A note on series of positive numbers

Jorge Jiménez Urroz
Departamento de Matemática Aplicada IV
Universidad Politecnica de Catalunya
Barcelona, 08034, España
jjimenez@ma4.upc.edu

July 8, 2009

Abstract

We provide a new criterion for convergence of series of positive numbers.

1 Introduction

The study of series appear everywhere in Analysis. And the first issue is to know whether the series is convergent or not. Most of the times we need to appeal to absolute convergence and, in this way, we end up by trying to understand series of positive numbers. There are several criteria to decide if a series of positive numbers is convergent or not, however most of them seems to have two similar characteristics: first, they come, in one way or another, from the Comparison Principle. In certain sense, one could think that this fact is limiting our study of convergence of series of positive numbers. Second, none of them gives equivalent conditions. For example D'Alambert's, Cauchy's or Raabe's criteria fail when the corresponding limit is 1.

Here we present a new criterion which gives an equivalent condition for the convergence of a series of positive numbers, and also it does not comes from the Comparison Principle.

Theorem 1 *Let $a_k > 0$ for $k \geq 0$, and $S_N = \sum_{k \leq N} a_k$. Then,*

1. *$\sum a_k$ converges if and only if $\sum \frac{a_k}{S_k}$ does.*
2. *$\sum \frac{a_k}{S_k(\log(S_k+1))^2}$ is always convergent.*

Remark. The "difficult" implication of part (1) of the Theorem is already proved in a paper by Abel in 1828, [1]. Dini in 1867 in [3] improved his result and obtained the convergence of the series $\sum_{n \neq N} \frac{a_n}{S_n^\alpha}$ for $\alpha > 1$. However, their proofs, more involved than the showed here, lose enough so it is not achieved the second part of Theorem 1.

2 Proofs.

Before proving the theorem, we should add some remarks. First let us say that Theorem 1 is in fact a criterion for series of positive numbers. Indeed, otherwise it could happen that $\frac{a_k}{S(k)}$ is not well defined for infinitely many k. But even though this is not the case, one can not ensure the result. Let us for example consider $a_1 = 1$ and $a_k = 3(-1/2)^{k-1}$ for any $k \geq 2$. Then $\sum a_k$ is convergent by Leibnitz's criterion. However, for any $k \geq 2$, $S(k) = \sum_{j=1}^k a_j = \frac{1}{3}a_k$, and so $\sum \frac{a_k}{S(k)}$ is divergent. Also, we should note that the second part of the theorem would not remain true by removing 1 from the logarithm. To see this, consider $a_k = \frac{1}{k(k+1)}$. Then, $S_k = 1 - \frac{1}{k+1}$, $|\log S_k| < \frac{2}{k+1}$, and so $\sum \frac{a_k}{S_k(\log(S_k)^2} > \frac{1}{4}\sum \frac{1}{1-\frac{1}{k+1}}$, is a divergent series. Notice that in this case S_k is convergent. Clearly this is the only case in which adding 1 to the argument of the logarithm is an important matter.

2.1 Proof of the Theorem.

If $S = \sum a_k$ is convergent, both results in the theorem are trivial. Indeed, note that in this case $S_k > S - \varepsilon$ for any $\varepsilon > 0$ and k sufficiently large depending on ε. Then, we assume S is divergent.

• Part (*1*). We have to prove that $\sum \frac{a_k}{S_k}$ is divergent.

