

Moment density of Zipoy's dipole solution

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Abstract. Zipoy used spheroidal coordinates to construct a family of static axisymmetric gravitational fields included in Weyl's class. We calculate the mass moment source for the dipole solution using techniques previously developed for magnetic and angular momentum densities and compare it with the Newtonian analogue studied by Bonnor and Sackfield.

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1. Introduction

In [1] Bonnor and Sackfield applied potential theory to analyse some metrics of the family obtained by Zipoy [2]. They calculated the mass density for the monopole solution from the discontinuities of the derivatives of the Weyl potential [3], taken as a Newtonian potential, and also they calculated the moment density for the dipole solution from the discontinuity of the potential itself, considering these jumps as surface sources for the gravitational fields. While their result for the monopole solution was also grounded on the thin surface layer formalism presented by Israel in [4], and therefore it had an interpretation within the framework of general relativity, their analysis of the dipole solution was carried out using only a Newtonian flat spacetime analogue, concluding that a sheet of dipoles was the only source present. In this paper a justification for that result will be given using a modification suited for mass dipoles of the formalism derived previously in [5] and [6] for rotation and electromagnetic moments in general relativity.

Section 2 is devoted to the Weyl static axisymmetric family of vacuum solutions of the Einstein equations, using in its derivation the differential form approach applied in [7] and [8] to vacuum fields. Zipoy's spheroidal metrics are dealt with in section 3. In section 4 use is made of the formalism introduced previously to provide a way of calculating dipole densities out of the discontinuities of the metric functions. This method is applied to the Zipoy dipole solution in section 5. A brief discussion of the results is provided in section 6.

2. The Weyl static family

The Weyl static axisymmetric vacuum solutions were discovered soon after the publication of the general theory of relativity [3]. It is the general solution for metrics with this symmetry and allows the construction of metrics from solutions of a flat spacetime Laplace equation.

We shall introduce the relevant equations for the problem in the form of a exterior system. Our starting point is an orthonormal vierbein $\{\theta^0, \theta^1, \theta^2, \theta^3\}$ such that the metric takes this form:

$${}^4g = -\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3. \tag{1}$$

The forms θ^0 and θ^1 lie on the space spanned by the orbits of the two Killing fields, ∂_t and ∂_ϕ , while the other two forms, θ^2 and θ^3 , lie on the orthogonal space.

The equations stating that the frame is torsion-free (2)–(5), together with its integrability conditions (6)–(7) and Einstein’s equations (8)–(9) can be shown to be [7, 8]:

$$d\theta^0 = a \wedge \theta^0 \tag{2}$$

$$d\theta^1 = (b - a) \wedge \theta^1 \tag{3}$$

$$d\theta^2 = -\nu \wedge \theta^3 \tag{4}$$

$$d\theta^3 = \nu \wedge \theta^2 \tag{5}$$

$$db = 0 \tag{6}$$

$$da = 0 \tag{7}$$

$$d * b + b \wedge * b = 0 \tag{8}$$

$$d * a + b \wedge * a = 0 \tag{9}$$

The symbol $*$ stands for the Hodge duality operator in the θ^2 – θ^3 space, that is $*\theta^2 = \theta^3$ and $*\theta^3 = -\theta^2$. The equations governing the connection ν in the θ^2 – θ^3 space have not been written, since they can be integrated by quadratures after the previous exterior system has been solved, and are not of interest for our purposes here.

The equations (2)–(8) can be formally integrated to yield the metric in canonical coordinates $\{t, \phi, z, \rho\}$, and therefore the meaning of the 1-form b as the differential of the logarithm of the volume of the θ^0 – θ^1 space is clear:

$$a = dU \quad b = d \ln \rho \quad * b = -\rho^{-1} dz \tag{10}$$

$${}^4g = -e^{2U} dt^2 + e^{-2U} \{ \rho^2 d\phi^2 + e^{2k} (d\rho^2 + dz^2) \}. \tag{11}$$

The remaining equation, (9), is analogous to the Raychaudhuri equation for the acceleration of a fluid. Therefore a can be viewed as an acceleration 1-form. After substitution of (10) it takes the form of a reduced three-dimensional Laplace equation:

$$d * dU + \rho^{-1} d\rho \wedge * dU = 0. \tag{12}$$

Or in explicit Weyl canonical coordinates:

$$U_{\rho\rho} + U_{zz} + \frac{1}{\rho} U_\rho = 0. \tag{13}$$

As we had already stated, Newtonian potential functions, U , yield after substitution in the metric (11) exact solutions of Einstein’s equations. But the analogy finishes here and no further interpretation can be derived from this fact. As an example, the classical potential for a finite rod with constant density produces the Schwarzschild metric, the relativistic monopole solution.

If we want the metric to be asymptotically flat, then U should be at most monopolar at infinity, that is:

$$U = -\frac{m}{r} + O\left(\frac{1}{r^2}\right). \tag{14}$$

We shall restrict ourselves to these metrics from now on.

