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Nonsingular G_2 stiff fluid cosmologies

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In this paper we analyze Abelian diagonal orthogonally transitive space–times with spacelike orbits for which the matter content is a stiff perfect fluid. The Einstein equations are cast in a suitable form for determining their geodesic completeness. A sufficient condition on the metric of these space–times is obtained, that is fairly easy to check and to implement in exact solutions. These results confirm that nonsingular space–times are abundant among stiff fluid cosmologies. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1705715]

I. INTRODUCTION

After the discovery of the first nonsingular perfect fluid cosmological model by Senovilla,[1] the possibility of constructing regular cosmologies was renewed. The interest for regular cosmologies had stifled for nearly 30 years due to the powerful singularity theorems (cf., for instance, Refs. 2 and 3), which seemed to preclude such space–times under very general requirements, such as chronology protecting, energy and generic conditions. The open way to regular cosmologies was found in the violation of some technical premises of the theorems. For instance, in Ref. 4 it was shown that the Senovilla space–time did not possess a compact achronal set without edge and could not have closed trapped surfaces.

However, the first results were not encouraging. The extension of the Senovilla solution to a family of space–times left the set of regular models limited to a zero-measure subset surrounded by space–times with Ricci and Weyl curvature singularities.[5] During the following decade only a few new nonsingular cosmologies were added to the list.[6]

Another estrategy to approach singularities arose with the publication of regularity theorems.[7–9] Whereas singularity theorems stated general sufficient conditions for the appearance of singularities, these theorems aimed the contrary, namely particular conditions to achieve regular space–times.

The application of the conclusions of Ref. 8 to a restricted family of stiff fluids provided an unexpected result. The set of known nonsingular perfect fluid cosmologies was enlarged with a huge family depending on two nearly arbitrary functions.[10]

The purpose of this paper is the extension of those results to determine which space–times among Abelian diagonal orthogonally transitive space–times with spacelike orbits and with a stiff fluid as matter content are nonsingular. Instead of restricting to an integrable family of solutions of the Einstein equations, we analyze the whole set of diagonal cylindrical stiff fluid space–times with a spacelike transitivity surface element.

With this aim in mind we write in Sec. II the Einstein equations for such space–times and we cast them in a form suitable for the application of the theorems. The analysis of the restrictions imposed by regularity conditions is done in Sec. III. Finally in Sec. IV we check the possibility of constructing regular space–times with nonvaninishing matter scalar space averages on Cauchy hypersurfaces in order to support the validity of a regularity conjecture by Senovilla.[11]

II. EQUATIONS FOR G_2 STIFF FLUID SPACE–TIMES

As it has been stated in the introduction, we shall focus on space–times endowed with an Abelian orthogonally transitive group of isometries G_2 acting on spacelike surfaces, since this is the framework where most nonsingular space–times have been found so far. We further impose that generators for the group can be found that are mutually orthogonal. We follow Ref. 8 for writing the Einstein equations for such space–times using a formalism based on differential forms.

If the generators of the isometry group are chosen to be $\{\xi,\eta\}$, we may write an orthonormal tetrad, $\{\theta^0,\theta^1,\theta^2,\theta^3\}$, where just θ^2 and θ^3 lie in $\mathrm{lin}\{\xi,\eta\}$. We may impose that these 1-forms be Lie-invariant under the isometry group.[12] The metric is written as

$$ds^2=-\theta^0\otimes\theta^0+\theta^1\otimes\theta^1+\theta^2\otimes\theta^2+\theta^3\otimes\theta^3. \tag{1}$$

Making use of the spacelike congruence for θ^2 and its kinematical quantities, we may define the tetrad basis according to the vanishing torsion equations,

$$d\theta^0=\nu\wedge\theta^1, \tag{2a}$$

$$d\theta^1=\nu\wedge\theta^0, \tag{2b}$$

$$d\theta^2=\alpha\wedge\theta^2, \tag{2c}$$

$$d\theta^3=(\beta-\alpha)\wedge\theta^3, \tag{2d}$$

where ν is just a connection in the $\theta^0-\theta^1$ subspace, α is an "acceleration" for θ^2 and β is related to the expansion of the surface element in the $\theta^2-\theta^3$ subspace, since $d(\theta^2\wedge\theta^3)=\beta\wedge\theta^2\wedge\theta^3$.

