AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 7742–7758.
DOI:10.3934/math.2022435
Received: 13 December 2021
Revised: 07 February 2022
Accepted: 10 February 2022
Published: 17 February 2022

Survey

On the reach and the smoothness class of pipes and offsets: a survey

Javier Sánchez-Reyes[1,*] **and Leonardo Fernández-Jambrina**[2]

[1] IMACI, ETS Ingeniería Industrial, Campus Universitario, Edificio Politécnico, Universidad de Castilla-La Mancha, 13071-Ciudad Real, Spain

[2] Dep. de Matemática e Informática Aplicadas a las Ingenierías Civil y Naval, Universidad Politécnica de Madrid, Av. de la Memoria 4, 28040-Madrid, Spain

* **Correspondence:** Email: Javier.SanchezReyes@uclm.es; Tel: +34-926-295463.

Abstract: Pipes and offsets are the sets obtained by displacing the points of their progenitor S (i.e., spine curve or base surface, respectively) a constant distance d along normal lines. We review existing results and elucidate the relationship between the smoothness of pipes/offsets and the reach R of the progenitor, a fundamental concept in Federer's celebrated paper where he introduced the family of sets with positive reach. Most CAD literature on pipes/offsets overlooks this concept despite its relevance, so we remedy this deficiency with this survey. The reach admits a geometric interpretation, as the minimal distance between S and its cut locus. For a closed S, the condition $d < R$ means a singularity-free pipe/offset, coinciding with the level set at a distance d from the progenitor. This condition also implies that pipes/offsets inherit the smoothness class C^k, $k \geq 1$, of a closed progenitor. These results hold in spaces of arbitrary dimension, for pipe hypersurfaces from spines or offsets to base hypersurfaces.

Keywords: cut locus; level set; offset; pipe; progenitor; reach; smoothness class; spine
Mathematics Subject Classification: 53-A04, 53-A05, 65-D17

1. Introduction: pipe and offset surfaces

1.1. Pipe and offset surfaces as locus of points at constant distance along normal lines

Computing the surface at a constant distance from a curve or another surface is a fundamental operation in Geometry Processing [1], which plays a key role in engineering and manufacturing [2, 3], or architectural design [4, 5]. These surfaces also appear in the description of wavefronts propagating in a homogeneous medium.

Pipe and offset surfaces can be regarded as the 3D generalization of planar offset curves (Figure 1). Given a distance d and a *central curve* S (also called *spine* [6]), a *pipe* or *tube* S_d is defined as

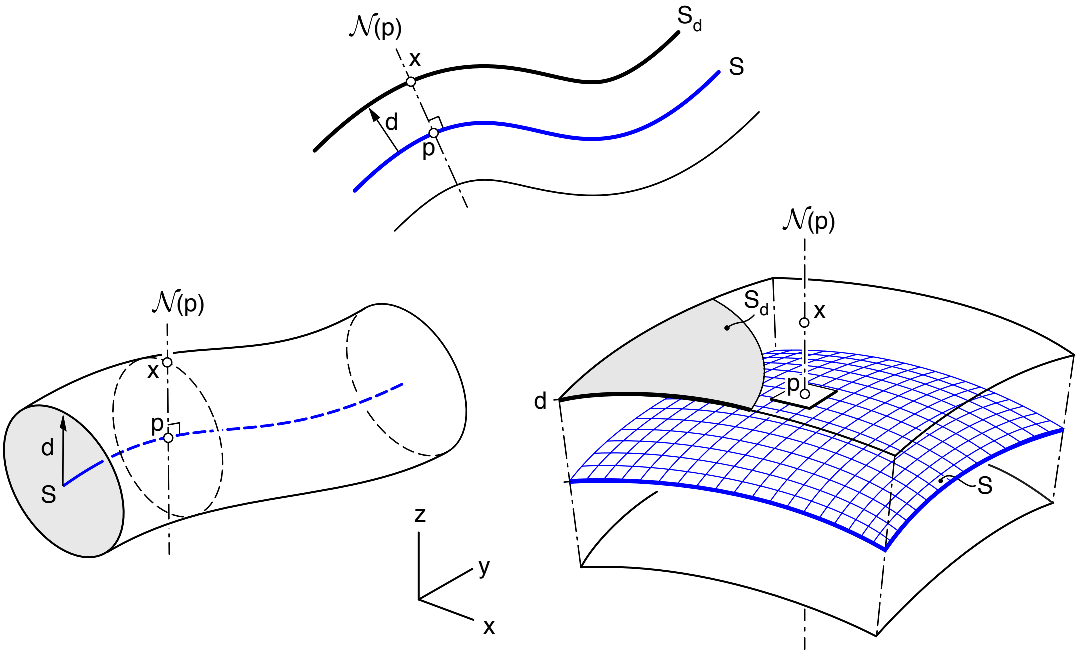

Figure 1. Planar offset curve and its 3D generalizations (pipe and offset surfaces).

the locus of points x at a distance d from points $p \in S$ along normal lines $\mathcal{N}(p)$ to S. By replacing the spine with a *base surface* S (also known as *progenitor*), its *offset* S_d (dubbed *parallel surface* in classical differential geometry [7, 8]) is defined in the same manner. These two families of surfaces are intimately related, as offsetting a pipe yields another pipe with the same spine. Their extension to a general n-dimensional space are both called *offset*. Also, throughout this paper, the unifying term *progenitor* will be employed to denote the spine, base surface, or the base curve when dealing with offsets to planar curves.

This explicit definition of pipes/offsets allows their construction and visualization. However, it requires a *regular* progenitor S, in the sense that S does not display singularities. Indeed, for pipes, we assume a spine S without cusps or self-intersections (Figure 2), so that we can define a tangent vector at each point of S and its the complementary normal plane. Similarly, the above definition of offset surfaces assumes a progenitor S with no singularities, such as cuspidal edges, cusps, or self-intersections, so that at each point of S a tangent plane is well-defined, and hence it makes sense to speak of normal directions. Observe that, if a progenitor surface S is connected and orientable, its offset consists of two components [8].

As an alternative to this explicit definition, we could characterize an offset implicitly for a general set S of points, not necessarily a regular curve or surface, as the level set at constant distance d from S. This survey employs this implicit form conceptually to determine the offset smoothness and will reveal the precise connection between the two offset versions (implicit and explicit), issues not clearly addressed in the existing literature.

1.2. Singularities

In general, a pipe or offset S_d is topologically more complex than its progenitor S since singularities may appear as d grows. The computation and detection of such singularities is a well-studied topic

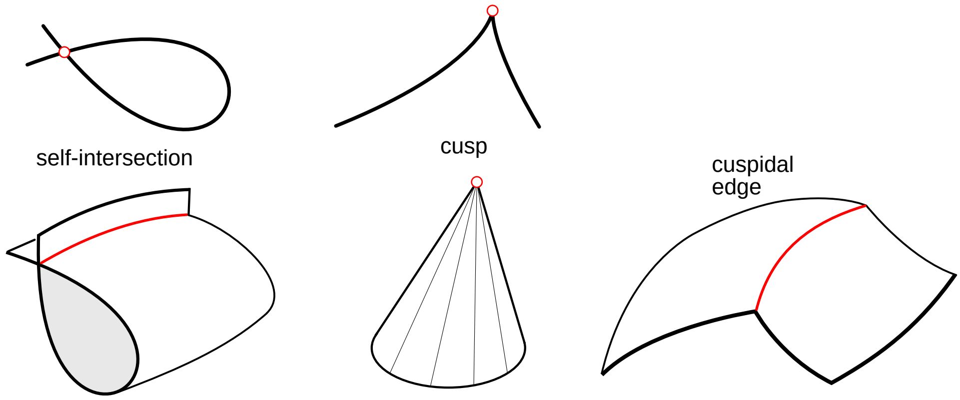

Figure 2. Singularities precluding a curve or surface from being regular.

