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Flux through a Möbius strip?

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Abstract
Integral theorems such as Stokes' and Gauss' are fundamental in many parts of physics. For instance, Faraday's law allows computing the induced electric current on a closed circuit in terms of the variation of the flux of a magnetic field across the surface spanned by the circuit. The key point for applying Stokes' theorem is that this surface must be orientable. Many students wonder what happens to the flux through a surface when this is not orientable, as it happens with a Möbius strip. On an orientable surface one can compute the flux of a solenoidal field using Stokes' theorem in terms of the circulation of the vector potential of the field along the oriented boundary of the surface. But this cannot be done if the surface is not orientable, though in principle this quantity could be measured on a laboratory. For instance, checking the induced electric current on a circuit along the boundary of a surface if the field is a variable magnetic field. We shall see that the answer to this puzzle is simple and the problem lies in the question rather than in the answer.

Keywords: Stokes theorem, Moebius strip, Faraday's law, flux, circulation

1. Introduction

The Möbius strip [1] has attracted the interest of researchers and academics due to its fascinating geometric properties [2]. In spite of its name [3], it was not discovered first by August Ferdinand Möbius, but independently by Johann Benedict Listing [4], the father of modern topology.

The construction is fairly simple: starting with a rectangular piece of paper, one can join two opposite edges in order to form a cylinder. But if before joining the opposite edges we twist the rectangle 180º we obtain this reknowned one-sided surface.

The strip is a non-orientable surface and for this reason it does not have an outer and an inner side as usual surfaces, such as the sphere, the plane or the cylinder. It is a one-sided surface and this fact has suggested many applications in engineering [2]. For instance, for designing audio and film tapes which could record longer, since they could be used on the only, but

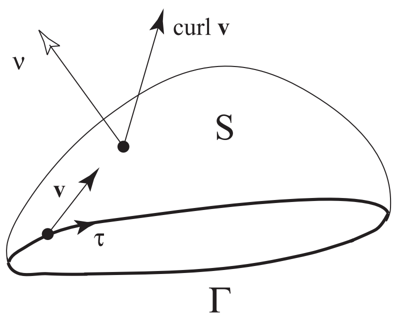

Figure 1. Stokes' theorem: we have a surface S with unitary normal ν, bounded by a closed curve Γ with tangent field τ. We may compute the flux of the vector field curl $\mathbf{v}$ either by summing up the contributions of $\langle \text{curl}\,\mathbf{v}, \nu\rangle$ on the surface S or the contributions of $\langle \mathbf{v}, \tau\rangle$ along the curve Γ.

double length, side. For the same reason, it has been used in printing tapes for printers and old typewriters. There are also Möbius' strips in luggage conveyor belts in airports in order to double their useful life. And a resistor with this shape was patented [5, 6], made up of two conductive layers and filled with a dielectric material, preventing residual self-inductance. There are even aromatic molecules in organic chemistry with this shape [7]. And we cannot forget that it is part of the universal recycling symbol, formed by three green arrows.

But besides academics and engineers, the Möbius strip has attracted the attention of many science students. Just check for 'flux across a Möbius strip' and you will obtain thousands of results in your favourite search engine. We focus on their interest in this issue as target for this paper, as well as their teachers'.

The reason for this is that integral theorems such as Stokes' just can be applied to orientable surfaces [8], relating the flux of the curl of a vector field across a surface with its circulation along the boundary of the surface (see figure 1).

One might think this is a tricky question, since the answer is negative: it just cannot be calculated. But there are experiments in physics where one could think this question could have a meaning.

Consider for instance a circuit attached to the boundary of a Möbius strip. According to Faraday's law, the flux of a variable magnetic field across the surface induces an electric current on the circuit. One can measure the electromotive force on the circuit, but in principle Faraday's law cannot be applied to calculate it with the flux across the surface. This is the issue we would like to clarify in this paper.

But before providing a solution to this puzzle, we need to recall some useful concepts. In section 2 we review the concepts of flux and circulation before stating Stokes' theorem. In section 3 we describe the Möbius strip as a non-orientable surface. As it was expected, the calculations performed on the Möbius strip and on its boundary do not coincide, as Stokes' theorem is not applicable, as we show in section 4. But a simple solution to this issue is provided in section 5. A final section of conclusions is included at the end of the paper.

