

## Geodesic completeness of orthogonally transitive cylindrical space–times

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In this paper a theorem is derived in order to provide a wide sufficient condition for an orthogonally transitive cylindrical space–time to be singularity free. The applicability of the theorem is tested on examples provided by the literature that are known to have regular curvature invariants. © 1999 American Institute of Physics. [S0022-2488(99)02708-5]

### I. INTRODUCTION

The issue of establishing whether a Lorentzian manifold is geodesically complete is not in principle a simple one since there is no Hopf–Rinow theorem that could settle the matter as it happens in the Riemannian case. One could think that regularity of the curvature invariants might be helpful, but there are known examples of space–times with regular invariants, such as Taub-NUT,<sup>1</sup> that enclose geodesics that are not complete in their affine parametrization due to a phenomenon called imprisoned incompleteness.

Taking completeness of causal geodesics ( $g$  completeness) as a definition of the absence of singularities (no observer in free fall leaves the space–time in a finite proper time), one can resort to many theorems in the literature (cf. Refs. 1 and 2, and references therein) in order to determine whether a space–time is singular. But on the contrary, theorems that provide large families of nonsingular space–times are not very usual<sup>3,4</sup> and in principle the proof of geodesic completeness involves cumbersome calculations.<sup>5</sup>

Instead of dealing with general Lorentzian manifolds, we shall approach orthogonally transitive cylindrical space–times<sup>6</sup> since they have provided many examples of regular manifolds (cf. Refs. 7 and 8) in inhomogeneous cosmology.<sup>9</sup>

Our aim will be the generalization of the theorem on diagonal orthogonally transitive cylindrical space–times in Ref. 4 to nondiagonal models and thereby comprise all known nonsingular cylindrical perfect fluid space–times in the literature.

First we shall show that the second-order system of geodesic equations can be reduced by the use of constants of motion to three first-order equations plus two quadratures. This fact will simplify the analysis of the prolongability of the geodesics and will enable us to write a sufficient condition for completeness of orthogonally transitive cylindrical space–times in a theorem.

### II. GEODESIC EQUATIONS

We shall write the metric of an orthogonally transitive cylindrical space–time in a chart using isotropic coordinates  $t$ ,  $r$  for the subspace orthogonal to the orbits of the isometry group and coordinates  $\phi$ ,  $z$  adapted to the commuting generators of the group of isometries. The metric shall be determined by four functions,  $g$ ,  $f$ ,  $A$ ,  $\rho$ , of the coordinates  $t$ ,  $r$ ,

$$ds^2 = e^{2g(t,r)}\{-dt^2 + dr^2\} + \rho^2(t,r)e^{2f(t,r)}d\phi^2 + e^{-2f(t,r)}\{dz + A(r,t)d\phi\}^2 \quad (1)$$

and we shall assume that these functions are  $C^2$  in their range,

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$$-\infty < t, z < \infty, \quad 0 < r < \infty, \quad 0 < \phi < 2\pi. \tag{2}$$

The axis will be located where the norm of the axial Killing field vanishes,

$$0 = \Delta = g(\xi, \xi) = \rho^2(t, r)e^{2f(t, r)} + e^{-2f(t, r)}A^2(r, t), \tag{3}$$

which means that both  $A$  and  $\rho$  must vanish on the axis, since  $f$  is a smooth function.

Since the choice of isotropic coordinates is not unique, we can take advantage of this freedom to have  $r=0$  as the equation for the axis. In order to avoid conical singularities, the usual requirement<sup>10</sup> will be imposed in order to have a well-defined axis,

$$\lim_{r \rightarrow 0} \frac{g(\text{grad } \Delta, \text{grad } \Delta)}{4\Delta} = 1. \tag{4}$$

Denoting by a dot differentiation with respect to the affine parameter, two of the four second-order geodesic equations,