[2, 9]. These singularities alter the original shape of S, so Alcázar and Sendra [10], and Alcázar [11], for plane algebraic curves and rational algebraic surfaces, respectively, put forward the topological concept of *good local behavior*, meaning that offsetting preserves the local shape. In the case of offsets to plane algebraic curves, this concept extends to *good global behavior* [12].

In a real design scenario, we need the upper bound of d that guarantees the regularity of S_d. Before the advent of CAGD (Computer Aided Geometric Design), Federer [13] published a seminal work on sets with positive reach that deserves renewed attention. He extended the concept of normal vectors to deal with convex sharp edges and vertices, which enriches the admissible geometries for explicit offset construction. He showed that offsetting *any* closed set S yields a C^1-smooth surface, for distances $d < R$, where R denotes the so-called *reach* of the set S. In a CAGD setting, Maekawa et al. [6], and Wallner et al. [14] compute this bound for C^2 pipes and offsets, respectively, by carefully analyzing all possible global and self-intersections. This bound plays a fundamental role in the construction of reliable (topologically equivalent) approximations to S_d [6, 15] and, in the case of the offset to compact surfaces, it coincides with minimal distance to the cut locus of the surface, i.e., the set of points equally distant from at least two points on the surface. Thus, these works show how to compute the reach (in the closed case), albeit without employing this terminology.

1.3. Higher-order smoothness

Although a rich literature exists on detecting singularities of pipes and offsets, surprisingly, their general smoothness has received scant attention. Our contribution focuses on clarifying how it depends on that of the progenitor by bringing together existing yet dispersed concepts, making them more accessible, and deriving the relevant results the applied mathematics community needs.

The most remarkable result regarding the smoothness of offset surfaces is due to Hermann [16]. In CAGD, surfaces are usually defined parametrically in a piecewise manner, and their smoothness is characterized by the so-called *geometric continuity* G^k [17], which means the agreement of i-th derivatives ($i = 0, \ldots, k$) at the joints of adjacent C^k patches, after suitable reparameterization. For a piecewise base surface composed of C^∞ patches, joined with geometric continuity G^k along a common boundary, such as non-degenerate NURBS [18, 19], Hermann proves that the offset inherits the G^k continuity. His approach relies on checking the existence of the above-mentioned reparameterization, yet without explicitly constructing it.

To derive a more general result, we assume no particular representation (implicit or parametric) of the progenitor, vindicate Federer's results, and extend them to C^k progenitors, so the paper is arranged as follows. In Section 2, we introduce the different geometries (sets of points) required for a unified handling of progenitors and resulting pipes and offsets. We begin with the classical concept of smooth submanifolds and extend them to sets of positive reach in Federer's framework involving the cone of normal vectors. Next, in Section 3, we define the untrimmed offset as the set obtained by displacing points of S a constant distance d along normal lines, and the trimmed offsets as the level set at distance d. In Section 4, we review the concept of reach and associated results noting that, if $d < R$, trimmed and untrimmed offset coincides. In Section 5, we employ existing results on the smoothness of the distance function to a closed C^k submanifold to show that, if $d < R$, then S_d inherits the class C^k of S. Finally, conclusions are drawn in Section 6.

2. Submanifolds and their generalization

2.1. Submanifolds of class C^k

Rather than in the constructional way of geometric continuity, we will characterize the smoothness of pipe/offset surfaces by defining them as a subset of points fulfilling certain conditions. For this purpose, we could employ the classical concept [20] of *surface of class* C^k in $\mathbb{R}^3$, or, for a more comprehensive scope, that of *submanifold of class* C^k [21], in Euclidean space $\mathbb{R}^n$:

Definition 1. Let m, n be integers such that $1 \leq m < n$. A subset $S \subset \mathbb{R}^n$ is a *m-dimensional submanifold of class* C^k, $k \geq 1$, if either of the following equivalent conditions [21] is fulfilled:

(i) For each point $x \in S$ there is an open neighborhood $U \subset \mathbb{R}^n$ and a homeomorphism $X : V \subset \mathbb{R}^m \to S \cap U$ of class C^k such that the differential dX has rank $= m$ for all points in U.

(ii) For each point $x \in S$ there is an open neighborhood U and a local function $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-m}$ of class C^k such that $F^{-1}(0) = S \cap U$ and the differential dF has rank $= n - m$ for all points over $S \cap U$.

Characterization (i) requires the existence of a *local parametrization* X of class C^k around every point $x \in S$, whereas (ii) requires that S be defined locally by an *implicit equation* $F(x) = 0$, with F of class C^k.

Remark 1. For $k = 1$, the first characterization reduces to that of a regular surface. Therefore, this concept of class extends the notion of regularity.

Remark 2. The concept of C^k class, stronger than that of C^k curves or surfaces employed in CAGD, encompasses the case of regular G^k piecewise surfaces: according to characterization (i), they are of class C^k.

Remark 3. For hypersurfaces ($m = n - 1$), in characterization (ii) dF simplifies to $dF = \nabla F$, with $\nabla F \neq 0$. Later, we will come across the so called hypersurfaces of *class* $C^{1,1}$ [23], weaker than C^2, but stronger than C^1:

Definition 2. A $C^{1,1}$ *hypersurface* is a $(n-1)$-dimensional submanifold characterized by a $C^{1,1}$ implicit function F, i.e, such that ∇F is 1-Lipschitz.

CAGD applications such as solid modeling [22] deal with a region of $\mathbb{R}^n$, rather than a hypersurface. The closed set bounded by a hypersurface (without boundary) corresponds to the concept of *n-dimensional manifold with boundary*. A general and formal definition for submanifolds of arbitrary

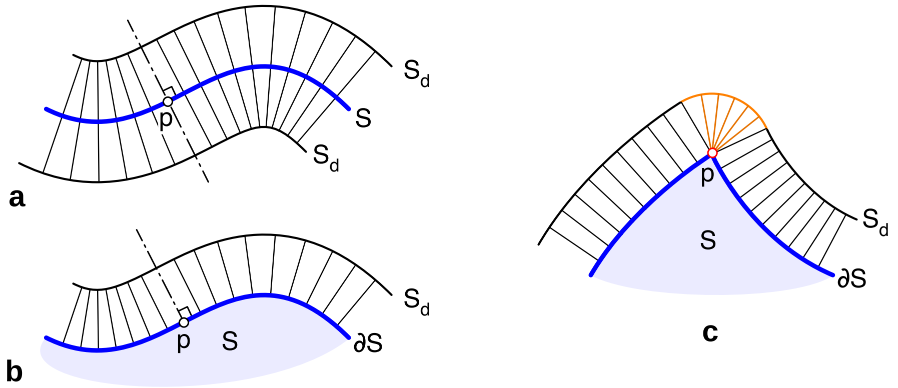

Figure 3. Example of sets S with positive reach: a) Submanifold. b) n-dimensional submanifold S with boundary ∂S. c) Set with convex sharp edges or vertices.

dimension with boundary is found in the classical book [21]. The particular case of a n-dimensional manifold with boundary is, in essence, a set locally diffeomorphic to the half-space $\mathbb{H}^n = [0, \infty) \times \mathbb{R}^{n-1}$.

