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Behavior of Friedmann-Lemaître-Robertson-Walker Singularities

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Abstract In Stoica (Int. J. Theor. Phys. **55**, 71–80, 2016) a regularization procedure is suggested for regularizing Big Bang singularities in Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes. We argue that this procedure is only appliable to one case of Big Bang singularities and does not affect other types of singularities.

Keywords Cosmology · Big bang · Singularities

Singularities are present in most physically relevant cosmological models and challenge the validity of General Relativity as the theory of gravitation. It is conjectured that such singularities will be removed or appeased by a quantum theory of gravity or by corrections to Einstein's theory.

In General Relativity the universe is a spacetime endowed with a metric and at cosmological scale it is assumed to be homogeneous and isotropic. Hence the metric in coordinates t, r, θ, ϕ is given by

$$ds^2 = -dt^2 + a^2(t) \left(dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right), \tag{1}$$

with the usual ranges for spherical coordinates. The only free function is the scale factor $a(t)$.

We consider here just flat cosmological models since they appear to be favored by observations.

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Assuming that the universe is filled by a perfect fluid with energy density ρ and pressure p, the equations which relate them to the scale factor $a(t)$ are Friedmann's equations

$$\rho = \frac{3\dot{a}^2}{a^2}, \qquad p = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}. \tag{2}$$

The problem is solved if we know the equation of state for the fluid, $p = p(\rho)$ or equivalently the barotropic index $w = p/\rho$.

For instance, if w is constant, we can integrate Friedmann's equations

$$a(t) = a_0 t^{2/3(1+w)}, \tag{3}$$

up to a constant a_0, obtaining power-law models. They are valid for epochs with nearly constant equation of state. For these models, the energy density and the pressure take the form

$$\rho(t) = \frac{4}{3(1+w)^2 t^2}, \qquad p(t) = \frac{4w}{3(1+w)^2 t^2}. \tag{4}$$

We see that for these models both energy density and pressure blow up as t^{-2} at $t = 0$ in coordinate time. It is the Big Bang singularity.

For instance, in a radiation dominated epoch, $w = 1/3$, the scale factor behaves as $t^{1/2}$, whereas in a pressureless dust dominated epoch, $w = 0$, it behaves as $t^{2/3}$. The case $w = -1$ corresponds to a dark energy dominated epoch and has a different solution,

$$a(t) = a_0 e^{\sqrt{\Lambda/3}t}, \qquad \rho = \Lambda = -p,$$

in terms of the cosmological constant Λ.

Stoica [1] suggests that Big Bang singularities may be removed by replacing the energy density and the pressure by their *densitized* versions,

$$\tilde{\rho} = \rho\sqrt{-g}, \qquad \tilde{p} = p\sqrt{-g}, \tag{5}$$

where $g = a(t)^3 r^2 \sin\theta$ is the determinant of the metric (1).

The key feature of the result is that the products ρa^3, $p a^3$ are expected to be smooth instead of blowing up at Big Bang. According to [1]:

Theorem 1 *If a is a smooth function, then the densities $\tilde{\rho}$, $\tilde{p}$ are smooth (and therefore nonsingular), even at moments t_0 when $a(t_0) = 0$.*

Confronting this result with cosmological power-law models, we notice that for them the products ρa^3, $p a^3$ behave as $t^{-2w/(1+w)}$. Hence, $\tilde{\rho}$ and $\tilde{p}$ do not blow up at $t = 0$ if and only if the exponent $-2w/(1+w)$ is positive, that is, if and only if $w \in [-1, 0]$, that is, quintessence cosmological models.

Classical equations of state fall out of this result: For them energy conditions [2] impose that $w \in [0, 1]$. The only case for which $\tilde{\rho}$, $\tilde{p}$ are regular at $t = 0$ is the one of presureless dust, $w = 0$. This is to be expected, since for these models the scale factor is not even a C^1 function, since behaves as a power of time t with exponent lower than one.

Although they are not considered in [1], the same happens with Big Rip singularities [3]. These are strong singularities [4] and correspond to power-law models with negative exponent, that is, $w < -1$.