The integrability conditions for these equations are easily obtained by exterior differentiation of the system,

$$d\beta=0, \tag{3a}$$

$$d\alpha=0. \tag{3b}$$

Finally, Einstein field equations are written in terms of these differential forms as an exterior system,

$$d*\alpha+\beta\wedge*\alpha=(\tfrac{1}{2}T-T_{22})\,\theta^0\wedge\theta^1, \tag{4a}$$

$$d*\beta+\beta\wedge*\beta=(T_{11}-T_{00})\,\theta^0\wedge\theta^1, \tag{4b}$$

$$d\nu+\alpha\wedge*\alpha-\beta\wedge*\alpha=\tfrac{1}{2}(T_{00}-T_{11}+T_{22}+T_{33})\,\theta^0\wedge\theta^1, \tag{4c}$$

$$d*\tilde\beta+\beta\wedge*\tilde\beta+2(\alpha-\beta)\wedge*\tilde\alpha+2\nu\wedge\tilde\beta=(T_{00}+T_{11})\,\theta^0\wedge\theta^1, \tag{4d}$$

$$d\tilde\beta+\beta\wedge\tilde\beta+2(\alpha-\beta)\wedge\tilde\alpha+2\nu\wedge*\tilde\beta=2T_{01}\theta^0\wedge\theta^1, \tag{4e}$$

for a matter content defined by the energy-momentum tensor $T=T_{ab}\theta^a\otimes\theta^b$.

The tilde denotes a reflection in the $\theta^0-\theta^1$ subspace, that is, if $\alpha=a\,\theta^0+b\,\theta^1$, then $\tilde\alpha=a\,\theta^0-b\,\theta^1$. The $*$ denotes the Hodge duality operator in the same subspace, $*\alpha=-a\,\theta^1-b\,\theta^0$.

Integration of the first Bianchi equations (3a) and (3b),

$$\alpha=-dU, \tag{5a}$$

$$\beta=\frac{d\ln\rho}{\rho}, \tag{5b}$$

allows integration of Cartan equations in terms of two functions, $z,\ \phi$,

$$\theta^2=e^{-U}\,dz,\quad \theta^3=\rho e^U\,d\phi, \tag{6}$$

that we take as coordinates in order to write the metric in a conventional form,

$$g=e^{2K}(-dt^2+dr^2)+e^{-2U}\,dz^2+\rho^2e^{2U}\,d\phi^2. \tag{7}$$

The coordinates are adapted to the Killing fields, so that $\xi=\partial_z,\ \eta=\partial_\phi$. The nonignorable coordinates $t,\ r$ are chosen so that the metric is isotropic in the subspace spanned by θ^0 and θ^1,

$$\theta^0=e^K\,dt,\quad \theta^1=e^K\,dr. \tag{8}$$

The range for these coordinates is the usual one,

$$-\infty<t,\quad z<\infty,\quad 0<r<\infty,\quad 0<\phi<2\pi, \tag{9}$$

if we require the space–time to be cylindrically symmetric. The remaining metric functions, K, U, and ρ, depend just on t and r.

The connection in this case is $\nu=*dK$.

This is the general framework for an orthogonally transitive diagonal space–time with space-like orbits. If the matter content is a perfect fluid with 4-velocity u, pressure p, and density μ, the Bianchi equations for such energy-momentum tensor,

$$T=\mu u\otimes u+p\,(g+u\otimes u), \tag{10}$$

may be written in compact expressions involving the kynematical 1-forms,

$$du+\frac{1}{\mu+p}\mathrm{d}p\wedge u=0, \tag{11a}$$

$$d*u+\left(\beta+\frac{d\mu}{\mu+p}\right)\wedge *u=0, \tag{11b}$$

which state that the fluid is irrotational.