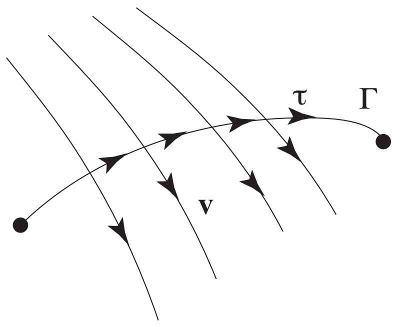

Figure 2. Circulation of the vector field **v** along the curve Γ: the circulation of the field **v** along the curve Γ is calculated summing up the contributions of $\langle \mathbf{v}, \tau\rangle$ along the curve Γ.

2. Stokes' theorem

Before recalling Stokes' theorem, there are a few definitions we need to recall: the circulation of a vector field along a curve and the flux of a vector field across a surface. This can be reviewed in your favourite Vector Calculus book. I have chosen [8] for its nice examples relating physics and mathematics.

The line integral or circulation of a vector field along a curve is the generalisation of the concept of the work done by a force along a trajectory.

Let us consider a continuous vector field **v** and a curve Γ oriented by its tangent field of velocities τ: that is, we specify if the curve is followed onwards or backwards. If the points on the curve Γ are parameterised by $\gamma(t) = (x(t), y(t), z(t))$, $t \in [a, b]$, the velocity of this parameterisation is given by $\tau\,(\gamma(t)) = \gamma'(t)$, where the $'$ denotes derivation with respect to time t.

We define the **line integral or circulation of v along Γ** as the sum of the projections of **v** along τ at the points on the curve,

$$\mathcal{C}_{\mathbf{v},\Gamma} := \int_\Gamma \left\langle \mathbf{v}, \frac{\tau}{\|\tau\|}\right\rangle \mathrm{d}s = \int_a^b \langle \mathbf{v}, \tau\rangle_{\gamma(t)}\,\mathrm{d}t, \tag{1}$$

taking into account that the length element of a parameterised curve is $\mathrm{d}s = \|\gamma'(t)\|\mathrm{d}t$. The $\langle , \rangle$ stands for the scalar or inner product, whereas $\|\|$ stands for the length of a vector.

We see that this definition does not change on changing the parameterisation of the curve, but it depends on the orientation of the curve. That is, it is the same no matter how fast we follow the curve. But if we follow the curve the other way round, the circulation changes by a sign (see figure 2).

On the other hand, the flux integral of a vector field across a surface is also suggested by examples in Mechanics, Electromagnetism and Fluid Mechanics [9]: the flux of a gravitational field across a closed surface is related to the mass contained inside, the flux of a electrostatic field is related to the total charge inside the surface and the variation of the flux of a magnetic field across a surface is related to the electromotive force induced on the boundary of the surface.

Let us consider a compact surface S and a continuous vector field **v**. The orientation of the surface is given by a continuous unitary vector field ν normal to S at every point. For a closed surface, we have just two choices: a vector field pointing inwards or outwards. If such a vector

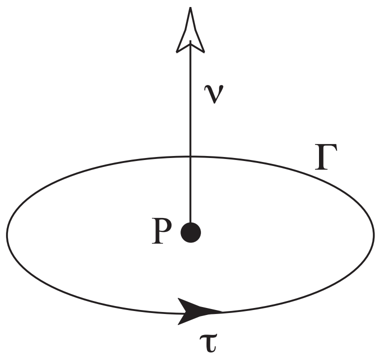

Figure 3. Orientation of circumference S^1 induced by the one on the disk D^2: if we set our right thumb along the normal ν to the disk, our fingers show the orientation for τ along the boundary curve Γ.

field exists, the surface is called **orientable**. The **flux of v across *S*** is defined as the sum of the projections of $\mathbf{v}$ along ν at the points of the surface,

$$\Phi_{\mathbf{v},S} := \int_S \langle \mathbf{v}, \nu \rangle \, \mathrm{d}S, \tag{2}$$

where $\mathrm{d}S$ is the area element of the surface.

