type IV singularities. They also comprise the kind of singularities discussed in [26] and [27], which include w-singularities, for which all derivatives of the energy density and pressure are regular, but with just diverging barotropic index.

(ii) For $\eta_0 = -2$, ρ vanishes at $t = 0$ as t^2, but p remains finite, whereas w diverges. They are a special case of generalized sudden singularities.

(iii) For $\eta_0 \in (-2, -1]$, ρ vanishes at $t = 0$, but p diverges. These are sudden or type II singularities [20].

(iv) For $\eta_0 \in (-1, 0)$, ρ, p and w diverge at $t = 0$. These are type III, big freeze of finite scale factor singularities.

(v) For $\eta_0 = 0$, both ρ and p diverge at $t = 0$ as t^{-2} and $w \simeq -1 + 2h_0/3$ is finite, corresponding to models of the form $a(t) \simeq t^{1/h_0}$. These produce classical big bang/big crunch singularities if h_0 is positive and big rip or type I singularities if h_0 is negative.

(vi) For $\eta_0 > 0$, ρ and p diverge at $t = 0$ as $t^{-2(\eta_0+1)}$ and w tends to the value -1. The possibility of singularity has not been considered before in the previous frameworks. The reason for this is that it cannot be embedded in the classifications in [19] and [33], since the scale factor (exponential of rational functions) does not accept convergent power expansions, generalized or not, with a finite number of terms with negative powers, though $x(t)$ does. We name them "grand rip" or "grand bang/crunch," depending on the behavior of the scale factor at the singularity. We analyze these in detail in Sec. IV.

These results are summarized in Table I, where we have related the first exponent in the generalized power expansion of $h(t)$ at the singularity to the values of the scale

TABLE I. Expansions of q and w at t_s vs possible singularities.

η_0	a_s	ρ_s	p_s	w_s	Sing.
$(-\infty, -2)$	finite	0	0	∞	IV or V
-2	finite	0	finite	∞	IV
$(-2, -1]$	finite	0	∞	∞	II
$(-1, 0)$	finite	∞	∞	∞	III
0	$0/\infty$	∞	∞	finite	big crunch/rip
$(0, \infty)$	$0/\infty$	∞	∞	-1	grand crunch/rip

factor, the energy density, the pressure and the barotropic index and to the type of singularity.

III. BEHAVIOR AT INFINITE TIME

In addition to this analysis of singularities at a finite coordinate time t, we can take into account what happens at $t = \infty$. It is not pointless, since it has been shown [29] that there are geodesics in FLRW spacetimes which reach $t = \infty$ in a finite proper time. As we have already pointed out, the analysis for $t = -\infty$ is entirely similar.

For this analysis we consider now asymptotic expressions for $h(t)$ for large t. We take then $t_0 = \infty$ in (3). Asymptotic expressions for the scale factor, the energy density and pressure take the form

$$a(t) = \exp\left(-\int \left(\int_t^\infty h(t)dt + K_1\right)^{-1} dt\right),$$

$$\rho(t) = 3\left(\int_t^\infty h(t)dt + K_1\right)^{-2},$$

$$p(t) = \frac{3(h(t) - 1)}{(\int_t^\infty h(t)dt + K_1)^2}$$

and if the constant $K_1 = 0$, ρ and p diverge at infinity.

Of course, these expressions are valid only if the integral

$$\int_t^\infty h(t)dt \tag{4}$$

is finite. With this we guarantee that $K_1 = \sqrt{3}\rho(\infty)^{-1/2}$, which is useful for keeping control of the asymptotic behavior of the energy density. Otherwise, we would have to resort to expressions (3) for large t.

For having a finite integral (4) we need $h(t) \to 0$ for large t, though it is not a sufficient condition. For instance, $h(t) = 1/t$ tends to zero, but its integral diverges for large t. Combining finiteness of (4) and asymptotic behavior of $h(t)$ leads in principle to several cases:

(1) Finite $\int_t^\infty h(t)dt$: This happens when $h(t)$ decreases faster than $1/t$. We consider first this case.