2.2. A more general framework: cone of normal vectors and sets with positive reach

The concept of submanifold allows a unified handling of smooth progenitor curves and surfaces, and hence of the resulting pipes and offsets. However, it excludes progenitors with singularities, a severe restriction in solid modeling. For instance, a simple polyhedral solid does not qualify as a submanifold. Federer's work provides a more general setting by extending the concept of tangent and normal lines [13, 24]:

Definition 3. The *tangent cone of vectors* $\mathrm{Tan}(p, S)$ at each point $p \in S$ of a given set $S \subset \mathbb{R}^n$ consists of the limits of all (secant) vectors originating from p and passing through a sequence of points $p_i \in S \setminus p$ that converges to p. The corresponding dual *cone of normal vectors* $\mathrm{Nor}(p, S)$ consists of all vectors v such that $\langle u, v \rangle \leq 0$ whenever $u \in \mathrm{Tan}(p, S)$.

This elegant framework extends beyond the submanifold realm to the so-called sets with positive reach. Although the concept of reach will be formally in Section 4, we advance that these sets are characterized [25, 26] by admitting a value $R > 0$ such that we can roll up a ball of a radius $d < R$ all over the boundary ∂S. Thus, they encompass the following closed sets $S \in \mathbb{R}^n$ (Figure 3):

a) Traditional submanifolds S (without boundary): The cone $\mathrm{Nor}(p, S)$ coincides with the customary normal direction that, in Definition 1, the conditions on the rank guarantee.
b) n-dimensional submanifolds S with boundary ∂S: The cone consists of the outward normal vectors to ∂S.
c) Closed sets whose boundary ∂S may contain convex sharp edges and vertices: At these singularities, the cone (highlighted in orange in Figure 3c) fills the gaps between the normal directions of adjacent faces. In [27], the term *normal pyramid* is employed instead to denote this cone.

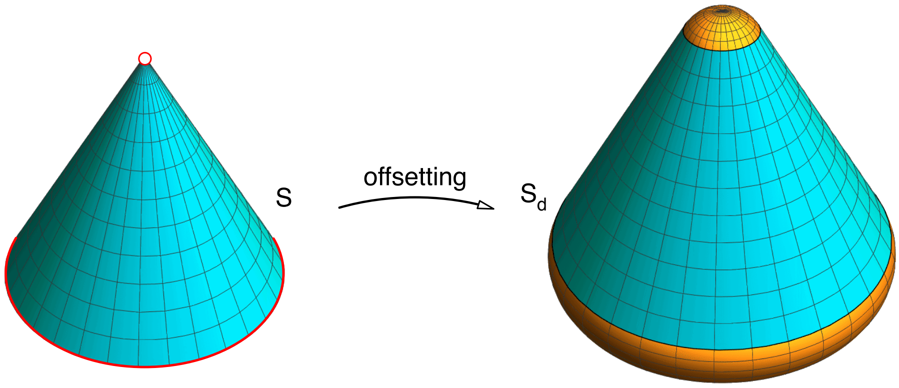

Figure 4. Offsetting a solid cone S with a sharp edge and vertex.

3. Untrimmed offsets vs. trimmed offset (level set of the distance function)

In the general framework of Section 2.2, from Definition 3 we can construct the *normal bundle* of a given set S:

$$\nu(S) := \{(p, v) \in \mathbb{R}^n \times \mathbb{R}^n : \ p \in S \text{ and } v \in \mathrm{Nor}(p, S)\}. \tag{3.1}$$

and the map σ:

$$\sigma : \nu(S) \to \mathbb{R}^n, \quad \sigma(p, v) = p + v. \tag{3.2}$$

We assume that $\forall p \in S$ the cone $\mathrm{Nor}(p, S)$ does not degenerate to a null vector $v = 0$, so that the above concepts make sense. Consequently, we exclude sets whose boundary contains sharp concave edges or vertices.

Now, we are ready to give a formal definition of the explicit offset, dubbed *untrimmed* in contrast to its *trimmed* (level set) sibling:

Definition 4. The *untrimmed offset* S_d at distance d to S is the set constructed by displacing all points $p \in S$ a constant distance d along normal lines $\mathcal{N}(p)$:

$$S_d := \{\sigma(p, v) \in \mathbb{R}^n \ : \ (p, v) \in \nu(S) \text{ and } \|v\| = d\}. \tag{3.3}$$

Remark 4. This definition encompasses pipes S_d, generated from a submanifold S of dimension $m = 1$ (a curve).

Remark 5. When S is a solid with convex sharp edges and vertices, such as the cone of Figure 4, S_d (3.3) includes the resulting vertex and edge offsets. These components (highlighted in orange) fill the gaps between face offsets (blue), making up the topology Farouki [28] identified.

Since the map σ (3.2) is C^{k-1} [29], we could jump to the conclusion that a C^k submanifold S generates another C^{k-1} submanifold S_d (3.3). However, on the one hand we must constrain values d to ensure that S_d actually meets the definition of submanifold, because S_d may have singularities, such as the self-intersections of the inner component of the planar offset in Figure 5a. These singularities substantially alter the shape of the original progenitor S. On the other hand, this class can be improved to C^k.

To characterize the smoothness of S_d (3.3), the key idea is using an alternative offset construction, what Farouki [30, 31] calls *trimmed* or *true* offset:

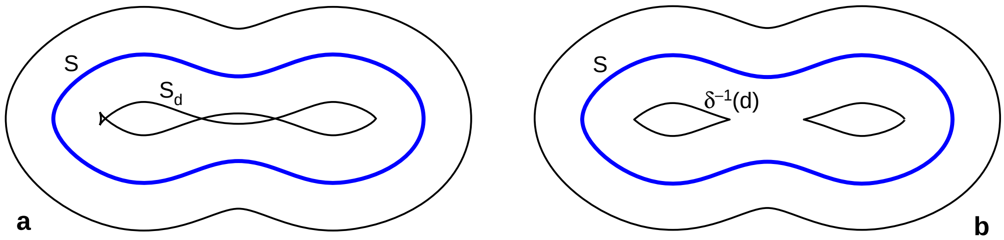

Figure 5. a) Untrimmed offset S_d vs. b) Trimmed offset (level set).

Definition 5. Given a non-empty set S, its *trimmed offset* at distance d is the level set $\delta^{-1}(d, S)$ of the *distance function* δ to S:

$$\delta : \mathbb{R}^n \to \mathbb{R}, \quad \delta(x, S) = \inf_{p \in S} \|x - p\|, \tag{3.4}$$

When no ambiguity is possible, we drop the symbol S in (3.4) and upcoming definitions. The catch is that, once again, $\delta^{-1}(d)$ may display singularities (Figure 5b). Furthermore, $\delta^{-1}(d)$ does not always coincide with its untrimmed counterpart S_d (3.3), since S_d may contain points x at distance $\delta(x, S) < d$, points trimmed away [12,32–35] in $\delta^{-1}(d)$. In the example of Figure 5b, the inner component of $\delta^{-1}(d)$ would even vanish for a large d. The concept of *reach* of a set, reviewed in the upcoming Section, will elucidate the connection between the offsets S_d and $\delta^{-1}(d)$, i.e., Definitions 4 and 5.