We might choose $\theta^0=u$ for writing the Einstein equations, as it was done in Ref. 10, but since we aim full generality, we shall not follow that way and explore arbitrary possibilities of alignment for this 1-form. Preserving the unitarity of u, we may parametrize it in terms of a function ξ,

$$u=-\theta^0\cosh\xi-\theta^1\sinh\xi, \tag{12}$$

so that the Einstein equations for a perfect fluid take the following form:

$$U_{tt}-U_{rr}+\frac{1}{\rho}(U_t\rho_t-U_r\rho_r)=\frac{p-\mu}{2}e^{2K}, \tag{13a}$$

$$\rho_{tt}-\rho_{rr}=(\mu-p)\rho e^{2K}, \tag{13b}$$

$$K_t\rho_r+K_r\rho_t=\rho_{tr}+U_t\rho_r+U_r\rho_t+2\rho U_tU_r+e^{2K}\rho\frac{\mu+p}{2}\sinh 2\xi, \tag{13c}$$

$$K_t\rho_t+K_r\rho_r=\frac{\rho_{tt}+\rho_{rr}}{2}+U_t\rho_t+U_r\rho_r+\rho\left(U_t^2+U_r^2+e^{2K}\frac{\mu+p}{2}\cosh 2\xi\right), \tag{13d}$$

$$K_{rr}-K_{tt}+\frac{U_r\rho_r-U_t\rho_t}{\rho}+U_r^2-U_t^2=\frac{\mu+p}{2}e^{2K}, \tag{13e}$$

and the energy-momentum conservation laws yield

$$K_r-\xi_t+\frac{p_r\cosh^2\xi+(\mu_t-p_t)\sinh\xi\cosh\xi-\mu_r\sinh^2\xi}{\mu+p}+\frac{\rho_t\cosh\xi-\rho_r\sinh\xi}{\rho}\sinh\xi=0, \tag{14a}$$

$$K_t-\xi_r+\frac{\mu_t\cosh^2\xi+(p_r-\mu_r)\sinh\xi\cosh\xi-p_t\sinh^2\xi}{\mu+p}+\frac{\rho_t\cosh\xi-\rho_r\sinh\xi}{\rho}\cosh\xi=0. \tag{14b}$$

The system of equations becomes much simpler if we restrict to stiff fluids, $\mu=p$,

$$U_{tt}-U_{rr}+\frac{1}{\rho}(U_t\rho_t-U_r\rho_r)=0, \tag{15a}$$

$$\rho_{tt}-\rho_{rr}=0, \tag{15b}$$

$$\frac{K_t\rho_r+K_r\rho_t}{\rho}=\frac{\rho_{tr}+U_t\rho_r+U_r\rho_t}{\rho}+2U_tU_r+e^{2K}p\sinh 2\xi, \tag{15c}$$

$$\frac{K_t\rho_t+K_r\rho_r}{\rho}=\frac{\rho_{tt}+\rho_{rr}}{2\rho}+\frac{U_t\rho_t+U_r\rho_r}{\rho}+U_t^2+U_r^2+e^{2K}p\cosh 2\xi, \tag{15d}$$

$$K_{rr}-K_{tt}+\frac{U_r\rho_r-U_t\rho_t}{\rho}+U_r^2-U_t^2=p\,e^{2K}, \tag{15e}$$

$$K_r-\xi_t+\frac{p_r}{2p}+\frac{\rho_t\cosh\xi-\rho_r\sinh\xi}{\rho}\sinh\xi=0, \tag{15f}$$

$$K_t-\xi_r+\frac{p_t}{2p}+\frac{\rho_t\cosh\xi-\rho_r\sinh\xi}{\rho}\cosh\xi=0. \tag{15g}$$

The reason why the stiff fluid equations are easy to integrate is that the metric functions U, ρ decouple from the pressure, which only appears in the equations for the conformal factor K. Therefore the stiff fluid case is fairly similar to vaccuum and can be generated from this one.

