Gen. Relativ. Gravit. (2005) 37(2): 421–424
DOI 10.1007/s10714-005-0030-2

COMMENTS

L. Fernández–Jambrina

Singularity-free cosmological solutions with non-rotating perfect fluids

Received: 2 June 2004 / Published online: 2 March 2005

Abstract A conjecture stated by Raychaudhuri which claims that the only physical perfect fluid non-rotating non-singular cosmological models are comprised in the Ruiz–Senovilla and Fernández–Jambrina families is shown to be incorrect. An explicit counterexample is provided and the failure of the argument leading to the result is explicitly pointed out.

Keywords Singularity theorem · Cosmological model

Since the publication of the first non-singular cosmological model with a realistic equation of state [1], much effort has been devoted to either produce new regular models or to prove that they are a set of measure zero in some sense. Raychaudhuri [2] attempts to settle the issue by proving the following conjecture:

Conjecture The only solutions to Einstein equations that fulfill the following conditions,

1. Non-singularity: the curvature and physical scalars are regular in the whole spacetime and do not blow up at infinity.
2. Non-rotation: the vorticity of the cosmological fluid is zero.
3. Perfect fluid: the matter content of the spacetime is a perfect fluid. Therefore the energy–momentum is $T = (p + \rho)u \otimes u + p\, g$, where u is the velocity of the fluid, p is the pressure, ρ is the density and g is the metric.
4. Cosmology: there is fluid throughout the space which fulfills the energy conditions $0 \leq p \leq \rho$. Discontinuities are excluded.

L. Fernández–Jambrina (✉)
E.T.S.I. Navales, Universidad Politécnica de Madrid, Arco de la Victoria s/n, 28040 Madrid, Spain
E-mail: lfernandez@etsin.upm.es

5. $\partial p/\partial \rho$ is positive everywhere.
6. The acceleration of the fluid is hypersurface-orthogonal.

are those included in the Ruiz–Senovilla [3] and Fernández–Jambrina [4] families.

An infinite family of counter-examples for this claim is supported by the models in [5] and [6],

$$\mathrm{d}s^2 = \mathrm{e}^{2K}(-\mathrm{d}t^2 + \mathrm{d}r^2) + \mathrm{e}^{-2U}\mathrm{d}z^2 + \rho^2\mathrm{e}^{2U}\mathrm{d}\phi^2, \tag{1}$$

which correspond to a cylindrical cosmological model with matter content due to a stiff perfect fluid, $\rho = p$,

$$p = \alpha \mathrm{e}^{-2K}, \tag{2}$$

with $\alpha = \mathrm{const.} > 0$

The coordinates are comoving since the velocity of the fluid is just

$$u = \mathrm{e}^{-1/2\,K}\partial_t. \tag{3}$$

The metric function U is a solution of the reduced two-dimensional wave equation,

$$U_{tt} - U_{rr} - \frac{U_r}{r} = 0, \tag{4a}$$

$$K_t = U_t + 2rU_tU_r, \tag{4b}$$

$$K_r = U_r + r\left(U_t^2 + U_r^2\right) + \alpha r, \tag{4c}$$

and K is obtained by a quadrature when U is known.

Therefore, the general solution for this problem may be obtained from the Cauchy problem for the wave equation, for initial data $U(r,0) = f(r)$, $U_t(r,0) = g(r)$,

$$U(r,t)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^1\mathrm{d}\tau\frac{\tau}{\sqrt{1-\tau^2}}\left\{tg(v)+f(v)+tf'(v)\frac{t\tau^2+r\tau\cos\phi}{v}\right\}, \tag{5}$$

where $v = \sqrt{r^2 + t^2\tau^2 + 2rt\tau\cos\phi}$.

These models are non-singular provided $U|_{r=0}$ does not decrease too fast for large values of $|t|$,

$$U(0,t) = \int_0^1 \mathrm{d}\tau \frac{\tau}{\sqrt{1-\tau^2}}\left\{tg(|t|\tau) + f(|t|\tau) + |t|\tau f'(|t|\tau)\right\} \geq -\frac{1}{2}\ln|t| + b. \tag{6}$$

The fluid invariants can be shown to be regular and vanish at spatial and time infinity. The same happens with the curvature invariants.

It is obvious that these simple models fulfill Raychaudhuri's requirements: They are non-singular and non-rotating, the pressure and the density are positive at every point of the spacetime and they are related by a state of equation.

However, they do not belong to the Ruiz–Senovilla or Fernández–Jambrina family. In fact, they are a generalization of the latter family.

What is wrong then in Raychaudhuri's result? The most obvious failure in the reasoning leading to his claim lies at the onset of his line of thought.