4. Reach of a set

4.1. Definitions. Reach as shortest distance to the cut locus

Definition 6. Given a set S, we denote $\mathcal{U}(S)$ the set of all $x \in \mathbb{R}^n$ with the *unique nearest point property*, i.e., $\forall x \in \mathcal{U}(S)$, there is a unique point $\pi(x, S) \in S$ such that $\delta(x) = \|x - \pi(x, S)\|$. The map π is called the *nearest point function*, or *projection* onto S:

$$\pi : \mathcal{U}(S) \to S. \tag{4.1}$$

Definition 7. For points $p \in S$, if $B_r(p)$ denotes the open $n-$ball of radius r centered at p, then the *local* reach(p, S) around p and *global* reach(S) are defined as:

$$\text{reach}(p, S) := \sup\{r : B_r(p) \subset \mathcal{U}(S)\}, \qquad R = \text{reach}(S) := \inf_{p \in S} \text{reach}(p, S). \tag{4.2}$$

We say that a set S has positive reach if $R > 0$. Geometrically [25, 26], as already noted, this property means that we can roll up a ball of radius at most R all over the boundary ∂S.

All sets with positive reach are necessarily closed [13], so, from now on, we restrict ourselves to closed sets S. Thus, R admits a more familiar interpretation [41,42], using the complement $\mathbb{R}^n \setminus \mathcal{U}(S)$, i.e., the set of points having more than one nearest point. This complement set corresponds to the so-called *cut locus* $\mathcal{L} = \mathcal{L}(S)$ (without its limits points) [14, 25, 36, 37]. In terms of $\mathcal{L}$, the local and global reach (4.2) are given by (Figure 6a):

$$\text{reach}(p, S) = \delta(p, \mathcal{L}), \qquad R = \inf_{p \in S, q \in \mathcal{L}} \|p - q\|. \tag{4.3}$$

Consequently, positive reach is tantamount to saying that $\mathcal{L}(S)$ does not touch S [25].

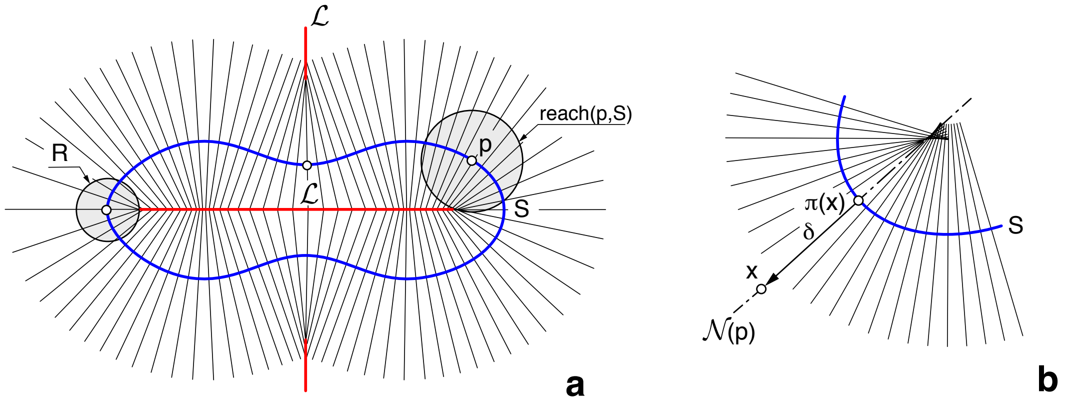

Figure 6. a) Local reach(p, S) and global reach R. b) Projection $\pi(x)$.

4.2. *Condition for the coincidence of the untrimmed offset S_d with the level set*

Equipped with the above definitions, we show that the reach R determines when S_d and the level set $\delta^{-1}(d)$ coincide. For closed sets, Theorem 4.8 in [13] furnishes two fundamental results:

(i) The projection (4.1) occurs always along a normal line $\mathcal{N}(p)$ to S, as Figure 6b sketches:

$$x - \pi(x) = v \in \mathrm{Nor}(\pi(x), S), \quad \forall x \in \mathcal{U}(S). \tag{4.4}$$

(ii) Moving from p along $\mathcal{N}(p)$ up to a distance reach(p, S), all points x project onto p, and hence the distance $\delta(x, S)$ is precisely along $\mathcal{N}(p)$. In terms of the map σ (3.2):

$$\begin{matrix} \pi(\sigma(p, v)) = p \\ \delta(\sigma(p, v)) = \|v\| \end{matrix}, \quad \text{for } \|v\| < \mathrm{reach}(p, S). \tag{4.5}$$

Result (i) means that $\delta^{-1}(d) \subset S_d$, whereas (ii) implies that, for $d <$ reach(S), then $S_d \subset \delta^{-1}(d)$ and $S_d \subset \mathcal{U}(S)$. Let us summarize these relationships:

Theorem 1. *Let S be a closed set with* reach$(S) = R > 0$. *Then, for $d \in (0, R)$, its untrimmed offset S_d (3.3) at distance d coincides with the level set $\delta^{-1}(d)$, and $S_d \subset \mathcal{U}(S)$.*

Remark 6. This Theorem can be regarded as the generalized version, to sets with positive reach, of Corollary 8 in [15], formulated for the offset to compact C^2 surfaces. Sakkalis et al. [15] do not employ the terminology *reach* to denote the upper bound of the size of the neighborhood enjoying the unique nearest points property. However, and more importantly, they give a practical recipe for its numerical computation.

4.3. *Reach of the set S bounded by ∂S*

In general, both the local and global reach improve when we consider a set S instead of its boundary ∂S (Figure 7a). Indeed ([13], Remark 4.2), for points $p \in \partial S$:

$$\mathrm{reach}(p, S) \geq \mathrm{reach}(p, \partial S) \;\Rightarrow\; \mathrm{reach}(S) \geq \mathrm{reach}(\partial S). \tag{4.6}$$

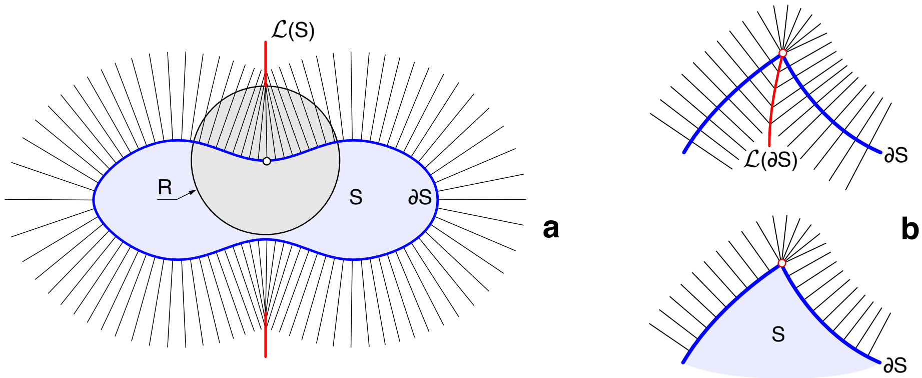

Figure 7. a) Reach R of a closed set S bounded by ∂S. b) Effect of a sharp edge or vertex.

A sharp edge or vertex in ∂S (Figure 7b) implies that $\mathcal{L}(\partial S)$ touches ∂S, so reach$(\partial S) = 0$, whereas S may still have positive reach. If S is the interior region bounded by ∂S, then, in terms of cut loci, $\mathcal{L}(S)$ contains only the exterior of $\mathcal{L}(\partial S)$, i.e., excluding its interior component, which coincides with the *medial axis* of S [25, 36, 37]. The medial axis can hence be defined as the closure of the locus of centers of maximal balls inscribed in S [2, 31, 36, 37]. The exterior component of $\mathcal{L}(\partial S)$ is involved in trimming the outer offsets to the solid S, whereas offsets inside the pocket S are trimmed against the medial axis [31, 38–40]. A warning: in [41, 42], and also in [43] in a submanifold context, the term *medial axis* is used instead to denote the cut locus, but the terminology we follow is more widespread [2, 25, 31, 36–40, 44, 45].