A further simplification can be obtained if we take ρ as coordinate. This is fully compatible with an isotropic parametrization, since equation (15b),

$$0=d*\beta+\beta\wedge*\beta=d\left(\frac{*d\rho}{\rho}\right)+\frac{d\rho\wedge*d\rho}{\rho^2}=\frac{d*\mathrm{d}\rho}{\rho},$$

states that $*d\rho$ is also an exact differential form.

We take $\rho=r$ as a spatial coordinate, since every known nonsingular solution has a surface element with spacelike gradient. With this choice of coordinates the differential system becomes even simpler,

$$U_{tt}-U_{rr}-\frac{U_r}{r}=0, \tag{16a}$$

$$K_t=U_t+2rU_tU_r+e^{2K}pr\sinh 2\xi, \tag{16b}$$

$$K_r=U_r+r(U_t^2+U_r^2)+e^{2K}pr\cosh 2\xi, \tag{16c}$$

$$K_{rr}-K_{tt}+\frac{U_r}{r}+U_r^2-U_t^2=pe^{2K}, \tag{16d}$$

$$K_r-\xi_t+\frac{p_r}{2p}-\frac{\sinh^2\xi}{r}=0, \tag{16e}$$

$$K_t-\xi_r+\frac{p_t}{2p}-\frac{\sinh\xi\cosh\xi}{r}=0. \tag{16f}$$

The integrability condition, $K_{rt}=K_{tr}$, for (16b) and (16c) requires that a combination of functions be an exact differential form,

$$dH=e^{2K}rp(\sinh 2\xi\,dt+\cosh 2\xi\,dr), \tag{17}$$

from which we can read ξ and the pressure, if K is known,

$$\tanh 2\xi=\frac{H_t}{H_r},\quad |p|=\frac{e^{-2K}}{r}\sqrt{H_r^2-H_t^2}. \tag{18}$$

The integrability of dH is also a consequence of the energy-momentum conservation equations (16e) and (16f).

For consistency these expressions imply that the gradient of H be spacelike and that H_r be positive in order to have positive pressure.

The simple case, $\xi=0$, for which u is parallel to the time direction corresponds to $H=\gamma r^2/2$, where γ is a positive constant.

The remaining system of differential equations,

$$U_{tt}-U_{rr}-\frac{U_r}{r}=0, \tag{19a}$$

$$H_{rr}-H_{tt}=\frac{\sqrt{H_r^2-H_t^2}}{r}, \tag{19b}$$

$$K_t=U_t+2rU_tU_r+H_t, \tag{19c}$$

$$K_r=U_r+r(U_t^2+U_r^2)+H_r, \tag{19d}$$

is formed by a reduced wave equation in polar coordinates for U on the plane and a nonlinear wave equation for H. Once these equations are solved, we are left with a quadrature for K. The integrability of this quadrature is guaranteed by the other equations.

As it has already been stated, these equations are pretty similar to those of vaccuum. The only diference is the additional conformal factor defined by H.

Regularity of the metric at the axis $r=0$ is already implicit in the equations, provided that metric functions are regular. Following Ref. 13, we have a regular axis whenever

$$\lim_{r\to 0}\frac{\langle \mathrm{grad}\,\Delta,\mathrm{grad}\,\Delta\rangle}{4\Delta}=e^{2(U-K)}|_{r=0}=1, \quad \Delta=\langle \partial_\phi,\partial_\phi\rangle=r^2e^{2U}. \tag{20}$$

But according to Eqs. (17), (19c), and (19d), K and U are equal at the axis, except for a constant, since

$$K_r(t,0)=U_r(t,0), \quad K_t(t,0)=U_t(t,0), \tag{21}$$

if pressure and K are regular functions, so that $H_t(t,0)=0=H_r(t,0)$, and therefore condition (20) is fulfilled either by taking the constant equal to zero or conveniently rescaling the angular coordinate.