If the surface is closed, the orientation of the surface is taken as positive when ν points out of the surface. For a closed surface then, the flux is positive if more field lines go out of the surface than enter the surface.

If the surface is open, we can choose either orientation for it. But the chosen orientation for S induces an orientation for its boundary Γ, as we see in figures 1 and 3: if our right thumb points as the normal vector ν, our fingers show the way the boundary Γ is to be followed. This convention is necessary to avoid amibiguities on stating Stokes' theorem.

For explicit calculations, we usually need a parameterisation for the points on the surface S. This is a function, with certain restrictions [10], $g : D \in \mathbb{R}^2 \to \mathbb{R}^3$, such that $g(u, v) = (x(u, v), y(u, v), z(u, v)) \in S$. That is, we describe the points of S using curvilinear coordinates u, v.

The lines of constant u, parameterised by $g(u_0, v)$ and the lines of constant v, parameterised by $g(u, v_0)$, are called coordinate lines of the parameterisation g of S. Since these lines are contained on the surface, their velocities,

$$\mathbf{X_u}(u, v) = \frac{\partial g(u, v)}{\partial u}, \qquad \mathbf{X_v}(u, v) = \frac{\partial g(u, v)}{\partial v},$$

are tangent vector fields to the surface S and their vector product $\mathbf{X_u} \times \mathbf{X_v}$ defines a normal vector field to the surface S. Hence, a unitary normal vector field is

$$\nu(u, v) = \frac{\mathbf{X_u} \times \mathbf{X_v}}{\|\mathbf{X_u} \times \mathbf{X_v}\|},$$

but we could have chosen the opposite one, just exchanging the order of the coordinates.

If the unitary normal vector field is provided this way, since the surface element in such parameterisation is

$$\mathrm{d}S = \|\mathbf{X_u} \times \mathbf{X_v}\| \, \mathrm{d}u \, \mathrm{d}v,$$

the flux may be computed as

$$\Phi_{\mathbf{v},S} = \int_D \langle \mathbf{v}, \mathbf{X_u} \times \mathbf{X_v} \rangle \, \mathrm{d}u \, \mathrm{d}v = \int_D \begin{vmatrix} v^x v^y & & v^z \\ \dfrac{\partial x(u,v)}{\partial u} & \dfrac{\partial y(u,v)}{\partial u} & \dfrac{\partial z(u,v)}{\partial u} \\ \dfrac{\partial x(u,v)}{\partial v} & \dfrac{\partial y(u,v)}{\partial v} & \dfrac{\partial z(u,v)}{\partial v} \end{vmatrix}_{g(u,v)} \mathrm{d}u \, \mathrm{d}v.$$

It can be seen that this expression is independent of the chosen parameterisation, except for the sign due to the choice of orientation.

For instance, a sphere of radius R can be parameterised using the colatitude angle θ and the azimuthal angle ϕ,

$$g(\theta, \phi) = (R \sin\theta \cos\phi, R \sin\theta \sin\phi, R \cos\theta), \quad \theta \in (0, \pi), \ \phi \in (0, 2\pi),$$

with some degeneracy, since $g(0, \phi) = (0, 0, R)$ is the north pole of the sphere for all values of ϕ and $g(\pi, \phi) = (0, 0, -R)$ is the south pole of the sphere for all values of ϕ.

The lines of constant θ, parameterised by $g(\theta_0, \phi)$, are the parallels of the sphere and the lines of constant ϕ, parameterised by $g(\theta, \phi_0)$, are the meridians of the sphere.

Now we are ready to state Stokes' theorem. Integral theorems such as Green's, Gauss' and Stokes' theorems are fundamental in physics, mainly in Fluid Mechanics and Electromagnetism, since they relate integrals of a field in a region with integrals on its boundary. In this sense, they may be viewed as a way to reduce the dimensions of the integral, but the physical consequences are far deeper. This is most relevant for conservative and solenoidal fields, which can be written respectively as the gradient or the curl of a potential.