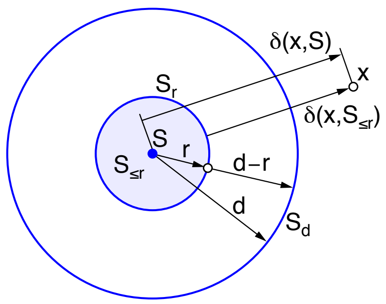

Figure 8. Set $S_{\leq r}$ bounded by a tube S_r.

A remarkable case is the generalized tubular neighborhood

$$S_{\leq r} := \{x : \delta(x, S) \leq r\}, \quad r < \text{reach}(S), \tag{4.7}$$

called *r-offset* in [41, 42]. Corollary 4.9 in [13] relates the properties of $S_{\leq r}$ and S:

$$\begin{aligned} \delta(x, S_{\leq r}) = \delta(x, S) - r \quad \text{whenever} \quad \delta(x, S_{\leq r}) \geq r, \\ \text{reach}(S_{\leq r}) \geq \text{reach}(S) - r. \end{aligned} \tag{4.8}$$

In particular, we are interested in the case where S is a spine and hence $S_{\leq r}$ is bounded by a pipe S_r, as Figure 8 illustrates at a normal section through a point $p \in S$. Combining relationships (4.8) and Theorem 1, we formalize the relationship between pipes and their offsets mentioned in the Introduction:

Lemma 1. *Let $S_{\leq r}$ be the r-offset (4.7) bounded by the pipe S_r. Then, the offset to $S_{\leq r}$ at a distance $(d - r)$, with $r < d <$ reach(S), coincides with the level set $\delta^{-1}(d - r, S_{\leq r})$ and the pipe $S_d = \delta^{-1}(d, S)$.*

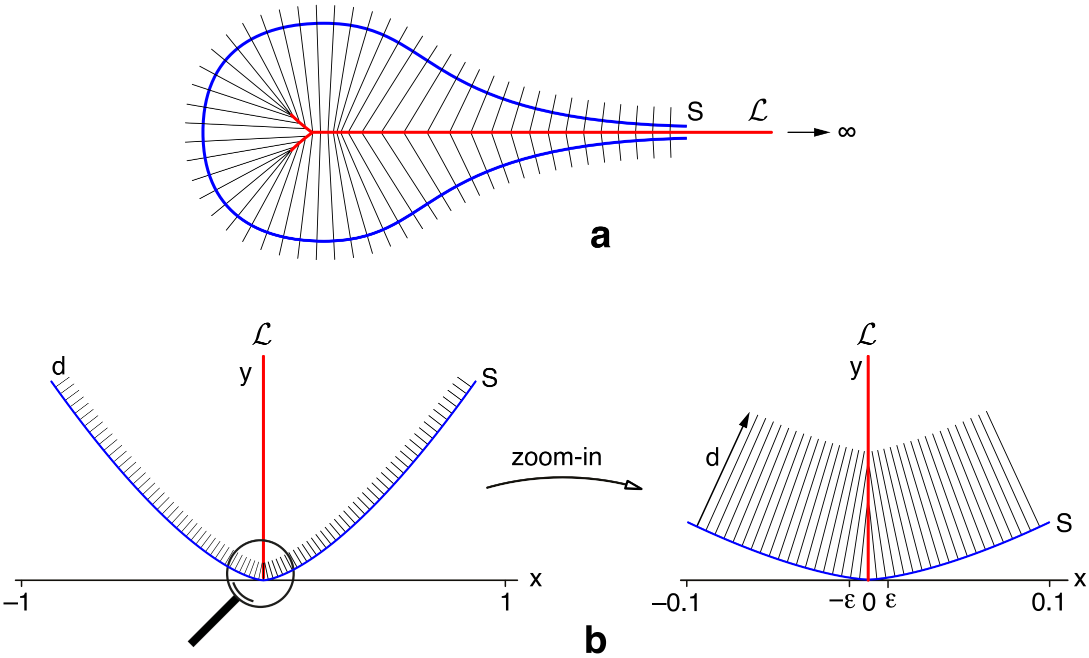

Figure 9. Examples of curves S with zero reach: a) Unbounded case with convergence at both sides to a common asymptote. b) C^1 graph of $y(x) = |x|^{3/2}$.

4.4. Reach of submanifolds

We recall the well-known result on the reach of general m-dimensional submanifolds [26, 29, 46]:

Lemma 2. *Compact C^2 submanifolds have positive reach $R > 0$.*

Violating either of the conditions required in this Lemma may lead to a zero reach:

a) The planar curve S of Figure 9a, adapted from [21], illustrates the unbounded case. As S tapers off to infinity along an asymptote contained in $\mathcal{L}$, the points $p \in S$ get closer and closer. Consequently, $\text{reach}(p, S) \to 0$ and $R = 0$.
b) Figure 9b shows that a C^1 curve may have points p with zero reach, contrary to intuition. Consider the graph of the C^1 function $y(x) = |x|^{3/2}$, an example taken from [46, 47] and clearly a C^1 curve. If we draw the normal lines to S of constant length $d > 0$ (upper side), no matter how small d is, if we zoom in near the origin O, there exists an interval $x \in [-\epsilon, \epsilon]$ such that the normals at $f(x)$ intersect. Therefore, O does not admit any neighborhood with the unique nearest point property, i.e., $\text{reach}(O, S) = 0$. Equivalently by (4.3), $\mathcal{L}$ (the non-negative vertical semi-axis) touches O.

Remark 7. By inequality (4.6), Lemma 2 carries over to n-dimensional submanifolds S with compact C^2 boundary.

Remark 8. For compact hypersurfaces S, Lucas [23] proved that the C^2 condition can be relaxed, since S is of positive reach if and only if it enjoys $C^{1,1}$ continuity.

5. Smoothness of pipes/offsets

As a consequence of Theorem 1, given a closed set S the function

$$F(x) = \delta(x) - d \tag{5.1}$$

characterizes S_d as a submanifold according to Definition 1(ii). Since $\nabla F = \nabla\delta$, the smoothness of the distance function δ carries over to S_d. Whereas Federer [13] already analyzed the differentiability of the distance function δ to general closed sets, the smoothness of higher derivatives is well-studied in the setting of submanifolds of class C^k. Thus, we proceed with a separate analysis for general closed sets, which include closed C^1 submanifolds, and smoother C^k submanifolds, $k \geq 2$.

5.1. *General closed sets*

For general closed sets, it is well-known that $\nabla\delta(x)$ enjoys Lipschitzian character everywhere but $S \cup \mathcal{L}(S)$, as Federer [13] (Theorem 4.8) proved:

Theorem 2. *Let $S \subset \mathbb{R}^n$ be a nonempty closed set. Then, on* $\mathrm{Int}(\mathcal{U}(S) \setminus S)$ *the distance (3.4) $\delta(x)$ to S is a $C^{1,1}$ function with gradient:*

$$\nabla\delta(x) = \frac{x - \pi(x)}{\delta(x)} \neq 0, \quad x \in \mathrm{Int}(\mathcal{U}(S) \setminus S). \tag{5.2}$$

In a CAGD setting, Wolter [36] rederived this result, without reference to Federer's work. For $d < R$, Theorem 2 implies a $C^{1,1}$ function F (5.1) and, by invoking Definition 2, the $C^{1,1}$ class of the trimmed offset $\delta^{-1}(d)$. Since trimmed and untrimmed offset S_d coincide (Theorem 1), S_d inherits this class [13, 42]:

Corollary 1. *Let S be a closed set with positive reach $R > 0$. Then, its offset S_d at distance $d \in (0, R)$ is a $C^{1,1}$ hypersurface.*

Remark 9. In the case of a closed set S with sharp edges and vertices, hence not eligible as a smooth submanifold, Corollary 1 implies that offsetting with $d < R$ improves the smoothness. Figure 4 illustrates this property with a solid cone S, which enjoys $R = \infty$ for being a convex set [13], thereby admitting a $C^{1,1}$ offset S_d for any d.