3. Zipoy's metric

Zipoy's metrics belong to Weyl class [2] and are constructed from solutions of the Laplace equation in oblate spheroidal coordinates $\{r, \theta\}$ instead of using pseudo-cylindrical coordinates $\{\rho, z\}$:

$$z = r \cos \theta \quad \rho = \sqrt{r^2 + a^2} \sin \theta \tag{15}$$

where a is a constant and the new coordinates range as usual:

$$0 \leq r < \infty \quad 0 \leq \theta \leq \pi. \tag{16}$$

Note that events with coordinates $(t, \phi, r = 0, \theta)$ and $(t, \phi, r = 0, \pi - \theta)$ have the same canonical coordinates. Therefore we shall identify them instead of attempting more complicated interpretations.

In Bonnor and Sackfield's paper, θ is a latitude angle instead of a co-latitude, as it is taken to be in this paper, and we also make use of the radius r instead of the coordinate u related to the previous ones by $r = a \sinh u$. We have done this since their interpretation as asymptotic spherical coordinates is simpler.

The reduced Laplace equation in these coordinates takes the following form:

$$(r^2 + a^2)U_{rr} + 2rU_r + U_{\theta\theta} + \cot \theta U_\theta = 0. \tag{17}$$

We can construct solutions for this equation of the form $U_n = R_n(r)P_n(\cos \theta)$ where P_n stands for the Legendre polynomials. The index n runs from zero to infinity allowing us to classify the Newtonian potentials U and the metrics derived from them using a multipole expansion. We shall be concerned with the solution for $n = 1$, Zipoy's dipole solution.

4. Dipole sources

Since the aim of this paper is just the interpretation of Zipoy's dipole solution, we shall restrict ourselves to asymptotically dipolar static axisymmetric vacuum solutions of the Einstein equations. For this we mean that the metric function U must be:

$$U = \frac{M \cos \theta}{r^2} + O\left(\frac{1}{r^3}\right). \tag{18}$$

where M is the total mass dipole.

Now, we shall proceed as in [5] and [6] and consider the two formal expressions for the acceleration form arising from the integration of equations (7) and (9):

$$a = dU = -\rho^{-1} * d\Lambda. \tag{19}$$

The asymptotic behaviour of U imposes a condition at infinity for the newly introduced function Λ :

$$\Lambda = -\frac{M \sin^2 \theta}{r} + O\left(\frac{1}{r^2}\right). \tag{20}$$

In order to have a dipole source for the gravitational field, it is assumed that the function U is discontinuous across a closed surface S . This condition on S means no restriction, since the difference between the values of U on either side of the surface can be null on some parts of it.

Taking into account both expressions for the acceleration, we integrate the scalar product of a and the differential of the Weyl coordinate z with a factor e^U . The domain of integration will be the space V_3 orthogonal to θ^0 , whose metric is $g = {}^4g + \theta^0 \otimes \theta^0$. The square root of the determinant of this metric is $\sqrt{g} = \rho e^{2k-3U}$.

$$\begin{aligned}
 0 &= \int_{V_3} \sqrt{g} e^U \langle [a + *(a)], dz \rangle dx^1 dx^2 dx^3 \\
 &= \int_{V_3} \sqrt{g} e^U \langle [dU + \rho^{-1} * d\Lambda], dz \rangle dx^1 dx^2 dx^3.
 \end{aligned}
 \tag{21}$$

The Weyl coordinate z satisfies the following differential equation arising from (8), since $\sqrt{{}^4g} = e^U \sqrt{g}$:

$$\partial_\mu \{ \sqrt{g} e^U g^{\mu\nu} \partial_\nu z \} = 0.
 \tag{22}$$

Therefore the integrand in (21) can be shown to be a total derivative and the expression can be reduced to a surface integral.

$$0 = \int_{V_3} \partial_\mu \{ \sqrt{g} U e^U g^{\mu\nu} \partial_\nu z + [\mu\nu] \Lambda \partial_\nu z \} dx^1 dx^2 dx^3.
 \tag{23}$$

Actually, since U is discontinuous, the integral has to be split into two pieces, V_3^+ and V_3^- , respectively the outer and inner sections of V_3 referred to the surface S . The boundary of V_3^- is formed by the surface S and the boundary of V_3^+ consists of the same surface and the 2-sphere at infinity $S^2(\infty)$. The integral over $S^2(\infty)$ can be straightforwardly carried out, taking into account the asymptotic conditions (18) and (20) and that the metric is assumed to be asymptotically flat:

Hence the integral defining the total mass dipole is just:

$$4\pi M = \int_{S^+} dS^+ \{ U e^U g^{\mu\nu} n_\mu \partial_\nu z \}^+ - \int_{S^-} dS^- \{ U e^U g^{\mu\nu} n_\mu \partial_\nu z \}^-.
 \tag{24}$$

Because the metric is discontinuous—the potential U is also part of the metric and of the unitary normal, n , to the surface S —it seems, in principle, not possible to reduce the previous equation to one integral of the jump of U . But it can be shown, for instance, that, if the discontinuity takes places on a disk on the plane $z = 0$, metric factors like e^U cancel each other, and therefore the differential element of mass dipole is perfectly defined.