In this paper we are interested in Stokes' theorem, which relates the flux integral of the curl of a vector field across a surface with the circulation of the field along the boundary of the surface. It may be stated as follows:

Stokes' theorem: let S be a smooth, compact, oriented surface, bounded by a curve Γ. Let $\mathbf{v}$ be a smooth vector field. The flux of the curl of $\mathbf{v}$ across S, $\Phi_{\mathrm{curl}\mathbf{v},S}$ and the circulation or $\mathbf{v}$ along Γ, $\mathcal{C}_{\mathbf{v},\Gamma}$ are related by

$$\Phi_{\mathrm{curl}\,\mathbf{v},S} = \mathcal{C}_{\mathbf{v},\Gamma}, \tag{3}$$

where the orientation for Γ is the one induced by the orientation of S.

The curl is a differential vector operator,

$$\mathrm{curl}\,\mathbf{v} = \begin{vmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_x} \\ \partial_x & \partial_y & \partial_z \\ v^x & v^y & v^z \end{vmatrix},$$

for a vector field $\mathbf{v} = v^x \mathbf{e_x} + v^y \mathbf{e_y} + v^z \mathbf{e_z}$ with coordinates (v^x, v^y, v^z) in the orthonormal trihedron $\{\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\}$ of unitary vectors along the respective axes X, Y, Z.

Stokes' theorem provides a nice interpretation for the curl of a vector field $\mathbf{v}$ at a point P. Let us consider a small disk D^2, bounded by a circumference S^1 of radius ε centered at P with unitary normal ν parallel to curl $\mathbf{v}(P)$ (see figure 3).

At lowest order, if the radius ε is small, we can take curl $\mathbf{v}$ as constant on the disk,

$$\mathcal{C}_{\mathbf{v},S^1(\varepsilon)} = \Phi_{\mathrm{curl}\,\mathbf{v},D^2(\varepsilon)} \approx \pi\varepsilon^2 \, \|\mathrm{curl}\,\mathbf{v}(x_0, y_0, z_0))\,\|,$$

and so we may view the curl of $\mathbf{v}$ at a point P as the density of circulation of this field on the orthogonal plane, since

$$\|\mathrm{curl}\,\mathbf{v}(P)\| = \lim_{\varepsilon\to 0}\frac{\mathcal{C}_{\mathbf{v},S^1(\varepsilon)}}{\pi\varepsilon^2}.$$

Hence, the curl of a field shows the existence of closed field lines or whirlpools (finite circulation) around a point. Besides, its direction provides the orientation of these whirlpools. This is related to the fact that solenoidal fields are generated by currents instead of charges.

One typical example of application of Stokes' theorem is Faraday's law, one of Maxwell's laws for Electromagnetism [8], which relates the electrical field $\mathbf{E}$ with the magnetic field $\mathbf{B}$ through

$$\mathrm{curl}\,\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}. \tag{4}$$

If we calculate the circulation of the electric field along a closed curve Γ, after applying Stokes' theorem to a surface S bounded by Γ, we get

$$\mathcal{C}_{\mathbf{E},\Gamma} = \Phi_{\mathrm{curl}\,\mathbf{E},S} = -\Phi_{\frac{\partial\mathbf{B}}{\partial t},S} = -\frac{\partial\Phi_{\mathbf{B},S}}{\partial t},$$

using Faraday's law and taking out the derivative with respect to time.

If we think of the curve Γ as a closed circuit, the circulation of $\mathbf{E}$ is the electromotive force induced by the varying magnetic field. This is the simple principle which explains how electric motors work.

Another useful application of the theorem is the calculation of the flux of a solenoidal field $\mathbf{v} = \mathrm{curl}\,\mathbf{A}$, that is, of a vector field $\mathbf{v}$ endowed with a vector potential $\mathbf{A}$,

$$\Phi_{\mathbf{v},S} = \mathcal{C}_{\mathbf{A},\Gamma}, \tag{5}$$

so that it equals the circulation of its vector potential along the boundary of the surface.

According to this result, the flux of the solenoidal field $\mathbf{v}$ does not depend on the surface S, but just on its boundary Γ. If the surface is closed, there is no boundary and the flux of a solenoidal field across closed surfaces is always zero. For open surfaces, the flux is the same across *any* other surface bounded by Γ. This fact shall be useful for our purposes later on.