5.2. *Submanifolds of class $C^k, k \geq 2$*

Regarding the smoothness of the distance to C^k submanifolds, $k \geq 2$, at first glance, δ seems C^{k-1}, since δ is always continuous and expressible in terms of the direction normals to S. However, Gilbarg and Trudinger [48] first noted that, for hypersurfaces, δ is actually C^k in a certain neighborhood of S when $k \geq 2$. By incorporating Federer's work, Krantz and Parks [47] added the case $k = 1$, whereas Foote [29], with an elegant coordinate-free proof, extended the result to general submanifolds:

Lemma 3. *Let S be a compact C^k submanifold, with $k \geq 2$. Then, it admits a generalized tubular neighborhood $S_{\leq r} \subset \mathcal{U}(S)$ such that the distance $\delta(x, S)$ is C^k for $x \in$* $\mathrm{Int}(S_{\leq r} \setminus S)$.

Krantz and Parks provide another proof in [49] and, in Section 4.4 of their book [46], summarize the most relevant results on the smoothness of δ. Also, the survey by Jones et al. [25] on distance fields briefly reviews the differentiability of the distance function.

The main shortcoming of Lemma 3, limiting their applicability to characterize the smoothness of pipes/offsets, is that it does not specify the size r of the tubular neighborhood. However, when S is a C^k hypersurface, it can locally be expressed as the graph of a C^k function [21], which leads to the more specific result about r. Giusti [50] (Appendix B), adapting the original result by Gilbarg and Trudinger [48] and without citing Federer's paper [13], proves that $\delta(x, S)$ to a compact set S bounded by a C^k submanifold is C^k, for distances $\delta(x, S) < R$ such that points x have a unique nearest point and property (4.4) applies. Extending the argumentation to the boundary by considering its normal space in (4.4), since this bound is precisely $R = \text{reach}(S)$, we can rewrite Giusti's result in a more precise way by quantifying r:

Lemma 4. *Let S be a compact C^k hypersurface, with $k \geq 2$. Then, it has reach $R > 0$ and the distance $\delta(x)$ is C^k for points x such that $\delta(x) \in (0, R)$.*

Lemma 4 is still incomplete for our purposes, as it only applies to hypersurfaces, not to general m-dimensional submanifolds. To find a more powerful result, we make a short foray into Riemannian geometry [51] and extend our setting, replacing $\mathbb{R}^n$ and the Euclidean distance d with a general n-dimensional Riemannian space M and its induced inner metric. The concepts of submanifold, unique nearest point property [52], set of positive reach [53] and hence cut locus extend to this new framework. Moreover, Mantegazza and Mennucci [54] (Proposition 4.6, point 7) derive the equivalent of Theorem 2 for C^k submanifolds. For our purposes, where $M = \mathbb{R}^n$, their remarkable result particularizes as follows:

Theorem 3. *Let S be a closed C^k submanifold, with $k \geq 2$. Then, the distance $\delta(x)$ is C^k for points x everywhere but $\mathcal{L}(S) \cup S$.*

Remark 10. With respect to Lemmas 3,4, Theorem 3 relaxes the condition on S, from compact to closed. However, it no longer guarantees $R > 0$, as the unbounded curve in Figure 8a illustrates.

With a similar argumentation to that leading from Theorem 2 to Corollary 1, from Theorem 3 we derive the sibling of Corollary 1 for closed C^k progenitors:

Corollary 2. *Let S be a closed C^k progenitor with $k \geq 2$ and of positive reach $R > 0$. Then, its offset S_d at distance $d \in (0, R)$ is a C^k hypersurface.*

Remark 11. Corollary 2 applies, with improved reach (4.6), to the offset to an $n-$dimensional submanifold with C^k boundary. It also implies that offsetting with $d < R$ keeps the smoothness of S. Figure 10 illustrates this property in the case $k = 2$, with the C^2 tube S_d generated from a C^2 spine S, defined as a periodic cubic B-spline [3] from its control polygon. Figure 11 shows another example, namely offsetting a solid (i.e., a 3-dimensional submanifold) with C^2 boundary, using a commercial solid-modeling system. This operation yields a C^2 offset surface if $d < R$, whereas trying a value $d \geq R$ triggers a system error, warning that the resulting geometry is unsupported.

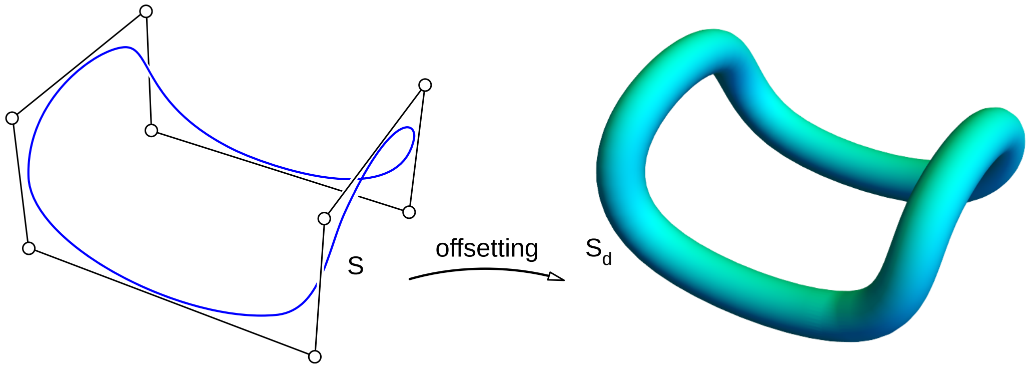

Figure 10. C^2 tube S_d constructed from a C^2 spine S (periodic cubic B-spline).

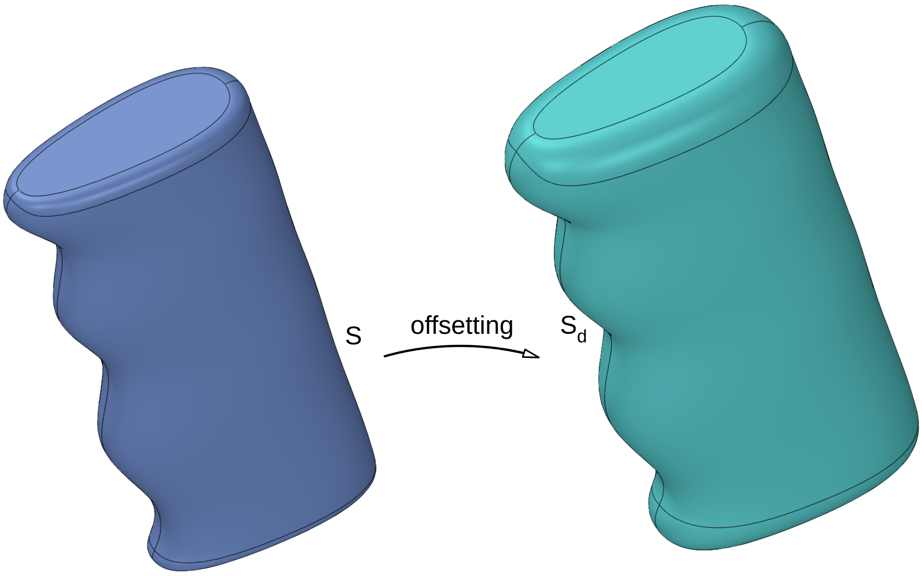

Figure 11. Offsetting a solid S bounded by a C^2 surface.