Since $h(t)$ tends to zero for large values of time, the asymptotic value of the barotropic index w is -1:

(a) If $h(t) > 0$ for large values of t, the scale factor decreases to zero at infinity as a negative exponential. It would be a sort of "little crunch." The asymptotic value $w_\infty = -1$ of the barotropic index is reached from above in this case. Since $a(t)$ is an integrable function at infinity, this case is included in the set of directional singularities described in [29], which are strong singularities, but only accessible for some observers.

(b) If $h(t) < 0$ for large values of t, the scale factor blows up at infinity exponentiallly. It is the little rip [30] or, for some choices of $h(t)$, the little sibling [32]. The asymptotic value $w_\infty = -1$ of the barotropic index is reached from below.

If we let $K_1 \neq 0$, the scale factor, the energy density and the pressure would be finite at infinity. The case $K_1 < 0$ would correspond to a pseudorip [31].

(2) Infinite $\int_t^\infty h(t)dt$: The expression for the scale factor (3), as well as the ones for the energy density and the pressure are valid with $K_1 \neq 0$. In this case both the energy density and the pressure tend to zero for large t. The sign of $h(t)$, as in the previous case, determines if the scale factor diverges or tends to zero. The asymptotic value of the barotropic index w_∞ is -1 if $h(t)$ tends also to zero. This leads to several subcases:

(a) $1/t \lesssim |h(t)| \to 0$ for large t: The asymptotic value of the scale factor is $w_\infty = -1$.

If $h(t)$ is negative for large t, the scale factor decreases exponentially as an integrable function. This means that noncomoving observers and lightlike geodesics [29] take finite normal time to reach time to reach $t = \infty$, which is a strong "directional singularity."

If $h(t)$ is positive for large t, the scale factor increases exponentially and so this case is similar to the "little rip," but with asymptotically vanishing energy density and pressure and approaching the asymptotic value $w_\infty = -1$ from above.

(b) $h(t) \sim K$ const. for large t: The asymptotic value of the scale factor is $w_\infty = -1 + K$ and the scale factor behaves as a power of time, $a(t) \sim t^{1/K}$, which is an integrable function for large t if $K \in (-1, 0)$, corresponding to a strong directional singularity at $t = \infty$. Otherwise, the scale factor diverges for $K > 0$ or tends to zero for $K \leq -1$, but without singularity.

(c) $|h(t)| \to \infty$ for large t: The barotropic index diverges and the scale factor is nonintegrable. There is no singularity in this case.

If $h(t)$ is positive for large t, the scale factor grows to a finite asymptotic constant value if $\int dt/\int h(t)dt$ converges. Otherwise, the scale factor diverges to infinity.

If $h(t)$ is negative for large t, the scale factor decreases to a finite asymptotic constant value if $\int dt/\int h(t)dt$ converges. Otherwise, the scale factor tends to zero.

These results are summarized in Table II, where the asymptotic behavior of $h(t)$ for large t is related to the asymptotic values of the scale factor, the energy density, the pressure and the barotropic index and to the type of singularity or future behavior.

IV. GRAND RIP AND GRAND BANG/CRUNCH SINGULARITIES

Let us take a look at the new family of singularities for $\eta_0 > 0$. First of all, we notice that pressure and energy

TABLE II. Asymptotic behavior of q and w at $t=\infty$ vs possible behaviors.

h	Signum (h)	K_1	a_∞	ρ_∞	p_∞	w_∞	Behavior
Finite $\int^\infty hdt$	$+$	0	0	∞	∞	-1	∞
	$-$	0	∞	∞	∞	-1	little rip/sibling
	$\pm$	positive	0	finite	finite	-1	nonsingular
	$\pm$	negative	∞	finite	finite	-1	pseudorip
$t^{-1} \lesssim h(t) \to 0$	$+$	any	∞	0	0	-1	little rip with 0 ρ and p
	$-$	any	0	0	0	-1	∞
K	$+$	any	∞	0	0	$-1+\mathrm{K}$	nonsingular
	$-$	$(-1,0)$	0	0	0	$-1+\mathrm{K}$	∞
	$-$	$(-\infty,-1]$	0	0	0	$-1+\mathrm{K}$	nonsingular
$\vert h(t)\vert \to \infty$, infinite $\int^\infty dt/\int h(t)dt$	$+$	any	∞	0	0	∞	nonsingular
$\vert h(t)\vert \to \infty$, infinite $\int^\infty dt/\int h(t)dt$	$-$	any	0	0	0	∞	nonsingular
$\vert h(t)\vert \to \infty$, finite $\int^\infty dt/\int h(t)dt$	$\pm$	any	finite	0	0	∞	nonsingular