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \tag{5}$$

can be integrated, taking into account that there are two first integrals of the geodesic motion associated with the generators of the isometries. These are the angular momentum around the axis,  $L$ , and the linear momentum along the axis,  $P$ , of a test particle of unit mass,

$$L = e^{2f(t, r)}\rho^2(t, r)\dot{\phi} + e^{-2f(t, r)}A(t, r)\{\dot{z} + A(t, r)\dot{\phi}\}, \tag{6}$$

$$P = e^{-2f(t, r)}\{\dot{z} + A(t, r)\dot{\phi}\}. \tag{7}$$

The affine parametrization is determined, up to an affinity of the real line, by the prescription

$$\delta = e^{2g(t, r)}\{\dot{t}^2 - \dot{r}^2\} - \{L - PA(t, r)\}^2 \rho^{-2}(t, r)e^{-2f(t, r)} - P^2 e^{2f(t, r)}, \tag{8}$$

where  $\delta$  is one for timelike, zero for null, and minus one for spacelike geodesics. Since we are dealing just with causal geodesics, for our purposes  $\delta$  will always be positive. After writing  $\dot{z}, \dot{\phi}$  as functions of  $L$  and  $P$ , the second-order equations in  $t$  and  $r$ ,

$$\begin{aligned} &\ddot{t} + g_t(t, r)\dot{t}^2 + 2g_r(t, r)\dot{t}\dot{r} + g_r(t, r)\dot{r}^2 - P^2 e^{2\{f(t, r) - g(t, r)\}}f_t(t, r) \\ &+ e^{-2\{f(t, r) + g(t, r)\}} \frac{\{L - PA(t, r)\}^2}{\rho^2(t, r)} \left\{ \frac{\rho_t(t, r)}{\rho(t, r)} + f_t(t, r) + \frac{PA_t(t, r)}{L - PA(t, r)} \right\}, \end{aligned} \tag{9}$$

$$\begin{aligned} &\ddot{r} + g_r(t, r)\dot{t}^2 + 2g_t(t, r)\dot{t}\dot{r} + g_t(t, r)\dot{r}^2 + P^2 e^{2\{f(t, r) - g(t, r)\}}f_r(t, r) \\ &- e^{-2\{f(t, r) + g(t, r)\}} \frac{\{L - PA(t, r)\}^2}{\rho^2(t, r)} \left\{ \frac{\rho_r(t, r)}{\rho(t, r)} + f_r(t, r) + \frac{PA_r(t, r)}{L - PA(t, r)} \right\}, \end{aligned} \tag{10}$$

can be rearranged in a form that will be useful afterwards,

$$\{e^{2g(t, r)}\dot{t}\} \cdot - e^{-2g(t, r)}F(t, r)F_t(t, r) = 0, \tag{11}$$

$$\{e^{2g(t, r)}\dot{r}\} \cdot + e^{-2g(t, r)}F(t, r)F_r(t, r) = 0, \tag{12}$$

$$F(t, r) = e^{g(t, r)} \sqrt{\delta + P^2 e^{2f(t, r)} + \{L - PA(t, r)\}^2 \frac{e^{-2f(t, r)}}{\rho^2(t, r)}}, \tag{13}$$

which have the same structure as in the diagonal case.

If at least one of the constants  $L, P, \delta$  is different from zero, the system is equivalent to three equations of first order for future-pointing (past-pointing) geodesics,

$$\dot{t} = \pm e^{-2g(t,r)} F(t,r) \cosh \xi(t,r), \tag{14}$$

$$\dot{r} = e^{-2g(t,r)} F(t,r) \sinh \xi(t,r), \tag{15}$$

$$\dot{\xi}(t,r) = -e^{-2g(t,r)} \{ \pm F_t(t,r) \sinh \xi(t,r) + F_r(t,r) \cosh \xi(t,r) \}, \tag{16}$$

parametrizing (8) by a function  $\xi(t,r)$ . More explicitly, the last equation takes the form,