The problem of obtaining solutions for H is solved by the Wainright–Ince–Marshman formalism.[14] Solutions to (19b) may be generated from solutions of the reduced wave equation on the plane with timelike gradient,

$$\sigma_{tt}-\sigma_{rr}-\frac{\sigma_r}{r}=0, \quad \sigma_t^2-\sigma_r^2>0, \tag{22}$$

by a quadrature identical to the one which defines $K-U$ in the vacuum case,

$$H_t=2r\sigma_t\sigma_r, \tag{23a}$$

$$H_r=r(\sigma_t^2+\sigma_r^2). \tag{23b}$$

The functions H generated by this mechanism have trivially a spacelike gradient and positive radial derivative. The fluid properties may be read directly from the generating function,

$$\tanh 2\xi=\frac{2\sigma_t\sigma_r}{\sigma_t^2+\sigma_r^2}, \quad p=e^{-2K}(\sigma_t^2-\sigma_r^2). \tag{24}$$

The function that generates the $\xi=0$ case is $\sigma=\sqrt{\gamma}t$.

Using this formalism, the remaining system of equations is formed by a quadrature and two reduced wave equations,

$$U_{tt}-U_{rr}-\frac{U_r}{r}=0, \tag{25a}$$

$$\sigma_{tt}-\sigma_{rr}-\frac{\sigma_r}{r}=0, \tag{25b}$$

$$K_t=U_t+2rU_tU_r+2r\sigma_t\sigma_r, \tag{25c}$$

$$K_r=U_r+r(U_t^2+U_r^2)+r(\sigma_t^2+\sigma_r^2). \tag{25d}$$

III. NONSINGULAR MODELS

We have obtained a fairly simple system of equations (25a)–(25c) that will be useful for analyzing the regularity of the solutions. Following Ref. 2 we take causal geodesic completeness as our definition for regularity.

Even if we have regular metric components, geodesic completeness of the space–time is not guaranteed and we have to check explicitly that every timelike and lightlike geodesic in the space–time can be extended to all values of the affine parameter, that is, in the parametrization for which the geodesic equations take the form

$$\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^k=0, \tag{26}$$

in terms of the Christoffel symbols.

Fortunately, results concerning causal geodesic completeness of diagonal Abelian orthogonally transitive space–times have already been obtained in Ref. 8. The conclusions of that paper may be condensed in two theorems. We follow the simplified version of Ref. 10.

Theorem 1: A diagonal Abelian orthogonally transitive space–time with spacelike orbits endowed with a metric in the form (7) with C^2 metric functions K,U,ρ, where ρ has a spacelike gradient, is future causally geodesically complete provided that along causal geodesics.

(1) For large values of t and increasing r,
  (a) $(K-U-\ln\rho)_r+(K-U-\ln\rho)_t\geqslant 0$, and either $(K-U-\ln\rho)_r\geqslant 0$ or $|(K-U-\ln\rho)_r|\lesssim(K-U-\ln\rho)_r+(K-U-\ln\rho)_t$,
  (b) $K_r+K_t\geqslant 0$, and either $K_r\geqslant 0$ or $|K_r|\lesssim K_r+K_t$,
  (c) $(K+U)_r+(K+U)_t\geqslant 0$, and either $(K+U)_r\geqslant 0$ or $|(K+U)_r|\lesssim(K+U)_r+(K+U)_t$.

(2) For large values of t, a constant b exists such that

$$\left.\begin{matrix} K(t,r)-U(t,r) \\ 2K(t,r) \\ K(t,r)+U(t,r)+\ln\rho(t,r)\end{matrix}\right\}\geqslant -\ln|t|+b.$$

Theorem 2: A diagonal Abelian orthogonally transitive space–time with spacelike orbits endowed with a metric in the form (7) with C^2 metric functions K,U,ρ, where ρ has a spacelike gradient, is past causally geodesically complete provided that along causal geodesics.

(1) For small values of t and increasing r,
  (a) $(K-U-\ln\rho)_r-(K-U-\ln\rho)_t\geqslant 0$, and either $(K-U-\ln\rho)_r\geqslant 0$ or $|(K-U-\ln\rho)_r|\lesssim(K-U-\ln\rho)_r-(K-U-\ln\rho)_t$.
  (b) $K_r-K_t\geqslant 0$, and either $K_r\geqslant 0$ or $|K_r|\lesssim K_r-K_t$.
  (c) $(K+U)_r-(K+U)_t\geqslant 0$, and either $(K+U)_r\geqslant 0$ or $|(K+U)_r|\lesssim(K+U)_r-(K+U)_t$.