density diverge as a power of coordinate time which is different from -2, which would be the case of big bang/crunch and big rip, but it can be as close to such value as desired if the exponent η_0 is small enough.

Second, whereas big bang/crunch and big rip have a different value of the barotropic index $w(0)$ depending on the equation of state, these singularities have the value $w(0)=-1$ regardless of the exponent η_0. Considering only the barotropic index, these singularities arise as small perturbations in coordinate time, $w(t)=-1+2h_0t^{\eta_0}/3$, $\eta_0>0$, around the de Sitter value. This does not mean of course that such perturbations are necessarily singular, since we have explicitly removed the constant K_1 in order to look for singular behavior.

The sign of the coefficient h_0 determines the type of singularity. Since

$$a(t) \simeq e^{-\mathrm{sgn}(h_0)\alpha/t^{\eta_0}}, \quad \alpha = \frac{\eta_0+1}{\eta_0|h_0|} > 0, \quad t>0,$$

we observe two kinds of behavior:

(i) $h_0>0$: In this case the exponential in (3) decreases for $t>0$ and the scale factor a tends to zero on approaching $t=0$ (Figure 1 left). This would be a sort of exponential big bang singularity, or big crunch if we exchange t for $-t$. Since h_0 is positive, the barotropic index w remains always under the phantom divide close to $t=0$. That is, the value $w=-1$ is approached from below. In order to pinpoint the differences and similarities with classical big bang and big crunch singularities, we may call them grand bang and grand crunch singularities.

(ii) $h_0<0$: On the contrary, the exponential increases for $t>0$, and the scale factor a diverges to infinity on approaching $t=0$ (Figure 1 right). We would have then a sort of exponential big rip at $t=0$, which we can locate in the future by exchanging t for $-t$. In this case the barotropic index w is always over the phantom divide and hence the value $w=-1$ is approached from above. As in the previous case, we may name them grand rip singularities.

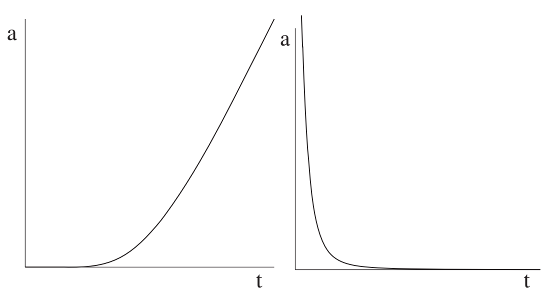

FIG. 1. Singularities at $t=0$ for $\eta_0=1$.

We may check the behavior of causal geodesics at these singularities [13]. We consider parametrized curves on a FLRW spacetime, $\gamma(\tau)=(t(\tau),r(\tau),\theta(\tau),\phi(\tau))$, and impose a normalization condition on the velocity $u(\tau)=\gamma'(\tau)$, depending on its causal type

$$\left.\begin{array}{ll}\text{Timelike:} & -1\\ \text{Lightlike:} & 0\\ \text{Spacelike:} & +1\end{array}\right\} = \varepsilon = \|\gamma'(\tau)\|^2 = -t'^2(\tau) + a^2(t(\tau))(r'^2(\tau)+r^2(\tau)(\theta'^2(\tau) + \sin^2\theta(\tau)\phi'^2(\tau)), \quad (5)$$

where the prime denotes derivation with respect to the parameter τ.