3. Möbius strip

As we mentioned in section 1, building a Möbius strip is fairly simple (see, for instance, page 106 in [10]). Let us consider a vertical segment $I = \{(R, 0, z) : z \in [-a, a]\}$ of length $2a$ and the circumference C of radius $R > a$ and center $(0, 0, 0)$, lying on the plane $z = 0$. If we rotate the segment I, keeping it vertical, along the circumference C, we would obtain a circular cylinder. But we allow the segment also to rotate upside down on travelling along C in such a way that the segment is always contained in the plane described by the Z axis and the radius of the circumference through the center of the segment (see figures 4 and 5). The resulting surface S is a Möbius strip (see figure 6), which may be parameterised in a simple fashion with such a geometric construction.

It seems reasonable to use as parameters the position of a point on the segment I, $r \in (-a, a)$ and the angle ϕ rotated by the center of the segment along the circumference.

When the center of the segment has rotated and angle ϕ along the circumference, the segment rotates an angle $\phi/2$ around its center. If the center of the segment had not rotated along

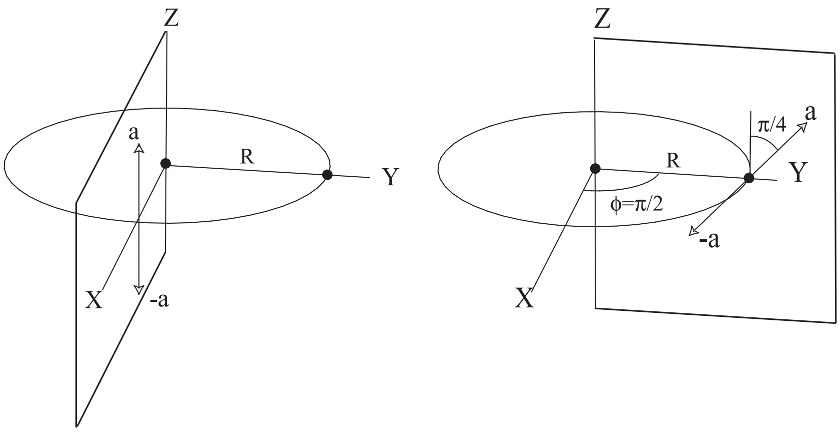

Figure 4. Initial location of the segment *I* and after its center rotates $\phi = \pi/2$.

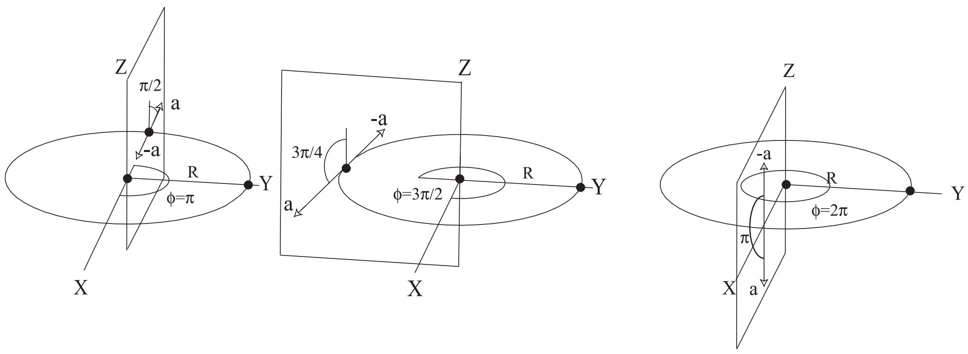

Figure 5. Location of the segment *I* after its center rotates $\phi = \pi, 3\pi/2, 2\pi$.

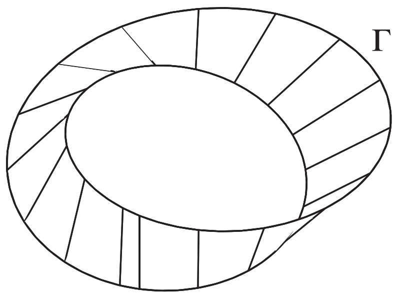

Figure 6. Möbius strip.

the circumference, it would have been parameterised as $(R + r\sin(\phi/2), 0, r\cos(\phi/2))$. But since it has rotated an angle ϕ along the circumference, we have

$$g(r,\phi)=\left(\left(R+r\,\sin\frac{\phi}{2}\right)\cos\,\phi,\left(R+r\,\sin\frac{\phi}{2}\right)\sin\,\phi,r\,\cos\frac{\phi}{2}\right),$$

for $r\in(-a,a)$, $\phi\in(0,2\pi)$ as a parameterisation for the Möbius' strip.