(2) For small values of t, a constant b exists such that

$$\left.\begin{matrix} K(t,r)-U(t,r) \\ 2K(t,r) \\ K(t,r)+U(t,r)+\ln\rho(t,r)\end{matrix}\right\}\geqslant -\ln|t|+b.$$

Therefore now we just have to verify under which conditions these theorems can be applied to stiff fluid space–times. Since the theorems do not make use of Einstein equations, it is expected that when we take them into account the conditions will not be so restrictive as they seem.

We begin with future-pointing geodesics. The first part of the theorem is a set of conditions on the derivatives of the metric functions.

(1)
  (a) From (19c) and (19d) we obtain

$$(K-U-\ln\rho)_t+(K-U-\ln\rho)_r=r(U_t+U_r)^2+H_t+H_r-\frac{1}{r}.$$

  The sum of the derivatives of H is always positive, since $H_r>|H_t|$ in order to have positive pressure. In fact, this is the $r(\sigma_t+\sigma_r)^2$ term in the Wainright–Ince–Marshman formalism. This expression is positive if either $|U_t+U_r|$ or H_t+H_r ($|\sigma_t+\sigma_r|$ in the Wainright–Ince–Marshman formalism) does not decrease as $1/r$ or faster for large values of t and r. That is, we need either U or H to overcome the negative term. Under such conditions, the second part of the premise,

$$(K-U-\ln\rho)_r=r(U_t^2+U_r^2)+H_r-\frac{1}{r}\geqslant 0,$$

  is also satisfied.

(b) Once (1) (a) is fulfilled, this condition,

$$K_t+K_r=U_t+U_r+r(U_t+U_r)^2+H_t+H_r>0,$$

is trivial, since the only possible negative contribution would be that of U_t+U_r and this is counteracted by H_t+H_r if it decreases as $1/r$ or faster, or by $r(U_t+U_r)^2$ if it does not. Following a similar line of thought we also conclude that

$$K_r=U_r+r(U_t^2+U_r^2)+H_r$$

is positive for large values of t and r.

(c) The last set of conditions on the derivatives,

$$(K+U)_t+(K+U)_r=2(U_t+U_r)+r(U_t+U_r)^2+H_t+H_r\geqslant 0,$$
$$K_r+U_r=2U_r+rU_r^2+H_r\geqslant 0,$$

is also a consequence of (1) (a). Therefore the first part of the theorem is satisfied if

$$\left.\begin{array}{c} r^{1-\varepsilon}|U_r+U_t| \\ \text{or} \\ r^{1-\varepsilon}(H_r+H_t) \end{array}\right\} \not\to 0 \qquad (27)$$

for large values of t and r. The conclusion for past-pointing geodesics is quite similar. We just have to change the sign of the time derivatives,

$$\left.\begin{array}{c} r^{1-\varepsilon}|U_r-U_t| \\ \text{or} \\ r^{1-\varepsilon}(H_r-H_t) \end{array}\right\} \not\to 0 \qquad (28)$$

for large values of r and small values of t. For instance, these restrictions are trivial for the $\xi=0$ case, since $H=\gamma r^2/2$ does not decrease.

(2) The dependence on the matter content of the space–time may be removed from these conditions, since we may write

$$K(t,r)=U(t,r)+\int_0^r dr'(r'U_r^2(t,r')+r'U_t^2(t,r')+H_r(t,r'))=U(t,0)+\int_0^r dr'\,K_r(t,r'), \qquad (29)$$

and according to (19d) or (25d) K_r is a positive term if the first part of the theorem is satisfied.