Geodesic curves have zero acceleration, $\nabla_u u \equiv 0$, where ∇ is the covariant derivative associated to the metric (1). However, in this simple case, there is no need to write down the whole system of second order differential equations [19]. Taking into account the symmetry of FLRW spacetimes, it suffices for our analysis to consider curves on the equatorial hypersurface $\theta=\pi/2$ with constant angle ϕ. Homogeneity of the spacetime implies that the linear momentum of geodesics is a conserved quantity,

$$P = u \cdot \partial_r = a^2(t) r'.$$

This equation together with the normalization condition (5) allow us to write the set of differential equations for geodesic motion as a first order system

$$r' = \frac{P}{a^2(t)}, \qquad t' = \sqrt{-\varepsilon + \frac{P^2}{a^2(t)}},$$

for the normal parameter τ.

The key equation is the second one, since the first equation can be integrated once $t(\tau)$ is known.

We analyze know whether causal geodesics are complete [13], that is, if the parameter τ can be extended from $-\infty$ to ∞.

It this happens, it would take an infinite normal time to reach the singularity, which would not be accessible along causal geodesics. This would mean that the worldlines of nonaccelerated observers traveling along them would not end up there.

The analysis of causal geodesics in FLRW spacetimes reduces to just three families of curves:

(i) Lightlike geodesics: $\varepsilon = 0$. These are readily solved,

$$t' = \frac{P}{a(t)} \Rightarrow \tau = \frac{1}{P}\int_0^t a(t) dt,$$

if the scale factor is integrable.

In our case, $a(t) \simeq e^{-\mathrm{sgn}(h_0)\alpha/t^{\eta_0}}$, the integral is convergent for positive h_0. This means that lightlike geodesics meet the singularity at $t = 0$ in a finite normal time τ. These geodesics are therefore incomplete.

On the contrary, for negative h_0, the integral is divergent and it takes an infinite normal time τ to reach $t = 0$. Hence in this case lightlike geodesics avoid reaching the singularity and are complete in that direction. This is similar to what it happens for big rip singularities [19].

(ii) Comoving timelike geodesics: $\varepsilon = -1$, $P = 0$. In this case we can take $\tau = t$ and in both cases these geodesics meet the singularity. They are incomplete.

(iii) Radial timelike geodesics: $\varepsilon = -1$, $P \neq 0$. For $h_0 > 0$, we have $a(t) \ll 1$ and hence

$$t' = \sqrt{1 + \frac{P^2}{a^2(t)}} \simeq \frac{P}{a(t)},$$

and so this case is similar to the lightlike one. They are incomplete.

For $h_0 < 0$, we have $a(t) \gg 1$ and $t' \simeq 1$ and we arrive at the same conclusions as in the comoving case. They are also incomplete.

Summarizing, all causal geodesics arrive at $t = 0$ in finite normal time and are thereby incomplete except for lightlike geodesics at the grand rip, which are complete and do not experience the singularity.

V. STRENGTH OF GRAND RIP AND GRAND CRUNCH SINGULARITIES

Finally, we can check if the strength of the new singularities at $t = 0$ is enough for tidal forces to distort extended bodies [34]. There are several criteria to determine this. All of them model the finite object at each point of a causal geodesic by a volume spanned by three independent Jacobi fields in the hyperspace which has as normal vector the velocity of the curve. Tipler's criterion [35] considers that a singularity is strong if such volume tends to zero on approaching the singularity along the geodesic, whereas Królak's criterion [36] just demands that the derivative of the volumen with respect to the normal parameter must be negative. Hence, there are singularities which are strong according to Królak's criterion, but weak according to Tipler's, for instance, type III or big freeze singularities [19]. Another criterion can be found in [37].

Dealing with Jacobi fields is burdensome, since it involves solving the Jacobi equation along geodesics. However, characterizations for lightlike geodesics and necessary and sufficient conditions for timelike geodesics for fulfillment of both criteria have been established [38] in terms of integrals of the Ricci and Riemann curvatures of the metric of the spacetime along these curves:

(i) Lightlike geodesics: According to Tipler's criterion a singularity is strong along a lightlike geodesic if and only if

$$\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' R_{ij} u^i u^j$$

blows up when the normal parameter τ approaches the value corresponding to the singularity.