$$\begin{aligned} \dot{\xi} = & - \frac{e^{-g}}{\sqrt{\delta + \Lambda^2 \rho^{-2} e^{-2f} + P^2 e^{2f}}} \left\{ \cosh \xi \left( \delta g_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r + P^2 e^{2f} q_r \right) \right. \\ & \left. \pm \sinh \xi \left( \delta g_t + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_t + P^2 e^{2f} q_t \right) \right\}, \end{aligned} \tag{17}$$

$$h = g - f - \ln \rho + \ln |\Lambda|, \quad \Lambda = L - PA, \quad q = g + f, \tag{18}$$

which will be useful for deriving prolongability conditions for causal geodesics. The minus (plus) sign corresponds to past-pointing (future-pointing) geodesics.

Note that the general equations are obtained from those of the diagonal case just replacing  $L$  by  $\Lambda$ , which can be therefore considered as a sort of ‘‘effective angular momentum’’ in the case where the Killing fields are not orthogonal.

### III. PROLONGABILITY OF THE GEODESICS

In this section we shall introduce two theorems on causal geodesic completeness of orthogonally transitive cylindrical space–times. Null coordinates,

$$u = \frac{t+r}{2}, \quad v = \frac{t-r}{2}, \tag{19}$$

will play an important role in the results.

**Theorem 1:** An orthogonally transitive cylindrical space–time endowed with a metric whose local expression in terms of  $C^2$  metric functions  $f, g, A, \rho$  is given by (1) such that the axis located at  $r=0$  has complete future causal geodesics if the following set of conditions is fulfilled:

(1) For large values of  $t$  and increasing  $r$ ,

$$(a) \quad \begin{cases} g_u \geq 0 \\ h_u \geq 0 \\ q_u \geq 0, \end{cases}$$

$$(b) \text{ either } \begin{cases} g_r \geq 0 \text{ or } |g_r| \leq g_u \\ h_r \geq 0 \text{ or } |h_r| \leq h_u \\ q_r \geq 0 \text{ or } |q_r| \leq q_u. \end{cases}$$

(2) For  $L \neq 0$  and large values of  $t$  and decreasing  $r$ ,

$$(a) \quad \delta g_v + P^2 e^{2f} q_v + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_v \geq 0,$$

$$(b) \text{ either } \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \leq 0 \text{ or } \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \leq \delta g_v + P^2 e^{2f} q_v + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_v.$$

(3) For large values of the time coordinate  $t$ , constants  $a, b$  exist such that

$$\text{that } \left. \begin{array}{l} 2g(t,r) \\ g(t,r) + f(t,r) + \ln \rho - \ln |\Lambda| \\ g(t,r) - f(t,r) \end{array} \right\} \geq -\ln|t+a| + b.$$

(4) The limit  $\lim_{r \rightarrow 0} (A/\rho)$  exists.

A theorem can be obtained for past-pointing geodesics just changing the sign of the time derivatives in the previous one.

**Theorem 2:** An orthogonally transitive cylindrical space-time endowed with a metric whose local expression in terms of  $C^2$  metric functions  $f, g, A, \rho$  is given by (1) such that the axis is located at  $r=0$  has complete past causal geodesics if the following set of conditions is fulfilled:

(1) For small values of  $t$  and increasing  $r$ ,

$$(a) \left\{ \begin{array}{l} g_v \leq 0 \\ h_v \leq 0 \\ q_v \leq 0, \end{array} \right.$$

$$(b) \text{ either } \left\{ \begin{array}{l} g_r \geq 0 \text{ or } |g_r| \leq -g_v \\ h_r \geq 0 \text{ or } |h_r| \leq -h_v \\ q_r \geq 0 \text{ or } |q_r| \leq -q_v. \end{array} \right.$$

(2) For  $L \neq 0$  and small values of  $t$  and decreasing  $r$ ,

$$(a) \delta g_u + P^2 e^{2f} q_u + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_u \leq 0$$

$$(b) \text{ either } \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \leq 0 \text{ or } \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \leq \left| \delta g_u + P^2 e^{2f} q_u + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_u \right|.$$

(3) For small values of the time coordinate  $t$ , constants  $a, b$  exist such that

$$\text{that } \left. \begin{array}{l} 2g(t,r) \\ g(t,r) + f(t,r) + \ln \rho - \ln |\Lambda| \\ g(t,r) - f(t,r) \end{array} \right\} \geq -\ln|t+a| + b.$$

(4) The limit  $\lim_{r \rightarrow 0} (A/\rho)$  exists.

