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ON "GOOD" HALF-INTEGRAL WEIGHT MODULAR FORMS

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1. Introduction and Statement of Results

If k is a positive integer, let $S_k(N)$ denote the space of cusp forms of weight k on $\Gamma_1(N)$, and let $S_k^{cm}(N)$ denote the subspace of $S_k(N)$ spanned by those forms having complex multiplication (see [Ri]). For a non-negative integer k and any positive integer $N \equiv 0 \pmod 4$, let $M_{k+\frac{1}{2}}(N)$ (resp. $S_{k+\frac{1}{2}}(N)$) denote the space of modular forms (resp. cusp forms) of half-integral weight $k+\frac{1}{2}$ on $\Gamma_1(N)$. Similarly, if $k \in \frac{1}{2}\mathbb{N}$, then let $M_k(N,\chi)$ (resp. $S_k(N,\chi)$) denote the space of modular (resp. cusp) forms with respect to $\Gamma_0(N)$ and Nebentypus character χ. Throughout this note we shall refer to classical facts which may be found in [Ko, Mi, S-S, Sh].

If $i = 0$ or 1, $0 \leq r < t$, and $a \geq 1$, then let $\theta_{a,i,r,t}(z)$ denote the Shimura theta function

$$(1) \qquad \theta_{a,i,r,t}(z) := \sum_{n \equiv r \pmod t} n^i q^{an^2}$$

(Note: $q := e^{2\pi i z}$ throughout). Each $\theta_{a,i,r,t}(z)$ is a holomorphic modular form of weight $i + \frac{1}{2}$. If $\Theta(N)$ is the set of modular forms generated by such functions of level dividing N, then the Serre-Stark Theorem [S-S] implies

$$(2) \quad \Theta(N) = M_{\frac{1}{2}}(N) \cup \left\{ \text{subspace of } M_{\frac{3}{2}}(N) \text{ spanned by those } \theta_{a,1,r,t}(z) \text{ on } \Gamma_1(N) \right\}.$$

If $g(z) \in M_{k+\frac{1}{2}}(N_1)$ and $h(z) \in \Theta(N_2)$, then let $g_h(n)$ denote the Fourier coefficient of q^n of the modular form

$$g(z) \cdot h(z) = \sum_{n=0}^{\infty} g_h(n) q^n.$$

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Moreover, let $G_h(z)$ denote the modular form

$$G_h(z) := \sum_{\gcd(n, N_1N_2)=1} g_h(n)q^n. \tag{3}$$

It follows from [Lemma 4, S-S] that $G_h(z)$ is a modular form on $\Gamma_1(N_1^2N_2^2)$ of integral weight $k+1$ or $k+2$.

Definition. *A modular form $g(z) \in M_{k+\frac{1}{2}}(N_1)$ is* **good** *if there is an integer N_2 and a function $h(z) \in \Theta(N_2)$ for which*

(i) *$G_h(z)$ is a nonzero cusp form.*
(ii) *$G_h(z) \notin S_{k+1}^{cm}(N_1^2N_2^2) \cup S_{k+2}^{cm}(N_1^2N_2^2)$.*

There have been a number of recent papers on the non-vanishing of Fourier coefficients of half-integral weight modular forms modulo primes ℓ (see [B2, J, O-S1]), and in this direction the first author and C. Skinner were able to prove the following theorem for "good" forms.

Theorem. [p. 454, O-S1] *Let $g(z) = \sum_{n=0}^{\infty} c(n)q^n \in M_{k+\frac{1}{2}}(N)$ be an eigenform whose coefficients are algebraic integers. If $g(z)$ is good, then for all but finitely many primes ℓ there are infinitely many square-free integers m for which $|c(m)|_\ell = 1$.*

Here $| \bullet |_\ell$ denotes an extension of the usual ℓ-adic valuation to an algebraic closure of $\mathbb{Q}$.

In [O-S1], the first author and Skinner made the following natural conjecture:

The "Good" Conjecture. [p. 468, O-S1] *Every form in $M_{k+\frac{1}{2}}(N)\setminus \Theta(N)$ is good.*

In this note we prove:

Theorem 1. *The "Good" Conjecture is true.*

In a recent preprint, W. McGraw [M] obtains another proof of Theorem 1.

To prove the conjecture, we employ a well known result of M.-F. Vignéras, the Fundamental Lemma from [pp. 653–654, O-S2], and Brun's sieve.