According to Królak's criterion the singularity is strong if and only if the integral,

$$\int_0^\tau d\tau' R_{ij} u^i u^j,$$

blows up when τ approaches the singularity.

In our case, $u = (t', r', \theta', \phi') = (P/a, P/a^2, 0, 0)$, integrals of

$$R_{ij} u^i u^j d\tau = 2P^2 \left(\frac{a'^2}{a^4} - \frac{a''}{a^3}\right) \frac{a dt}{P} \simeq \frac{2P \mathrm{sgn}(h_0) \alpha \eta_0 (\eta_0 + 1)}{t^{\eta_0 + 2}} e^{\mathrm{sgn}(h_0)\alpha/t^{\eta_0}} dt$$

blow up at $t = 0$ for all $h_0 > 0$ and hence these singularities are strong according to both criteria. For $h_0 < 0$ we already know that these geodesics do not even reach the singularity.

(ii) Timelike geodesics: For these curves [38] does not provide a characterization, but different necessary and sufficient conditions.

Following Tipler's criterion a singularity is strong along a timelike geodesic if

$$\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' R_{ij} u^i u^j$$

blows up on approaching the singularity.

Following Królak's criterion, the singularity is strong if the integral

$$\int_0^{\tau} d\tau' R_{ij} u^i u^j$$

blows up on approaching the singularity.

There are also necessary conditions, but we are not making use of them for our purposes.

For comoving geodesics, $u = (1, 0, 0, 0)$, integrals of

$$R_{ij} u^i u^j d\tau = -\frac{3a''}{a} dt \simeq -\frac{3\alpha^2 \eta_0^2}{t^{2\eta_0+2}} dt,$$

blow up for all $\eta^0 > 0$ and therefore singularities at $t = 0$ are strong.

For radial geodesics, $u = (\sqrt{1 + P^2/a^2}, \pm P/a^2, 0, 0)$, the analysis is similar,

$$R_{ij} u^i u^j d\tau = \frac{-\frac{3a''}{a} + 2P^2\left(\frac{a'^2}{a^4} - \frac{a''}{a^3}\right)}{\sqrt{1 + \frac{P^2}{a^2}}} dt \simeq \begin{cases} -\frac{3a''}{P} + 2P\left(\frac{a'^2}{a^3} - \frac{a''}{a^2}\right) & \text{if } a \to 0 \\ -\frac{3a''}{a} + 2P^2\left(\frac{a'^2}{a^4} - \frac{a''}{a^3}\right) & \text{if } a \to \infty. \end{cases}$$

For $h_0 > 0$, a, a'' tend to zero at $t = 0$, but the P term has been shown to be exponentially divergent.

For $h_0 < 0$, the integrals of the a''/a term have been shown to be divergent, though the P term tends to zero.

Hence, in both cases radial geodesics meet a strong singularity at $t = 0$.

Summarizing, singularities are strong for all geodesics except for lightlike geodesics in the $h_0 < 0$ case, which are not even incomplete.

VI. CONCLUDING REMARKS

We have shown that generalized power and asymptotic expansions of the barotropic index w and the deceleration parameter q in time coordinate are useful to classify most singular and nonsingular future behaviors of the Universe. In addition to well-known scenarios, another type of possible singular behavior is found for small corrections of $w = -1$ and $q = -1$ at a finite time. These singularities share many features of big rip or big bang/crunch singularities, depending on the sign of the perturbation, and so we have dubbed them, respectively, grand rip and grand bang/crunch singularities. They can appear just when the barotropic index and the deceleration parameter take the value of minus one. Both energy density and pressure diverge at the singularity as a negative power of coordinate time, which can be as close as desired to minus two. The scale factor does not admit power expansions around the singular value $t = 0$ with a finite number of terms with negative powers, not even in the case of vanishing $a(0)$. They are strong singularities, except for lightlike geodesics, which avoid the grand rip singularity.

Considering the asymptotic expansions at $t = \infty$, in addition to little rip and pseudorip behaviors, the only singularities that are found are directional singularities, which are experienced just by noncomoving observers and lightlike geodesics. As a novelty, they are also found for $w_\infty = -1$.

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