5. The dipole solution's source

The aim of this paper has been the interpretation of Zipoy's dipole metric with the formalism developed in the previous section. As has already been said, this metric arises from the

substitution of the dipolar solution of the Laplace equation in oblate spheroidal coordinates in the Weyl metric. This solution can be shown to be:

$$U = \gamma \left\{ 1 - \frac{r}{a} \arctan \left(\frac{a}{r} \right) \right\} \cos \theta \tag{25}$$

where γ is a constant.

It is easily seen that this function behaves for large values of r like a dipolar potential whose moment is $M = \frac{1}{3}\gamma a^2$.

$$U = \frac{1}{3}\gamma a^2 \frac{\cos \theta}{r^2} + O\left(\frac{1}{r^4}\right). \tag{26}$$

The remaining metric function, although it will not be needed, is:

$$e^{2k} = \left(\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \right)^{-\gamma^2} \exp \left\{ -U^2 \tan^2 \theta - \gamma^2 \sin^2 \theta \left[\arctan \left(\frac{a}{r} \right) \right]^2 \right\}. \tag{27}$$

As we have identified on the surface of null r coordinate points whose co-latitude is θ with those that have $\pi - \theta$, it happens that, when we reach that surface from the subspace $z \geq 0$, the potential takes the value $U = \gamma \cos \theta$, but if we arrive with $z \leq 0$, then it is equal to $U = -\gamma \cos \theta$, θ ranging from 0 to $\pi/2$ after the identification of events [1].

Therefore the potential meets a jump at the disk $z = 0$, $\rho \leq a$ ($r = 0$, $\theta \leq \pi/2$):

$$[U] = 2\gamma \cos \theta = 2\gamma \sqrt{1 - \frac{\rho^2}{a^2}}. \tag{28}$$

The disk $z = 0$ has the following surface element and normal:

$$dS = \rho e^{k-2U} d\rho d\phi \quad n = e^{U-k} \partial_z. \tag{29}$$

Therefore the product $e^U n^\mu \partial_\mu z dS$ does not depend on U and takes the same value on both sides of the disk S , as we anticipated in the previous section.

Hence the integral defining the total mass dipole can be written as:

$$M = \frac{1}{4\pi} \int_S dS [U] e^U n^\mu \partial_\mu z = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^a d\rho \rho 2\gamma \sqrt{1 - \frac{\rho^2}{a^2}} = \frac{1}{3}\gamma a^2. \tag{30}$$

The mass dipole takes the value expected from the asymptotic form of the metric and the differential element of the mass moment is then:

$$dM = \frac{1}{4\pi} \rho 2\gamma \sqrt{1 - \frac{\rho^2}{a^2}} d\phi d\rho \tag{31}$$

in full accordance with Bonnor and Sackfield's Newtonian result.

6. Discussion

In the previous section the differential surface element for the mass moment source of Zipoy's dipole metric is constructed from the discontinuity of the metric function U . This result coincides with the one obtained by Bonnor and Sackfield [1] using classical potential theory, since the curved spacetime elements in the integral (30) arising from the metric cancel each other. Therefore the full relativistic theory and its classical Newtonian approximation yield the same output and Bonnor and Sackfield's result is confirmed within the framework of general relativity.

Something similar happens with the angular momentum density for Kerr's metric [5], where oblate spheroidal coordinates are also employed. As stated in [1], the potential U is due only to the sheet of dipoles described in (31), that is, there is no contribution of higher order multipole sheets to the gravitational field, and therefore the source is fully described, at least in the Newtonian limit. Again, this is the case for Kerr's metric.

With this construction it has been shown that discontinuities of metrics under the conditions stated in section 4 can be related to the dipole moment calculated from the expansion of the metric at infinity. This sort of result cannot be obtained from the junction conditions [4], since the Einstein equations involve products of delta distributions. A different attempt at interpreting these discontinuities could be to consider them as the limit of coalescence of two sheets of monopoles of opposite sign; regularizing procedures such as the splitting point method could be employed then. However, it has to be taken into account that mass dipoles may have two different causes: one of them would be a choice of an origin of coordinates which does not coincide with the centre of mass, and therefore these dipole moments would not be rejectable from the physical point of view. The other possible origin involves negative masses, and would not be quite physical. In the latter case the discontinuities would be a good test of the physical content of the vacuum spacetime. Even if the discontinuities could be hidden by matching the solution to a three-dimensional material source, the conservation law (24) would still reveal the dipole structure.

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