The theorems in Ref. 4 can be seen to be subcases of the ones introduced here.

#### IV. PROOF OF THE THEOREMS

In order to achieve prolongability of causal geodesics we just have to impose that  $\dot{t}$  remains finite for finite values of the affine parameter. The radial velocity,  $\dot{r}$ , need not be considered since it cannot become singular if  $\dot{t}$  is not singular too. The other derivatives,  $\dot{z}$  and  $\dot{\phi}$ , are quadratures

of smooth functions of  $t$  and  $r$  and therefore they only may turn singular if  $t$  or  $r$  become so. We shall focus on future-pointing geodesics. The analysis for past-pointing geodesics is entirely similar.

A way of preventing the hyperbolic functions of  $\xi$  from becoming singular is to require that  $\dot{\xi}$  does not grow indefinitely. Therefore for large values of  $\xi$ ,  $\dot{\xi}$  must eventually become negative. Since the constants,  $L, P, \delta$  may vanish independently, one is not to expect compensations between them. Hence their respective terms in (17) have to become negative for large values of positive  $\xi$  as it is stated in conditions (1a) and (1b), taking into account that  $\cosh \xi = \sinh \xi + e^{-\xi}$  and that therefore the terms in the negative exponential of  $\xi$  (1b) need not be negative but just of the same order as those in (1a).

By imposing condition (4) in the theorem, we require that the geometry of the space-time in the vicinity of the axis is determined by  $\rho$  and not by  $A$ . Hence for negative decreasing  $\xi$  the axis could be singular only for geodesics with  $L \neq 0$ , since the terms  $A/\rho$  are finite at the axis. Geodesics with zero angular momentum just cross  $r=0$  and reappear with positive  $\xi$  and polar angle  $\phi + \pi$  and hence need not be taken into account. In this case we can therefore allow compensations between the nonzero  $L$  term and the other ones. Splitting  $\cosh \xi$  as  $e^{\xi} - \sinh \xi$ , the condition for  $\dot{\xi}$  to become positive for large values of  $t$  and negative  $\xi$  is stated in (2a) for the terms in  $\sinh \xi$  and in (2b) for the terms in  $e^{\xi}$ , that can be at most of the same order as the former ones since they are exponentially damped.

No further conditions need be imposed on  $\dot{\xi}$ . But  $\dot{t}$  could turn singular also for the  $e^{-2g}F$  term. A way of preventing it is to impose a growth slower than linear for  $\dot{t}$  due to each of the three terms  $(\delta, \Lambda, P)$  for large values of  $t$ . This is done in condition (3).

The condition on  $g$  must be refined since we have not yet considered the geodesics that cannot be parametrized by  $\xi$ . These are those with zero  $F$ , that is, null geodesics with zero  $\dot{z}$  and  $\dot{\phi}$ . Since  $\dot{t} = |\dot{r}|$  for such geodesics, the equations of geodesic motion,

$$\ddot{t} + g_t \dot{t}^2 + 2g_r \dot{t} \dot{r} + g_r \dot{r}^2 = 0, \tag{20}$$

$$\ddot{r} + g_r \dot{t}^2 + 2g_t \dot{t} \dot{r} + g_t \dot{r}^2 = 0, \tag{21}$$

can be reduced to a single one that can be integrated,

$$\dot{t} + 2g_t \dot{t}^2 + 2g_r \dot{t} \dot{r} = 0 \Rightarrow (e^{2g} \dot{t})' = K, \tag{22}$$

which, in order to have  $t$  extendible to arbitrary values of the affine parameter, can be controlled by imposing at most linear growth for  $\dot{t}$  as it is done in condition (3).