(a) The first condition is tautological since

$$K(t,r)-U(t,r)=\int_0^r dr'\,(r'U_r^2(t,r')+r'U_t^2(t,r')+H_r(t,r'))>0.$$

(b) For geodesics along the axis, this condition requires for large values of the time coordinate that

$$K(t,0)=U(t,0)\geqslant -\tfrac{1}{2}\ln|t|+b, \qquad (30)$$

and for general geodesics the only difference is the positive term in (29). Therefore (30) is the only restriction for all geodesics.

(c) The same restriction is achieved likewise when applied to the expression $K+U+\ln\rho$.

Therefore we are left with just three regularity conditions on the metric of an Abelian diagonal orthogonally transitive space–time with spacelike orbits and with a stiff perfect fluid as matter content. We may summarize these results in two theorems:

Theorem 3: A cylindrical space–time with a stiff perfect fluid as matter content, endowed with a metric in the form (7) with C^2 metric functions K,U,ρ is future geodesically complete if the gradient of the surface element is spacelike and

(1) For large values of t, $U(t,0)\geqslant -\frac{1}{2}\ln|t|+b$.
(2) Either $r^{1-\varepsilon}|U_r+U_t|$ or $r^{1-\varepsilon}(H_r+H_t)$ does not tend to zero for large values of t and r.

Theorem 4: A cylindrical space–time with a stiff perfect fluid as matter content, endowed with a metric in the form (7) with C^2 metric functions K,U,ρ is past geodesically complete if the gradient of the surface element is spacelike and

(1) For small values of t, $U(t,0)\geqslant -\frac{1}{2}\ln|t|+b$.
(2) Either $r^{1-\varepsilon}|U_r-U_t|$ or $r^{1-\varepsilon}(H_r-H_t)$ does not tend to zero for small values of t and large values of r.

For vaccuum space–times both theorems hold just dropping the conditions on the derivatives of H.

The restrictions imposed by both theorems in order to have a nonsingular space–time are rather simple to implement, since U is just a solution of the wave equation and H is related to another one. We may state that regularity conditions are quite weak for stiff fluids, since it is very easy to provide solutions that fulfill such requirements. For instance,

Corollary: A metric with arbitrary H and a function U which grows for large $|t|$ and for large r makes the spacetime geodesically complete.

It is not difficult to derive such functions. The solutions to the reduced wave equation in the plane can be written as solutions of the initial value problem,

$$U_{tt}-U_{rr}-\frac{U_r}{r}=0,$$
$$U(0,r)=f(r), \quad U_t(0,r)=g(r). \tag{31}$$

The solution to this problem can be written in closed form,[15]

$$U(x,y,t)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^t dR\,R\,\frac{g(x+R\cos\phi,y+R\sin\phi)}{\sqrt{t^2-R^2}}$$
$$+\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi}d\phi\int_0^t dR\,R\,\frac{f(x+R\cos\phi,y+R\sin\phi)}{\sqrt{t^2-R^2}}, \tag{32}$$

for initial data $U(x,y,0)=f(x,y)$, $U_t(x,y,0)=g(x,y)$, taking into account that f and g are to have circular symmetry.

If we split U in U_f and U_g, the terms depending, respectively, on the initial data for U and its derivative, we notice that U_f is even in the time coordinate whereas U_g is odd. This implies that if U_f satisfies the first condition in theorem 3 for future-pointing geodesics, it will fulfill it for past-pointing geodesics too. On the contrary, if U_g satisfies it for future-pointing geodesics, it may not fulfill it for past-pointing ones, unless it behaves for large values of $|t|$ slower than a logarithm. Therefore one is to require either that U_f dominates over U_g for large $|t|$ or that both terms behave slower than a logarithm in order to have nonsingular behavior.

IV. DISCUSSION

In this paper we have derived sufficient conditions for an Abelian diagonal orthogonally transitive space–time with spacelike orbits and with a stiff perfect fluid as matter content to be geodesically complete. One of the metric functions appears to be determinant for the regularity of the space–time. These conditions are easy to check and do not mean much restriction on these space–times.