2. Proof of Theorem 1

Here we begin by recalling a well-known result due to M.-F. Vignéras [V] (see [B1] for a new elementary proof).

Theorem 2. [Th. 3, V] *Suppose that $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is in $M_{k+\frac{1}{2}}(N)$. If there are finitely many square-free integers $d_1, d_2, \ldots, d_j$ such that $a(n) = 0$ for every n not of the form $d_i m^2$ with $1 \leq i \leq j$ and $m \in \mathbb{Z}^+$, then $f(z) \in \Theta(N)$.*

We begin by combining Theorem 2 and [Fund. Lemma, pp. 653–654, O-S2] to obtain a lower bound for the number of non-zero coefficients of any modular form $f(z) \in M_{k+\frac{1}{2}}(N, \chi)\setminus\Theta(N)$.

Theorem 3. *Suppose that* $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ *is a modular form in* $M_{k+\frac{1}{2}}(N,\chi)\backslash\Theta(N)$. *If* $f(z)$ *is an eigenform of the Hecke operators* $T(p^2)$ *for every prime* $p \nmid N$, *then*

$$\#\{n \leq X \ : \ a(n) \neq 0\} \gg_f \frac{X}{\log X}.$$

Proof. By [Lemma 8, S-S], we may assume that all of the Fourier coefficients $a(n)$ and the eigenvalues of the Hecke operators $T(p^2)$, for primes $p \nmid N$, are algebraic integers in a fixed number field K. Let v be a place in K over 2.

By Theorem 2 there are infinitely many square-free positive integers $d_1 < d_2 < \dots$ for which there are positive integers n with $a(d_in^2) \neq 0$. Let s_0 be the smallest integer for which there is a square-free integer $d > 1$, with $d \nmid N$, and a positive integer n for which $\mathrm{ord}_v(a(dn^2)) = s_0$. Moreover, let d_0 be such a d and let n_0 be a positive integer for which $\mathrm{ord}_v(a(d_0n_0^2)) = s_0$. Since $d_0 \nmid N$, there are square-free integers $D_0 > 1$ and D_1 for which $d_0 = D_0D_1$ and $D_1 \mid N$ and $\gcd(D_0, N) = 1$. Similarly, let m_0 and m_1 denote the unique positive integers for which $n_0 = m_0m_1$, $\gcd(m_0, N) = 1$, and every prime $p \mid m_1$ also divides N.

Now recall the action of the Hecke operators. If p is prime, then

$$f(z) \mid T(p^2) := \sum_{n=0}^{\infty} \left(a(p^2n) + \chi(p)\left(\frac{(-1)^k n}{p}\right)p^{k-1}a(n) + \chi(p^2)p^{2k-1}a(n/p^2) \right) q^n. \tag{4}$$

Suppose that d is a positive integer and $p \nmid N$ is a prime for which $p^2 \nmid d$. Since $f(z)$ is an eigenform, it is easy to see that $a(d) \mid a(dp^{2i})$. As a consequence, it turns out that $a(D_0D_1m_1^2) \neq 0$ and $\mathrm{ord}_v(a(D_0D_1m_1^2)) = s_0$.

If $p \mid N$ is prime, then by [Lemma 1, S-S] it is known that

$$f(z) \mid U(p) = \sum_{n=0}^{\infty} a(pn)q^n, \tag{5}$$

is a cusp form in $M_{k+\frac{1}{2}}(N, \chi \cdot (\frac{4p}{\bullet}))$. Therefore, if j is any positive integer for which every prime $p \mid j$ also divides N, then

$$f(z) \mid U(j) = \sum_{n=0}^{\infty} a(jn)q^n \in M_{k+\frac{1}{2}}(N, \chi \cdot \left(\frac{4j}{\bullet}\right)).$$

Now define $f_0(z) \in M_{k+\frac{1}{2}}(N, \chi \cdot (\frac{4D_1}{\bullet}))$ by

$$f_0(z) = \sum_{n=0}^{\infty} b(n)q^n := f(z) \mid U(D_1m_1^2) = \sum_{n=0}^{\infty} a(D_1m_1^2n)q^n.$$

By construction, we have that $b(D_0) = a(D_0 D_1 m_1^2) \neq 0$ and $\mathrm{ord}_v(b(D_0)) = s_0$.