6. Conclusions

To analyze the smoothness of a pipe or offset S_d, at distance d from a progenitor S, we have brought together and adapted existing concepts and results, especially those from Federer's pioneering paper [13] on sets with positive reach. This work is usually overlooked in the literature on CAGD, with few exceptions [41–43], as well as in classical books on differential geometry [7,8] when dealing with tubes and offsets. On the other hand, the mathematical research on the smoothness of the distance function to submanifolds does not focus on the construction of pipes/offsets and seems unaware of the CAGD applied results on their regularity.

The reach R of S, i.e., the minimal distance from S to its cut locus, plays the key role. For closed sets S with positive reach, the condition $d < R$ guarantees:

- The coincidence between two offset versions: the explicit untrimmed offset S_d (obtained by displacing points of S a constant distance d along normal lines) and the trimmed offset (level set $\delta^{-1}(d)$ of the distance δ to S).
- The containment of S_d in a neighborhood with the unique nearest point property, i.e., that any point has a unique point on the progenitor at minimal distance.

- That S_d is a $C^{1,1}$ hypersurface. If S is a submanifold of class C^k (or the region bounded by a closed C^k hypersurface), this class is inherited by the distance function to S and hence by S_d. Hence, offsetting with $d < R$ improves or keeps the smoothness of S.

Sets with $R > 0$ are hence those admitting a singularity-free offset for $d < R$. Along with compactness, a C^2 class of a submanifold implies $R > 0$, but, contrary to intuition, C^1 smoothness does not.

Unlike the most relevant previous work [16] on higher-order smoothness of offset surfaces, our results hold for general closed progenitors of class C^k, not necessarily piecewise made up of C^∞ patches with G^k joints along common boundaries. Also, they hold in Euclidean space $\mathbb{R}^n$ of arbitrary dimension n, for pipe or offset hypersurfaces S_d, at constant distance d from a progenitor spine or hypersurface.

Our discussion applies only to the *classical* offset to a hypersurface, that is, along its normal. We do not consider the so-called *generalized offsets* [55–58], i.e., with translation along a vector different from the normal. Admittedly, a generalized offset to S can be rewritten as a standard offset S_d (at constant distance d) to a new progenitor $\hat{S}$ [59,60]. However, $\hat{S}$ involves tangents and normals to S_d. Consequently, a progenitor S of class C^k generates a new $\hat{S}$ of class only C^{k-1}.

Acknowledgments

This research was supported by grant PID2019-104586RB-I00 funded by MCIN/AEI/10.13039/501100011033; grant SBPLY/19/180501/000247 by *Consejería de Educación Cultura y Deportes* (*Junta de Comunidades de Castilla-La Mancha*); and grant 2021-GRIN-31214 by *Univ. de Castilla-La Mancha*; co-financed by the ERDF (European Regional Development Fund). We are grateful to a referee for their valuable suggestions, which substantially improved this article.

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. R. E. Barnhill, *Geometry Processing for Design and Manufacturing*, Philadelphia: SIAM, 1992.
2. N. M. Patrikalakis, T. Maekawa, *Shape interrogation for Computer Aided Design and Manufacturing*, Berlin: Springer, 2002.
3. J. Hoschek, D. Lasser, *Fundamentals of Computer Aided Geometric Design*, Wellesley, MA: AK Peters, 1993.
4. H. Pottmann, A. Asperl, M. Hofer, A. Kilian, *Architectural geometry*, Exton: Bentley Institute Press, 2007.
5. H. Pottmann, M. Eigensatz, A. Vaxman, J. Wallner, Architectural geometry, *Comput. Graph.*, **47** (2015), 145–164. https://doi.org/10.1016/j.cag.2014.11.002
6. T. Maekawa, N. M. Patrikalakis, T. Sakkalis, G. Yu, Analysis and applications of pipe surfaces, *Comput. Aided Geom. Des.*, **15** (1998), 437–458. https://doi.org/10.1016/S0167-8396(97)00042-3
7. M. do Carmo, *Differential Geometry of Curves and Surfaces*, Upper Saddle River, NJ: Prentice-Hall, 1976.

8. A. Gray, E. Abbena, S. Salomon, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 3rd Ed., Boca Raton, FL: Chapman&Hall/CRC, 2006.
9. T. Maekawa, An overview of offset curves and surfaces, *Comput.-Aided Des.*, **31** (1999), 165–163. https://doi.org/10.1016/S0010-4485(99)00013-5
10. J. G. Alcázar, J. R. Sendra, Local shape of offsets to algebraic curves, *J. Symb. Comput.*, **42** (2007), 38–351. https://doi.org/10.1016/j.jsc.2006.12.001
11. J. G. Alcázar, Good local behavior of offsets to rational regular algebraic surfaces, *J. Symb. Comput.*, **43** (2008), 845–857. https://doi.org/10.1016/j.jsc.2008.04.001
12. J. G. Alcázar, Good global behavior of offsets to plane algebraic curves *J. Symb. Comput.*, **43** (2008), 659–680. https://doi.org/10.1016/j.jsc.2008.01.003
13. H. Federer, Curvature measures, *Trans. Amer. Math. Soc.*, **93** (1959), 418–491.
14. J. Wallner, T. Sakkalis, T. Maekawa, H. Pottmann, G. Yu, Self-intersections of offset curves and surfaces, *International Journal of Shape Modeling*, **7** (2001), 1–21. https://doi.org/10.1142/S0218654301000023
15. T. Sakkalis, T. J. Peters, J. Bisceglio, Isotopic approximations and interval solids, *Comput.-Aided Des.*, **36** (2014), 1089–1100. https://doi.org/10.1016/j.cad.2004.01.008
16. T. Hermann, On the smoothness of offset surfaces, *Comput. Aided Geom. Des.*, **15** (1998), 529–533. https://doi.org/10.1016/S0167-8396(98)00002-8
17. J. Peters, Geometric continuity, in: G. Farin, J. Hoschek, M.-S. Kim (Eds.), *Handbook of Computer Aided Geometric Design*, North Holland/Elsevier, (2002), 193–227. https://doi.org/10.1016/B978-044451104-1/50009-5
18. G. Farin, *NURBS: from Projective Geometry to Practical Use*, 2th ed., Natick, MA: AK Peters, 1999.
19. G. Farin, *Curves and surfaces for CAGD: a practical guide*, 5th ed., San Francisco, CA: Morgan Kaufmann, 2002.
20. M. Spiegel, *Differential Geometry*, New York: Schaum McGraw-Hill, 1969.
21. M. Berger, B. Gostiaux, *Differential Geometry: Manifolds, Curves and Surfaces*, New York: Springer, 1988.
22. V. Peters, Solid Modeling, in: G. Farin, J. Hoschek, M.-S. Kim (Eds.), *Handbook of Computer Aided Geometric Design*, North Holland/Elsevier, 2002. https://doi.org/10.1016/B978-044451104-1/50021-6
23. K. Lucas, Submanifolds of dimension $n - 1$ in E^n with normals satisfying a Lipschitz condition, Studies in Eigenvalue Problems, Technical Report 18, Department of Mathematics, University of Kansas, 1957, Sect. 2.
24. M. Ghomi, R. Howard, Tangent cones and regularity of real hypersurfaces, *J. Reine Angew. Math.*, **697** (2014), 221–247. https://doi.org/10.1515/crelle-2013-0091
25. M. W. Jones, J. A. Bærentzen, M. Sramek, 3D Distance Fields: A Survey of Techniques and Applications, *IEEE Trans. Vis. Comput. Graph.*, **12** (2006), 581–599. https://doi.org/10.1109/TVCG.2006.56