Let us start by observing that

$$\begin{aligned}
\log(S_K) &= \log\left(\prod_{k=1}^{K} \frac{S_k}{S_{k-1}}\right) = \sum_{k=1}^{K} \log\left(\frac{S_k}{S_{k-1}}\right) \\
&= -\sum_{k=1}^{K} \log\left(\frac{S_{k-1}}{S_k}\right) = -\sum_{k=1}^{K} \log\left(1 - \left(1 - \frac{S_{k-1}}{S_k}\right)\right) = -\sum_{k=1}^{K} \log\left(1 - \frac{a_k}{S_k}\right). \qquad (1)
\end{aligned}$$

If $a_k \neq o(S_k)$, the result is trivial, (observe that always $a_k < S_k$). Hence, we assume $\lim_{k\to\infty} \frac{a_k}{S_k} = 0$, and so, for $k > K_0$, $0 < \frac{a_k}{S_k} < \frac{1}{2}$. Then, the inequality

$$x < -\log(1-x) < 2x \qquad (2)$$

valid for any $0 < x < \frac{1}{2}$, gives us for any $K > K_0$ in (1),

$$\log(S_K) < -\sum_{k=1}^{K_0} \log\left(1 - \frac{a_k}{S_k}\right) + 2\sum_{j=K_0}^{K} \frac{a_k}{S_k},$$

and the result follows.

• Part (*2*). Since $\sum \frac{a_k}{S_k(\log(S_k+1))^2} < \sum \frac{a_k}{S_k(\log(S_k))^2}$, it is enough to prove convergence of the second series.

We now note, by the left part of (2), that for sufficiently large K_0 so that $S_k > 1$ for any $k \geq K_0$,

$$\begin{aligned}
\sum_{k\leq K} \frac{a_k}{S_k(\log(S_k))^2} &< \sum_{k\leq K_0} \frac{a_k}{S_k(\log(S_k))^2} - \sum_{K_0<k\leq K} \frac{1}{(\log(S_k))^2} \log\left(1 - \frac{a_k}{S_k}\right) \\
&= C_{K_0} + \sum_{K_0<k\leq K} \frac{1}{(\log(S_k))^2} \log\left(\frac{S_k}{S_{k-1}}\right) \\
&= C_{K_0} + \sum_{K_0<k\leq K} \frac{1}{(\log(S_k))^2} \int_{S_{k-1}}^{S_k} \frac{1}{t} dt \\
&< C_{K_0} + \sum_{K_0<k\leq K} \int_{S_{k-1}}^{S_k} \frac{1}{t\log(t)^2} dt = C_{K_0} + \int_{S_{K_0}}^{S_K} \frac{1}{t(\log(t))^2} dt \\
&< C_{K_0} + \frac{1}{\log(S_{K_0})}
\end{aligned}$$

and the result follows.

3 Examples

Corollary 2 $\sum_{k\leq K} \frac{1}{k}$ *diverges*

Proof: Trivial from Theorem 1 and the divergence of $\sum_{k\leq K} 1$.

Corollary 3 *The series* $\sum_n \frac{n^n}{n!e^n}$ *is divergent.*

Proof: For any $n \geq 1$, the inequality

$$e < (1 + \frac{1}{n})^{n+1},$$

follows from (2) in $x = \frac{1}{n+1}$. Hence $\frac{(n+1)^{n+2}}{(n+1)!e^{n+1}} > \frac{n^{n+1}}{n!e^n} > \cdots \geq \frac{1}{e}$, and so the series $\sum_n \frac{n^{n+1}}{n!e^n}$ is divergent. Moreover, $S(n) = \sum_{j=1}^n \frac{j^{j+1}}{j!e^j} > \frac{n}{e}$, and so

$$\sum_n \frac{n^n}{n!e^n} > \frac{1}{e} \sum_n \frac{n^{n+1}}{n!e^n S(n)}.$$

The result now follows by Theorem 1.