That is, $g(r,\phi)$ describes the position of the original point corresponding to $r\in(-a,a)$ after rotation of the segment by an angle $\phi/2$ and rotation of its center along the circumference by an angle ϕ.

Using the velocities of the coordinate lines,

$$\mathbf{X_r}(r,\phi)=\left(\sin\frac{\phi}{2}\,\cos\,\phi,\sin\frac{\phi}{2}\,\sin\,\phi,\cos\frac{\phi}{2}\right),$$

$$\mathbf{X}_{\boldsymbol{\phi}}(r,\phi)=\left(-\left(R+r\,\sin\frac{\phi}{2}\right)\sin\,\phi,\left(R+r\,\sin\frac{\phi}{2}\right)\cos\,\phi,0\right)$$
$$+\frac{1}{2}\left(r\,\cos\frac{\phi}{2}\,\cos\,\phi,r\,\cos\frac{\phi}{2}\,\sin\,\phi,-r\,\sin\frac{\phi}{2}\right),$$

we may obtain a normal vector field, $\mathbf{X_r}\times\mathbf{X}_{\boldsymbol{\phi}}$ to the strip at every point.

We notice that this normal vector field is not continuous: if we compare the expressions at the center of the segment, $r=0$, after completing a turn from $\phi=0$ to $\phi=2\pi$,

$$\mathbf{N}(0,0)=(0,0,1)\times(0,R,0)=(-R,0,0),$$

$$\mathbf{N}(0,2\pi)=(0,0,-1)\times(0,R,0)=(R,0,0),$$

the normal vector changes from pointing out of the center of the circumference to pointing towards the center, though the point on the strip is the same. Hence, the Möbius strip is not orientable.

The boundary Γ of the Möbius strip S is the curve described by both endpoints $\{-a,a\}$ of the segment on rotating. Or equivalently, since the endpoint a arrives at the original position of $-a$ after a whole turn, we may describe Γ by the motion of just the endpoint a after the segment travels twice along the circumference to end up at the original position,

$$\gamma(\phi)=\left(\left(R+a\,\sin\frac{\phi}{2}\right)\cos\,\phi,\left(R+a\,\sin\frac{\phi}{2}\right)\sin\,\phi,a\,\cos\frac{\phi}{2}\right),$$

for $\phi\in[0,4\pi]$.

4. Flux *across* a Möbius' strip

We are ready to perform some calculations on the strip and its boundary. For simplicity, we consider a simple constant vector field along the Z axis, $\mathbf{v}=(0,0,1)$. This field is solenoidal and a simple vector potential for it is $\mathbf{A}(x,y,z)=(-y/2,x/2,0)$. That is, $\mathbf{v}=\text{ curl }\mathbf{A}$.

The circulation of $\mathbf{A}$ along Γ, the boundary of the strip S is well defined, since it is an oriented curve, and may be readily computed.

We need the velocity of the parameterisation of Γ, with velocity,

$$\gamma'(\phi)=\left(-\left(R+a\,\sin\frac{\phi}{2}\right)\sin\,\phi,\left(R+a\,\sin\frac{\phi}{2}\right)\cos\,\phi,0\right)$$
$$+\frac{1}{2}\left(a\,\cos\frac{\phi}{2}\,\cos\,\phi,a\,\cos\frac{\phi}{2}\,\sin\,\phi,-a\,\sin\frac{\phi}{2}\right),$$

and the vector potential on the points of Γ in this parameterisation,

$$\mathbf{A}(x(r,\phi),y(r,\phi),z(r,\phi))=\frac{1}{2}\left(-\left(R+a\,\sin\frac{\phi}{2}\right)\sin\,\phi,\right.$$
$$\left.\left(R+a\,\sin\frac{\phi}{2}\right)\cos\,\phi,0\right).$$