There are also singularities with finite scale factor $a(0)$ [5, 6]. They are classified as Type II, III and IV in [7] and V in [8]. For them close to $t = 0$ the scale factor behaves as $a(t) \simeq a_0 + a_1 t^\alpha$, with $\alpha > 0$, and so the energy density and the pressure

$$\rho(t) \simeq 3a_1^2 \alpha^2 t^{2(\alpha-1)}, \qquad p(t) \simeq 2a_1 \alpha(1-\alpha) t^{\alpha-2}, \tag{6}$$

behave behave as their densitized versions. Multiplyling by $a(t)^3$ does no affect their regularity.

Completing the classification of cosmological singularities [9], we find Grand Bang and Grand Rip singularities. For them the scale factor vanishes or blows up exponentially, $a(t) \simeq a_0 e^{a_1 t^{-\alpha}}$, depending on the sign of a_1, negative por Grand Bang and positive for Grand Rip. In this case, both the energy density and the pressure diverge as a power of coordinate time,

$$\rho(t) \simeq 3a_1^2 \alpha^2 t^{-2(\alpha+1)}, \quad p(t) \simeq -3a_1^2 \alpha^2 t^{-2(\alpha+1)},$$

and hence multiplying by a vanishing exponential scale factor (Grand Bang) brings vanishing instead of diverging $\tilde{\rho}$ and $\tilde{p}$. For Grand Rip singularities (diverging exponential scale factor), $\tilde{\rho}$ and $\tilde{p}$ diverge.

Anyway, the use of non-diverging versions of energy density and pressure in Einstein's equations does not prevent the formation of singularities:

According to General Relativity, non-accelerated observers follow causal geodesics on a spacetime endowed with a metric which is a solution of Einstein's equations. Geodesics can be parametrized using as parameter the proper time τ, which is defined by the relation $d\tau^2 = -ds^2$.

Singularities appear in the form of incomplete causal geodesics [2], that is, causal geodesics which cannnot be defined for all values of $\tau \in \mathbb{R}$.

In the case of FLRW spacetimes, there are two types of geodesics [4]: radial and comoving geodesics.

Parametrizations of radial geodesics are obtained as solutions of first order equations,

$$t' = \sqrt{-\varepsilon + \frac{P^2}{a^2}}, \quad r' = \frac{P}{a^2}, \quad \theta' = 0, \quad \phi' = 0, \quad (7)$$

where the prime stands for derivation with respect to τ. The constant P is a conserved quantity of geodesic motion, which is the linear momentum of the geodesic, and ε takes the value one for timelike geodesics and zero por lightlike geodesics.

If the conserved quantity P is zero, we have timelike comoving geodesics. For these coordinate time is essentially proper time, since geodesics equations reduce to

$$t' = 1, \quad r' = 0, \quad \theta' = 0, \quad \phi' = 0. \quad (8)$$

It is clear that regularizing the energy density and the pressure as in [1] does not affect the formation of singularities according to the previous equations.

Even in the case of timelike comoving geodesics, we could think in principle that they are complete, since their parametrization is just $t = \tau +$ const. for all values of the proper time τ. But most curvature invariants, for instance the Ricci curvature $R = -\mu + 3p$, point out that the manifold does not include the locus $t = 0$, since the curvature becomes infinite there, regardless of the regularization.

Summarizing, we have shown that the regularization of the energy density and the pressure suggested by [1] does not allow the extension of the spacetime beyond the Big Bang singularity, since the scale factor of the universe $a(t)$ is not affected by this change and it is the only function determining the geometry of the universe (1). In fact, the scale factor (3) is not a C^1 function at $t = 0$ for Big Bang singularities and hence second order equations for it, such as Friedmann's equations (2), are ill-defined for $a(t)$. Causal geodesics still reach the singularity at $t = 0$ in finite proper time, where they meet a curvature singularity, since curvature invariants of the spacetime diverge as t^{-2}, pointing out that the geometry of the universe is singular there, regardless of the regularization. Besides, the rescaling (5) does

not produce regular versions of the energy density and the pressure in the case of Big Bang singularities, except for the case of a pressureless dust equation of state.

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