Corollary 4 *Let $f'(t) \geq 0$ a decreasing function, and $f(0) > 0$. Then, $\sum f'(n)$ diverges if and only if $\sum \frac{f'(n)}{f(n)}$ diverges. Moreover, $\sum \frac{f'(n)}{f(n)(\log f(n)+1)^2}$ always converges.*

Proof: Again, in the case when $\sum f'(n)$ is convergent, both results are trivial by noting that $f(n) \geq f(0)$, so we will assume $\sum_{n \leq N} f'(n) \to \infty$ with N. Let us prove the first part of Corollary 4. Now, since

$$S(n) = \sum_{1 \leq j \leq n} f'(j) < \int_0^n f'(t)dt = f(n) - f(0) < f(n), \tag{3}$$

we deduce that $f(n) \to \infty$ with n. Moreover,

$$S(n) > \int_1^n f'(t)dt = f(n) - f(1) > \frac{1}{2} f(n),$$

for n sufficiently large. Hence,

$$\sum_{n \leq N} \frac{f'(n)}{S(n)} < 2 \sum_{n \leq N} \frac{f'(n)}{f(n)},$$

and the result follows from Theorem 1.

The second part of Corollary 4 follows from the second part of Theorem 1 and (3).

Let $\log_1(x) = \log x$, and for any integer j, $\log_{j+1}(x) = \log(\log_j(x))$.

Corollary 5 *For any integer J, $\sum_k \frac{1}{k \prod_{j \leq J} \log_j k}$ is divergent. On the other hand $\sum_k \frac{1}{k \prod_{j \leq J} \log_j k (\log_{J+1} k)^2}$ is convergent.*

Proof: In Corollary 4, take $f_J(t) = \log_J(t)$. Since $f_J'(t) = \frac{1}{t \prod_{j \leq J-1} \log_j t} = \frac{f_{J-1}'(t)}{f_{J-1}(t)}$, we just have to use Corollary 2, Corollary 4, and apply induction. For the second part, use the second part of Corollary 4, (note that for any J and t sufficiently large depending on J, $\log_J t > \frac{1}{2} \log_J(t+1)$).in the Remarks

We include one final example just to show the wide range of applications of this criterion. We will use it to give a new proof of a well known fact in Analysis, consequence of $(L^2)^* = L^2$.

Corollary 6 *Let (X, μ) an space of measure with $\mu(X) < \infty$. Suppose $f : X \to \mathbb{R}$ is a measurable function such that*

$$\int_X |fg| d\mu < \infty,$$

for any $g \in L^2(X)$. Then, $f \in L^2(X)$.

Proof: Without lost of generality we can assume $f \geq 0$. By taking $g = 1$ we see that $f \in L^1(X)$. Let us call $A_k = \{x \in X \,:\, k \leq f(x) < k+1\}$. Then

$$\sum_{k \geq 0} k\mu(A_k) \leq \int_X f d\mu < \infty. \tag{4}$$

Now suppose $f \notin L^2(X)$. Then

$$\sum_{k\geq 0}(k+1)^2\mu(A_k) > \int_X f^2 d\mu = \infty,$$

and so, by (4)

$$\sum_{k\geq 0} k^2\mu(A_k) = \infty.$$

Now, let us call $S(k) = \sum_{j=1}^{k} j^2\mu(A_j)$, and consider $g(x) = \frac{k}{S(k)}$ for any $x \in A_k$. Then $g \in L^2(X)$ since

$$\int_X g^2 d\mu = \sum_{k\geq 0} \frac{k^2}{S^2(k)}\mu(A_k) < \infty$$

by the second part of Theorem 1, (note that $S_k > (\log(S_k+1))^2$ for k sufficiently large), meanwhile

$$\int_X fg d\mu > \sum_{k\geq 0} \frac{k^2}{S(k)}\mu(A_k) = \infty,$$

by the first part of Theorem 1. Hence, we get a contradiction and the result follows.

Clearly both, Theorem 1 and Corollary 4, seem to have a wide variety of applications, and we leave to the interested reader to find new ones.

Acknowledgment: This note started when I was trying to prove, in an elementary way, Corollary 6. I would like to thank J. L. Varona for pointing out the reference [4] in which [1], and [3] are mentioned. I also want to thank Santiago Egido for sharing with me the first example described at the beginning of Section 2.

References

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