Their inner product is just

$$\langle\mathbf{A}(\gamma\ (\phi)),\gamma'(\phi)\rangle=\frac{1}{2}\left(R+a\,\sin\frac{\phi}{2}\right)^2,$$

which makes the calculation of the circulation simple,

$$\mathcal{C}_{\mathbf{A},\Gamma}=\int_0^{4\pi}\langle\mathbf{A}(\gamma(\phi)),\gamma'(\phi)\rangle\,\mathrm{d}\phi=\frac{1}{2}\int_0^{4\pi}\left(R+a\,\sin\frac{\phi}{2}\right)^2\mathrm{d}\phi=2\pi R^2+\pi a^2. \tag{6}$$

But if we naively calculate the flux of $\mathbf{v}$ across the strip,

$$\Phi_{\mathbf{v},S}=\int_{-a}^{a}\mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi\,\langle\mathbf{v},\mathbf{X_r}\times\mathbf{X}_\phi\rangle=\int_{-a}^{a}\mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi\left(R+r\,\sin\frac{\phi}{2}\right)\sin\frac{\phi}{2}$$
$$=8Ra,$$

which of course does not provide the same result as the circulation of $\mathbf{A}$ along the boundary Γ, since the strip is not orientable and Stokes' theorem is not applicable.

However, there is a way to provide a meaning and an interpretation to the previous integral. If we cut the strip along the original segment at $\phi=0$, we obtain an oriented open strip, but its boundary is not Γ as one could expect, but the union $\tilde{\Gamma}$ of four pieces: the piece of Γ corresponding to $\phi\in(0,2\pi)$, the piece of Γ corresponding to $\phi\in(2\pi,4\pi)$ with reversed orientation and the original segment I counted twice to link both segments of Γ (see figure 7). Since I is orthogonal to $\mathbf{A}$, it does not contribute to the circulation,

$$\mathcal{C}_{\mathbf{A},\tilde{\Gamma}}=\int_0^{2\pi}\langle\mathbf{A}(\gamma(\phi)),\gamma'(\phi)\rangle\,\mathrm{d}\phi-\int_{2\pi}^{4\pi}\langle\mathbf{A}(\gamma(\phi)),\gamma'(\phi)\rangle\,\mathrm{d}\phi$$
$$=\frac{1}{2}\int_0^{2\pi}\left(R+a\,\sin\frac{\phi}{2}\right)^2\mathrm{d}\phi-\frac{1}{2}\int_{2\pi}^{4\pi}\left(R+a\,\sin\frac{\phi}{2}\right)^2\mathrm{d}\phi=8Ra,$$

which of course provides the same result as the flux across the open strip, since Stokes' theorem is applicable to this oriented surface.

Though of course it is not the result we are after, since we wish to recover the circulation of $\mathbf{A}$ along Γ, not the flux of curl $\mathbf{A}$ across a broken Möbius strip.

5. Circulation along the boundary of the strip

We have checked explicitly that the flux of a solenoidal field across a Möbius' strip and the circulation of its potential vector along the boundary of the strip are not the same, since Stokes' theorem cannot be applied to a non-orientable surface.

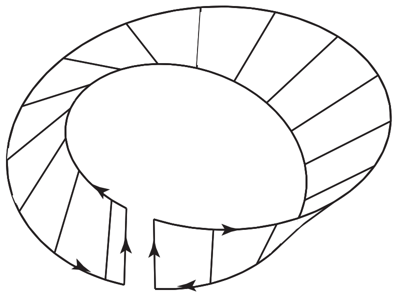

Figure 7. Open Möbius strip.

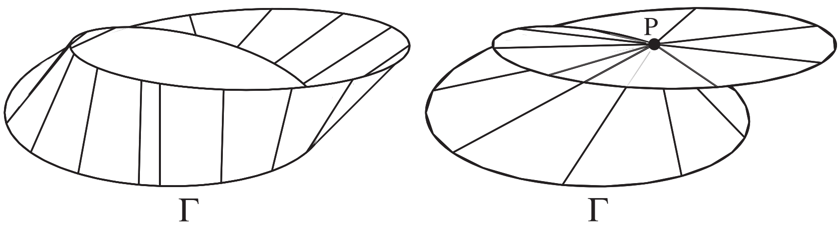

Figure 8. Möbius strip and orientable cone bounded by Γ: Both surfaces are bounded by the curve Γ. The cone is constructed by linking the point P with every point on Γ.