Also by construction, if there is an integer $s < s_0$ and an integer n for which $\mathrm{ord}_v(b(n)) = s$, then $\gcd(n, N) \neq 1$. This follows from the minimality of s_0. If this is the case, then define $f_1(z) \in M_{k+\frac{1}{2}}(N^2, \chi \cdot \left(\frac{4D_1}{\bullet}\right))$ (see [Lemma 4, S-S]) by

$$f_1(z) = \sum_{n=1}^{\infty} c(n) q^n := \sum_{\gcd(n,N)=1} b(n) q^n. \tag{6}$$

If there is no such s, then let $f_1(z) = \sum_{n=0}^{\infty} c(n) q^n := f_0(z)$.

In either case, $f_1(z) = \sum_{n=0}^{\infty} c(n) q^n$ is in $M_{k+\frac{1}{2}}(N^2, \chi \cdot \left(\frac{4D_1}{\bullet}\right))$ and has the property that s_0 is indeed the smallest integer for which there is an n with $\mathrm{ord}_v(c(n)) = s_0$. Moreover, the square-free integer D_0 which is coprime to N^2 is such an n. By the Fundamental Lemma [pp. 653–654, O-S2], if $f_1(z)$ is a cusp form, then

$$\#\{n \leq X \ : \ \gcd(n, N^2) = 1 \text{ and } a(D_1 m_1^2 n) = c(n) \neq 0\} \gg_{f_1} \frac{X}{\log X}.$$

Although the Fundamental Lemma is stated for eigenforms which are cusp forms, it is easy to modify the argument to apply to forms $f_1(z)$ which are not cuspidal. Following the proof of the Fundamental Lemma, consider the integer weight form

$$F(z) := f_1(z) \cdot \left(1 + 2\sum_{n=1}^{\infty} q^{n^2}\right),$$

and decompose it into a cusp form $C(z)$ and a linear combination of Eisenstein series $E(z)$. By construction, the coefficient of q^{D_0} in $F(z)$ has minimal 2-adic valuation s_0, and is determined by a linear combination of generalized divisor functions related to the Eisenstein series in $E(z)$ (see [Mi]) and the collection of 2-adic Galois representations associated to the newforms constituting $C(z)$. By Dirichlet's Theorem on primes in arithmetic progressions, the Chebotarev Density theorem, and the multiplicativity of the coefficients of newforms, it follows that a 'positive proportion' of the square-free integers D with the same number of prime factors as D_0 have the property that the coefficient of q^D in $F(z)$ have minimal 2-adic valuation s_0. As in the proof of the Fundamental Lemma, this implies that

$$\#\{1 \leq n \leq X \ : \ c(n) \neq 0\} \gg \frac{X}{\log X} (\log\log X)^{r-1}$$

where D_0 has exactly r prime factors. □

As a corollary, we obtain the following result (see [O] for a similar result).

Corollary 4. *If $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is a modular form in $M_{k+\frac{1}{2}}(N,\chi)\backslash\Theta(N)$, then*

$$\#\{n \leq X \ : \ a(n) \neq 0\} \gg_f \frac{X}{\log X}.$$

Proof. If $w = \sum_{n=0}^{\infty} a_w(n)q^n$ is a formal power series in q, then define

$$M_w(X) := \#\{0 \leq n \leq X \ : \ a_w(n) \neq 0\}.$$

Now suppose that $M_f(X) = o(X/\log X)$. In view of (4), it is easy to see that if $p \nmid N$ is prime, then

$$M_{f|T(p^2)}(X) \leq M_f(p^2 X) + 2M_f(X). \tag{7}$$

By (7), if $p \nmid N$ is prime, then $M_{f|T(p^2)}(X) = o(X/\log X)$.

If w_1 and w_2 are formal power series, then it is obvious that

$$M_{w_1+w_2}(X) \leq M_{w_1}(X) + M_{w_2}(X).$$

Therefore, if $\mathbb{T}$ is the Hecke algebra generated by the Hecke operators $T(p^2)$ and $\mathbb{X} = \mathbb{T}f$, then for every $u(z) \in \mathbb{X}$ we have that $M_u(X) = o(X/\log X)$.

Since $\mathbb{T}$ is commutative, every simple submodule of $\mathbb{X}$ is generated by an eigenform. If $u(z)$ is such an eigenform, then Theorem 3 contradicts the conclusion that $M_u(X) = o(X/\log X)$. Therefore, it must be that $M_f(X) \gg_f X/\log X$. □

Now we employ Brun's sieve to obtain an important technical result regarding the prime divisors of a shifted set of integers. As usual, $p^a || n$ means that a is the exact power of p dividing n.