26. C. Thäle, 50 years sets with positive reach - a survey -, *Surv. Math. Appl.*, **3** (2008), 123–165.
27. J. Wallner, Self-intersections and smoothness of general offset surfaces, *J. Geom.*, **70** (2001), 176–190. https://doi.org/10.1142/S0218654301000023
28. R. T. Farouki, Exact offset procedures for simple solids, *Comput. Aided Geom. Des.*, **2** (1985), 257–279. https://doi.org/10.1016/S0167-8396(85)80002-9
29. R. L. Foote, Regularity of the distance function, *Proc. Am. Math. Soc.*, **92** (1984), 153–155. https://doi.org/2045171
30. R. T. Farouki, C. A. Neff, Analytic properties of plane offset curves, *Comput. Aided Geom. Des.*, **7** (1990), 83–99. https://doi.org/10.1016/0167-8396(90)90023-K
31. R. T. Farouki, *Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable*, Berlin: Springer, 2008.
32. J.-K. Seong, G. Elber, M.-S. Kim, Trimming local and global self-intersections in offset curves/surfaces using distance maps, *Comput. Aided Geom. Des.*, **29** (2006), 555–564. https://doi.org/10.1016/j.cad.2005.08.002
33. J.-J. Kim, J. Lee, M.-S. Kim, G. Elber, Efficient offset trimming for planar rational curves using biarc trees, *Comput.-Aided Des.*, **29** (2012), 183–193. https://doi.org/10.1016/j.cagd.2012.03.014
34. J. G. Alcázar, J. Caravantes, G. M. Diaz-Toca, A new method to compute the singularities of offsets to rational plane curves, *J. Comput. Appl. Math.*, **290** (2015), 385–402. https://doi.org/10.1016/j.cam.2015.06.001
35. Q. Y. Hong, Y. Park, M.-S. Kim, G. Elber, Trimming offset surface self-intersections around near-singular regions, *Comput. Graph.*, **82** (2019), 84–94. https://doi.org/10.1016/j.cag.2019.05.016
36. F. E. Wolter, Cut Locus and Medial Axis in Global Shape Interrogation and Representation, technical report, MIT, 1993.
37. E. C. Sherbrooke, N.M. Patrikalakis, F.-E. Wolter, Differential and Topological Properties of Medial Axis Transforms, *Graphical Models and Image processing*, **58** (1996), 574–592. https://doi.org/10.1016/j.cag.2019.05.016
38. E. Kosinka, Z. Šír, C^2 Hermite interpolation by Minkowski Pythagorean hodograph curves and medial axis transform approximation, *Comput. Aided Geom. Des.*, **27** (2010), 631–643. https://doi.org/10.1016/j.cagd.2010.04.005
39. M. Bizzarri, M. Lávička, J. Vrček, Linear computational approach to interpolations with polynomial Minkowski Pythagorean hodograph curves, *J. Comput. Appl. Math.*, **361** (2019), 283–295. https://doi.org/10.1016/j.cam.2019.04.029
40. M. Bizzarri, M. Lávička, Interpolation of Hermite data by clamped Minkowski Pythagorean hodograph B-spline curves, *J. Comput. Appl. Math.*, **392** (2021). https://doi.org/10.1016/j.cam.2021.113469
41. F. Chazal, D. Cohen-Steiner, A. Lieutier, B. Thibert, Shape smoothing using double offsets, *SPM '07: Proceedings of the 2007 ACM symposium on Solid and physical modeling*, (2007), 133–158.
42. F. Chazal, D. Cohen-Steiner, A. Lieutier, Q. Mérigot, B. Thibert, Inference of Curvature Using Tubular Neighborhoods. In: Najman L., Romon P. (eds), Modern Approaches to Discrete Curvature. *Lecture Notes in Mathematics*, **2184** (2017), 133–158, Springer.

43. E. Horobeţ, M. Weinstein, Offset hypersurfaces and persistent homology of algebraic varieties, *Comput. Aided Geom. Des.*, **74** (2019), 101767. https://doi.org/10.1016/j.cagd.2019.101767

44. R. Ramamurthy, R. T. Farouki, Voronoi diagram and medial axis algorithm for planar domains with curved boundaries I. Theoretical foundations, *J. Comput. Appl. Math.*, **102** (1999), 119–141. https://doi.org/10.1016/S0377-0427(98)00211-8

45. R. Ramamurthy, R. T. Farouki, Voronoi diagram and medial axis algorithm for planar domains with curved boundaries — II. Detailed algorithm description, *J. Comput. Appl. Math.*, **102** (1999), 253–277. https://doi.org/10.1016/S0377-0427(98)00223-4

46. S. G. Krantz, H. R. Parks, *The Implicit Function Theorem: History, Theory and Applications*, Springer, 2003.

47. S. G. Krantz, H. R. Parks, Distance to C^k Hypersurfaces, *J. Differ. Equ.*, **40** (1981), 116–120.

48. D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Berlin: Springer, 1977.

49. S. G. Krantz, H. R. Parks, *The Geometry of Domains in Space*, Boston, USA: Birkhäuser, 1999.

50. E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Boston, USA: Birkhäuser, 1984.

51. S. Gallot, D. Hulin, J. Lafontaine, *Riemannian Geometry*, Berlin: Springer, 2004.

52. N. Kleinjohann, Nächste Punkte in der Riemannschen Geometrie, *Math. Zeit.*, **38** (1981), 327–344. https://doi.org/10.1007/BF01214610

53. V. Bangert, Sets with positive reach, *Arch. Math*, **38** (1982), 54–57. https://doi.org/10.1007/BF01304757

54. C. Mantegazza, A. C. Mennucci, Hamilton-Jacobi Equations and Distance Functions on Riemannian Manifolds, *Appl. Math. Optim.*, **47** (2003), 1–25. https://doi.org/10.1007/s00245-002-0736-4

55. R. Farouki, The approximation of non-degenerate offset surfaces, *Comput. Aided Geom. Des.*, **3** (1986), 15–43. https://doi.org/10.1016/0167-8396(86)90022-1

56. E. Arrondo, J. Sendra, J. R. Sendra, Parametric generalized offsets to hypersurfaces, *J. Symb. Comput.*, **23** (1997), 267–285. https://doi.org/10.1006/jsco.1996.0088

57. J. R. Sendra, J. Sendra, Algebraic analysis of offsets to hypersurfaces, *Math. Z.*, **234** (2000), 697–719. https://doi.org/10.1007/s002090050004

58. J. G. Alcázar, Local shape of generalized offsets to algebraic curves, *J. Symb. Comput.*, **47** (2012), 327–341. https://doi.org/10.1016/j.jsc.2011.12.001

59. X. Chen, Q. Lin, Properties of generalized offset curves and surfaces, *J. Appl. Math.*, **13** (2014), 124240. https://doi.org/10.1155/2014/124240

60. V. Bulut, A. Caliskan, The exchange variations of offset curves and surfaces, *Results Math.*, **67** (2015), 303–332. https://doi.org/10.1007/s00025-015-0445-3

AIMS Press