However, the circulation of the field along the boundary of the strip does have a physical meaning. As we have already mentioned, it could be the electromotive force induced on a circuit located along Γ by a varying magnetic field. Is it possible to calculate it using Faraday's law?

When written in this way we notice that the answer is simpler as expected when we formulated the question in terms of the flux across a Möbius' strip, which sounded more appealing. Our goal is not the flux, which is an auxiliary quantity, but the circulation or electromotive force, which is the one we can measure.

And again Stokes' theorem is of much help, since it can suggest the right answer to the right question. If we are interested in calculating the circulation $\mathcal{C}_{\mathbf{v},\Gamma}$, we notice that Stokes' theorem simply states that it can be done with the flux across *any* oriented surface bounded by Γ. That is, Möbius strip has Γ as boundary and has been useful for defining it, but that is all: the strip is a bad choice, since it is not an oriented surface. But we can use any other oriented surface with the same boundary, as suggested in exercise 7.30 in [11].

Cones are the simplest choice, since any closed curve without self-intersections can be the boundary of a cone. We take any point P in space as the vertex of the cone and draw the segments that link P with the points of Γ. The resulting surface is a cone bounded by Γ and is an oriented surface. The only issue is that we have to choose P so that the cone does not have self-intersections.

A simple choice for the vertex is $(-R, 0, 0)$ (see figure 8), the middle point of the horizontal segment on the strip at $\phi = \pi$.

A parameterisation for this cone $\tilde{S}$ is obtained by linear interpolation of a surface between the vertex, $\tilde{g}(0, \phi)$, and Γ, $\tilde{g}(1, \phi)$,

$$\tilde{g}(r, \phi) = (1 - r)(-R, 0, 0) + r\gamma(\phi), \quad r \in (0, 1), \ \phi \in (0, 4\pi),$$

with the corresponding velocities for the coordinate lines,

$$\mathbf{X_r}(r,\phi) = \gamma(\phi) - (R,0,0), \qquad \mathbf{X}_\phi(r,\phi) = r\gamma'(\phi),$$

allows calculation of the flux of $\mathbf{v}$ across the cone,

$$\begin{aligned}\Phi_{\mathbf{v},\tilde{S}} &= \int_0^1 \mathrm{d}r \int_0^{4\pi} \mathrm{d}\phi\, \langle \mathbf{v}, \mathbf{X_r} \times \mathbf{X}_\phi \rangle \\ &= \int_0^1 \mathrm{d}r \int_0^{4\pi} \mathrm{d}\phi \left(2R^2 \cos^2\frac{\phi}{2} + Ra\left(1 + 3\cos^2\frac{\phi}{2}\right)\sin\frac{\phi}{2} + a^2\sin^2\frac{\phi}{2}\right) r \\ &= 2\pi R^2 + \pi a^2,\end{aligned}$$

and obtain the same results as with the circulation (6), according to Stokes' theorem, since the cone is orientable.

Calculations provide the same result for any other choice of the vertex P of the cone.

6. Conclusions

In this paper we have provided a simple answer to the calculation of the flux of a vector field across a one-sided surface, where Stokes' theorem is not applicable.

We have shown that, though the question is ill posed, there is a way of restating the problem in order to provide a right answer, that is related to experiments we may perform in a laboratory.

It has been pointed out that the physically meaningful quantity is not the flux across the one-sided surface, but the circulation along the boundary of the surface. This quantity is not also meaningful, but can be measured, for instance, as the electromotive force along a circuit induced by a varying magnetic field.

In fact, once we focus in computing the circulation along the boundary, we notice that the one-sided surface is auxiliary and may be replaced by *any* other surface with the same boundary. If the chosen surface is orientable, this allows us to calculate the flux and the circulation an obtain the same result, according to Stokes' theorem. In fact, cones are always available for designing orientable surfaces with a given closed curve as boundary.

Summarising, the circulation of a vector field along the boundary of a Möbius strip, or any other one-sided surface, can be calculated using Stokes' theorem, though not using the Möbius strip, but any other surface with the same boundary.

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