### V. COMPLETENESS OF SEVERAL CYLINDRICAL MODELS

Since all known diagonal cylindrical perfect fluid models (Refs. 11–14) with regular curvature invariants have already been shown to be geodesically complete,<sup>4</sup> we shall only be concerned about the nondiagonal ones. To our knowledge there are just two, and both can be derived from Einstein space-times using the Wainwright–Ince–Marshman generation algorithm for stiff perfect fluids.<sup>15</sup>

(1) Mars:<sup>16</sup> It is the first known nonsingular nondiagonal cylindrical cosmological model in the literature. In another context it was previously published by Letelier.<sup>17</sup> In isotropic coordinates the metric functions can be written as

$$g(t,r) = \frac{1}{2} \ln \cosh(2at) + \frac{1}{2} \alpha a^2 r^2, \quad f(t,r) = \frac{1}{2} \ln \cosh(2at),$$

$$\rho(t,r) = r, \quad A(t,r) = ar^2, \tag{23}$$

where  $a$  is a constant and  $\alpha > 1$ . If  $\alpha = 1$  the pressure and the density of the fluid vanish.

All functions are even in  $t$  and therefore the analysis of past-pointing geodesics is equivalent to that of future-pointing ones and shall be omitted. The derivatives  $g_u, q_u$  in condition (1a) are positive for positive  $t$  whereas  $h_u$  also needs large radial coordinate.

The derivatives  $g_r, q_r$  in condition (1b) are positive regardless of  $t$  and  $h_r$  is positive for large  $r$ .

Conditions (2a), (2b) are satisfied when  $r$  is small and  $t$  is positive since the  $L$  term is dominant for small  $r$ .

The functions in condition (3) are all positive except for the  $\ln r$  term in  $h$  when  $r$  decreases. But this can be bounded by a logarithm of  $t$  and therefore the condition is fulfilled. The ratio  $A/\rho$  tends to zero for decreasing  $r$  and hence condition (4) in theorem 1 is satisfied. Hence this space-time is causally  $g$  complete.

(2) Griffiths–Bičák:<sup>18</sup> The previous model is comprised in this one for  $c=0$  after a redefinition of constants. The metric functions can be written as

$$g(t,r) = \frac{1}{2} \ln \cosh(2at) + \frac{1}{2} a^2 r^2 + \frac{1}{2} \Omega(t,r), \quad f(t,r) = \frac{1}{2} \ln \cosh(2at),$$

$$\rho(t,r) = r, \quad A(t,r) = ar^2,$$
(24)

where  $\Omega$  is a function that is obtained from a solution,  $\sigma$ , of the wave equation

$$\Omega_r = r(\sigma_t^2 + \sigma_r^2), \quad \Omega_t = 2r\sigma_t\sigma_r,$$

$$\sigma(t,r) = bt + \sqrt{2}c \sqrt{\frac{\sqrt{(\alpha^2 + r^2 - t^2)^2 + 4\alpha^2 t^2} + \alpha^2 + r^2 - t^2}{(\alpha^2 + r^2 - t^2)^2 + 4\alpha^2 t^2}}.$$
(25)

The analysis of the geodesics of this space-time can be dealt with easily since

$$\Omega_u = r(\sigma_t + \sigma_r)^2, \quad \Omega_v = -r(\sigma_t - \sigma_r)^2,$$
(26)

and therefore  $\Omega$  contributes with an additional positive term to conditions (1a), (1b) in theorem 1, that were already checked for Mars space-time. On the contrary,  $\Omega$  adds a negative term in conditions (2a), (2b) but it is negligible for small  $r$ .

Finally, the contribution of  $\Omega$  to  $g$  grows at most as  $\sqrt{t}$  for large  $t$  and is therefore negligible compared to the other terms in condition (3).

A similar reasoning is valid for theorem 2, although the metric function  $\Omega$  is not even in time.

Hence these space-times are geodesically complete.

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