The author claims that the line of maxima described by $\partial p/\partial r = 0$ is a constant r line and similarly the line of maxima described by $\partial p/\partial t = 0$ is a constant t line. The main assumption for such results is that the integrals of $\partial^2 p/\partial r^2$, $\partial^2 p/\partial t^2$ along an infinite path are necessarily infinite, but this is not true if the pressure decreases fast enough close to infinity.

This happens, for instance, when the pressure decreases exponentially. A simple and integrable example is provided by the solution of the Cauchy problem to the reduced wave equation with initial data $f(r) = \beta x^4$, $g(x) = \gamma x^2$, $\beta > 0$,

$$U(r,t) = \frac{2}{3}\gamma t^3 + \gamma t r^2 + \frac{8}{3}\beta t^4 + 8\beta t^2 r^2 + \beta r^4, \tag{7a}$$

$$\begin{aligned} K(r,t) = {} & \frac{512}{9}\beta^2 r^2 t^6 + \frac{64}{3}\beta\gamma r^2 t^5 + \left(\frac{448}{3}\beta^2 r^4 + \frac{8}{3}\beta + 2\gamma^2 r^2\right) t^4 \\ & + \left(\frac{2}{3}\gamma + \frac{112}{3}\beta\gamma r^4\right) t^3 + (8\beta r^2 + 2\gamma^2 r^4 + 64\beta^2 r^6) t^2 \\ & + \left(8\beta\gamma r^6 + \gamma r^2\right) t + \frac{1}{6}\gamma^2 r^6 + \beta r^4 + 2\beta^2 r^8 + \frac{1}{2}\alpha r^2, \end{aligned} \tag{7b}$$

and the subsequent quadrature for K.

If we analyze the lines of maxima for $p = \alpha e^{-2K}$ we find

$$\begin{aligned} \frac{1}{p}\frac{\partial p}{\partial r} = {} & -\frac{2}{9} r (1024\beta^2 t^6 + 384\beta\gamma t^5 + (5376\beta^2 r^2 + 36\gamma^2) t^4 \\ & +1344\beta\gamma r^2 t^3 + (144\beta + 72\gamma^2 r^2 + 3456\beta^2 r^4) t^2 \\ & +(18\gamma + 432\beta\gamma r^4) t + 9\alpha + 9\gamma^2 r^4 + 36\beta r^2 + 144\beta^2 r^6), \end{aligned} \tag{8}$$

that indeed the $r = 0$ line provides a maxima. On the other hand,

$$\begin{aligned} \frac{1}{p}\frac{\partial p}{\partial t} = {} & -\frac{2}{3}(8\beta r^4 + 32\beta t^2 r^2 + 4\gamma t r^2 + 1) \\ & (6\gamma t^2 + 3\gamma r^2 + 32\beta t^3 + 48\beta t r^2), \end{aligned} \tag{9}$$

it is clear that there is no constant t line of maxima.

This solution does not therefore exhibit any simple property of separability. Furthermore, it also contradicts the claim in Sect. 9 about space time reversibility of non-singular solutions. It is explicit that this solution is not time reversible. In fact, the models in [5] are generically non-reversible.

Finally, it is to be pointed out that the example in this work is just an easy integrable case. The features exhibited by this model are shared by the whole family, except for the most simple solutions, like [4].

Acknowledgements The present work has been supported by Dirección General de Enseñanza Superior Project PB98-0772. L.F.J. wishes to thank F.J. Chinea, L.M. González-Romero, F. Navarro-Lérida and M.J. Pareja for valuable discussions.

References

1. Senovilla, J.M.M.: New class of inhomogeneous cosmological perfect-fluid solutions without big-bang singularity. Phys. Rev. Lett. **64**, 2219–2221 (1990)
2. Raychaudhuri, A.K.: Singularity-free cosmological solutions with non-rotating perfect fluids. Gen. Relativ. Gravit. **36**, 343–359 (2004)
3. Ruiz, E., Senovilla, J.M.M.: General class of inhomogeneous perfect-fluid solutions. Phys. Rev. D **45**, 1995–2005 (1992)
4. Fernández–Jambrina, L.: Singularity-free cylindrical cosmological model. Class. Quantum Gravity **14**, 3407–3412 (1997)
5. Fernández–Jambrina, L., González-Romero, L.M.: A wide family of singularity-free cosmological models. Phys. Rev. D **66**, 024027–24031 (2002)
6. Fernández–Jambrina, L., González-Romero, L.M.: Nonsingular $G_{\{2\}}$ stiff fluid cosmologies. J. Math. Phys. **45**, 2113–2123 (2004)