Lemma 5. *Let ℓ be a fixed prime, and let $1 \leq r < t$ be integers for which* $\gcd(r,t) = 1$. *If A is a set of non-negative integers for which*

$$\#\{n \leq X \ : \ n \in A\} \gg \frac{X}{\log X},$$

then there is a positive integer E and at least one integer $n \in A$ with $n < \ell^E$ such that $p||(n + \ell^E)$ for some prime $p \equiv r \pmod{t}$.

Proof. If $\phi(\bullet)$ denotes the usual Euler phi-function, then define the polynomial $F(n)$ by

$$F(n) = (n+\ell)(n+\ell^2)\cdots(n+\ell^{\phi(t)+1}). \tag{8}$$

Let $\mathcal{A}_X$ denote the set of integers

$$\mathcal{A}_X := \{F(n) \ : \ n \leq X\}, \tag{9}$$

and let P_X denote the set

$$P_X := \{p \equiv r \pmod{t} \text{ prime} \ : \ \log^2 X < p < X\}. \tag{10}$$

It is easy to see that if X is sufficiently large, then every prime $p \in P_X$ has the property that the multiplicative order of ℓ in $(\mathbb{Z}/p\mathbb{Z})^\times$ is larger than $\phi(t)+1$. Therefore, if n is an integer and $p \in P_X$ is any prime for which $F(n) \equiv 0 \pmod{p}$, then there is exactly one integer $1 \leq i \leq \phi(t)+1$ for which

$$n + \ell^i \equiv 0 \pmod{p}. \tag{11}$$

Moreover, it is obvious that if $p \in P_X$, then there are $\phi(t)+1$ distinct residue classes $n \pmod{p}$ for which $F(n) \equiv 0 \pmod{p}$.

Now we consider the function $\mathcal{S}(\mathcal{A}_X, P_X, X)$ which is defined by

$$\mathcal{S}(\mathcal{A}_X, P_X, X) := \#\{1 \leq n \leq X \ : \ \gcd(F(n), p) = 1 \text{ for every } p \in P_X\}. \tag{12}$$

By a straightforward application of Brun's sieve method [Theorem 2.2, H-R] we find that

$$\mathcal{S}(\mathcal{A}_X, P_X; X) \ll X \prod_{p \in P_X} \left(1 - \frac{\phi(t)+1}{p}\right). \tag{13}$$

Using the well known fact [p. 605, R] that

$$\prod_{\substack{p \leq X \\ p \equiv r \pmod{t}}} \left(1 - \frac{1}{p}\right) \ll \frac{1}{(\log X)^{1/\phi(t)}},$$

it is easy to deduce

$$\mathcal{S}(\mathcal{A}_X, P_X; X) \ll \frac{X}{(\log X)^{1+1/2\phi(t)}}. \tag{14}$$

Therefore, if X is sufficiently large, then there are integers $n \in A$ with $n \leq X$ for which there is at least one prime $p \in P_X$ with $F(n) \equiv 0 \pmod{p}$. In particular, in view of (14) we find that

$$\#\{n \leq X \ : \ n \in A \text{ and } F(n) \equiv 0 \pmod{p} \text{ for some prime } p \in P_X\} \gg \frac{X}{\log X}. \tag{15}$$

However, the number of positive integers $n \leq X$ which are divisible by p^2 for some prime $p \in P_X$ is

$$\ll X \sum_{\log^2 X < p < X} \frac{1}{p^2} < \frac{X}{\log^2 X} \sum_{p < X} \frac{1}{p} \ll \frac{X}{(\log X)^{1+1/2}},$$

since $\sum_{p\leq X} 1/p \ll \log\log X$. Therefore, by (11) and (15) we find that the number of integers $n \leq X$ and $n \in A$ for which there is at least one prime $p \in P_X$ and an integer $1 \leq e \leq \phi(t)+1$ such that $p||n+\ell^e$ is $\gg X/\log X$.