This means that nonsingular space–times are not as scarce as it was thought, considering the reduced list of geodesically complete perfect fluid cosmologies in the literature. Further work is needed with more generic symmetries and matter contents in order to clarify the issue, since stiff perfect fluids are rather peculiar. They may be interpreted as a massless scalar field and they are the limiting case for which a barotropic perfect fluid with linear equation of state satisfies every energy condition. These space–times also fulfill the generic condition and are causally stable.

The latter assert is true since they possess a cosmic time, which is the coordinate t. This coordinate has a timelike gradient everywhere. Therefore,[2] these space–times satisfy weaker causality conditions. For instance, the chronology condition is true for them and no closed causal curves are possible.

As it was stated in the introduction, the existence of these nonsingular space–times is possible because they do not possess causally trapped sets. They obviously do not contradict then the singularity theorems. They just fall out of their scope.

Another interesting point that is worthwhile mentioning is that the regularity theorems appear to encourage a growing K for large values of $|t|$. This seems to support a conjecture that states that the spatial average value of the pressure in nonsingular space–times is zero,[11] since p decreases with large K according to (24),

$$p=e^{-2K}(\sigma_t^2-\sigma_r^2). \tag{33}$$

In our regular space–times, constant t sheets are Cauchy hypersurfaces and we may write the whole system of equations as an initial value problem for U, K, and H for any constant t. Without breaking the generality of the result, we may focus on $t=0$. The initial value problem can be stated as

$$U_{tt}-U_{rr}-\frac{U_r}{r}=0, \tag{34a}$$

$$\sigma_{tt}-\sigma_{rr}-\frac{\sigma_r}{r}=0, \tag{34b}$$

$$K_t=U_t+2r(U_tU_r+\sigma_t\sigma_r), \tag{34c}$$

$$U(0,r)=f(r), \quad U_t(0,r)=g(r), \tag{34d}$$

$$\sigma(0,r)=f_\sigma(r), \quad \sigma_t(0,r)=g_\sigma(r), \tag{34e}$$

$$K(0,r)=h(r), \tag{34f}$$

and the remaining equation in the system,

$$K_r=U_r+r(U_t^2+U_r^2+\sigma_t^2+\sigma_r^2), \tag{35}$$

is used to complete the initial data,

$$h(r)=U(0,0)+\int_0^r dr'\,K_r(0,r')=f(r)+\int_0^r dr'\,r'\{g(r)^2+f'(r)^2+g_\sigma(r')^2+f'_\sigma(r')^2\}. \tag{36}$$

In order to know the pressure on the hypersurface $t=0$ we just have to prescribe the initial data,

$$p(0,r)=e^{-2h(r)}\{g_\sigma(r)^2-f'_\sigma(r)^2\}. \tag{37}$$

We show that it must necessarily vanish at infinity if the space–time is causally geodesically complete.

If the term $g_\sigma^2 - f_\sigma'^2$ in the pressure does not tend to zero at infinity, the σ terms would contribute to h as r^2 (if $g_\sigma^2 + f_\sigma'^2$ tends to a constant) or greater. Unless a negative f overcomes this quadratic term, we would have a pressure decreasing as a Gaussian exponential and the average on $t=0$ would be zero.

But f cannot beat a quadratic term, because the f' term in the integral would mean a positive r^4 contribution to h, and we would have again a negative exponential. That is, if $g_\sigma^2 - f_\sigma'^2$ does not vanish at infinity, it grows much slower than the exponential term decreases and the pressure tends to zero.

The only possibility we have left then is a positive exponential. This means a negative h. If we want f to overcome just the f' term in h, we require $|f(r)| \leqslant \ln r$ for large values of r, a very narrow strip.

But we also need to keep under control the σ terms in h. They remain bounded for large values of r if $r^2(g_\sigma^2 + f_\sigma'^2)$ tends to zero. This means that the σ term in the pressure decreases faster than r^{-2}. Admitting that $h(r)$ might behave as $-\ln r$ for large r, the exponential in the pressure would be a r^2 term, that cannot compensate the σ term.

Therefore, pressure tends to zero for large r on constant time hypersurfaces, thereby supporting Senovilla's conjecture in Ref. 11.

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