To conclude the proof, we note that if $p||(n+\ell^e)$, then $p||(n+\ell^{E(j)})$ where $E(j) := e + p(p-1)(p(p-1)+1)^j$ and $j \geq 0$. To see this, note that $n+\ell^{E(j)} = n+\ell^e+(\ell^{E(j)}-\ell^e)$, $\ell^{p-1} \equiv 1 \pmod{p}$ and $\ell^{p(p-1)} \equiv 1 \pmod{p^2}$. Therefore if j is sufficiently large, then $n < \ell^E$. □

Proof of Theorem 1. Here we recall the essential facts regarding modular forms with complex multiplication (see [Ri]). If $\phi(z) = \sum_{n=1}^{\infty} a_\phi(n)q^n \in S_k(N,\chi)$ is a newform with complex multiplication by the imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, where d is the discriminant of K, then $d \mid N$, and if p is a prime for which $\left(\frac{d}{p}\right) = -1$, then $a_\phi(p) = 0$.

Now suppose that $F(z) = \sum_{n=1}^{\infty} a_F(n)q^n$ is an integer weight cusp form in $S_w(N,\psi)$. There are finitely many fundamental discriminants of imaginary quadratic fields, say $d_1, d_2, \ldots, d_j$ for which $d_i \mid N$. Therefore, it is easy to construct an arithmetic progression $r \pmod{t}$ with $\gcd(r,t)=1$ such that every prime $p \equiv r \pmod{t}$ has the property that $\left(\frac{d_i}{p}\right) = -1$ for each $1 \leq i \leq j$. Therefore, by the multiplicativity of the Fourier coefficients of newforms, $F(z)$ cannot be a linear combination of forms with complex multiplication if there is a positive integer n and a prime $p \equiv r \pmod{t}$ for which $p||n$ and $a_F(n) \neq 0$.

Now we prove Theorem 1 by considering two different cases.

Case I. Suppose that $g(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_{k+\frac{1}{2}}(N,\chi)\backslash\Theta(N)$. By Corollary 4, we know that

$$\#\{n \leq X \ : \ a(n) \neq 0\} \gg_g \frac{X}{\log X}.$$

Now let $\ell \mid 576N$ be prime, and let $r \mod t$ with $\gcd(r,t)=1$ be an arithmetic progression such that $\left(\frac{d_i}{p}\right) = -1$ for every prime $p \equiv r \pmod{t}$ and every fundamental discriminant of an imaginary quadratic field $d_i \mid 576N$. By Lemma 5, there exists an integer $n < \ell^E$ for which $a(n) \neq 0$, a prime $p \equiv r \pmod{t}$, and a positive integer E such that $p||n+\ell^E$.

Now consider the cusp form $g(z)\cdot\eta(24\ell^E z)$, where $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ denotes Dedekind's eta-function. It is well known that

$$\eta(24z) = q + \cdots \in S_{1/2}(576, \chi_{12}),$$

where χ_{12} is the non-trivial quadratic character with conductor 12. Obviously, $\eta(24\ell^E z) \in \Theta(576\ell^E)$, and so $g(z)\eta(24\ell^E z) \in S_{k+1}(576N\ell^E)$. The coefficient of $q^{n+\ell^E}$ of this form is $a(n) \neq 0$. Since every fundamental discriminant of an imaginary quadratic field $d \mid 576N\ell^E$ already divides $576N$, we find that $g(z)\eta(24\ell^E z)$ cannot be a linear combination of forms with complex multiplication (i.e., $g(z)$ is good).

Case II. Suppose that $g(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_{k+\frac{1}{2}}(N)\backslash\Theta(N)$. It is well known that if $w \in \frac{1}{2}\mathbb{Z}$, then

$$M_w(N) = \oplus_\chi M_w(N, \chi), \tag{16}$$

where the direct sum is over Dirichlet characters $\chi \mod N$. Therefore, we may decompose $g(z)$ as

$$g(z) = \sum_\chi \alpha_\chi g_\chi(z).$$

If χ is a character for which $\alpha_\chi g_\chi(z) \neq 0$, then by Case I there is a weight $1/2$ cusp form $\theta(z) \in S_{1/2}(N_2, \Psi)$ for which $g_\chi(z)\theta(z)$ is a weight $k+1$ cusp form which is not a linear combination of forms with complex multplication.

If χ_1 and χ_2 are distinct characters mod N, then $g_{\chi_1}(z)\theta(z)$ and $g_{\chi_2}(z)\theta(z)$ will lie in different spaces of weight $k+1$ cusp forms with Nebentypus. Therefore, it follows immediately that $g(z)\theta(z)$ is good. □

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