# Introduction to Cryptography

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An algorithm in $n$-variables of $k_i$ bits each is called a polynomial time algorithm if the number of bit operations is $O(\prod_{i=1..n} k_i^{d_i})$

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- Let $m = \prod p_i^{\alpha_i}$ If $(a, m) = 1$, $a^{L(m)} \equiv 1 \pmod{m}$, where $L(m) = \mathrm{lcm}\{\varphi(p_i^{\alpha_i})\}$

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• Multiplying two elements of $\mathbb{F}_q$ needs $O((\log q)^3)$ operations while $a^k$, for $a \in \mathbb{F}_q$ and $k \in \mathbb{Z}$ needs $O((\log q)^3 \log k^3)$

**Theorem (Quadratic Reciprocity law)**

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### Theorem (Quadratic Reciprocity law)

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Proof. Consider $p, q$ primes, and $G = \sum_{i=0}^{q-1}\left(\frac{i}{q}\right)\xi^i$, where $\xi \in \mathbb{F}_{p^k}$ is a $q$-th root of unity. Then,

$$G^p = \sum_{i=0}^{q-1}\left(\frac{i}{q}\right)\xi^{ip} = \left(\frac{p}{q}\right)\sum_{i=0}^{q-1}\left(\frac{ip}{q}\right)\xi^{ip} = \left(\frac{p}{q}\right)G$$

But also

$$G^p = (G^2)^{(p-1)/2}G = ((-1)^{(q-1)/2}q)^{(p-1)/2}G$$

which finish the result.

$P$ is the set of plaintext messages, $C$ is the set of ciphertext message. A cryptosystem is a (biyective) function $f : P \to C$ such that given $m \in P$, $c = f(m)$ is easy to compute, but $m = f^{-1}(c)$ is very hard, unless an extra information is provided, which is called the key.

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It is easy to generate hash values from input data and easy to verify that the data matches the hash, but hard to 'fake' a hash value to hide malicious data.

Good for ensuring data integrity. Any change made to the contents of a message will result in a different hash.

The same key is used to encrypt and decrypt the messages. It is also called symmetric encryption.

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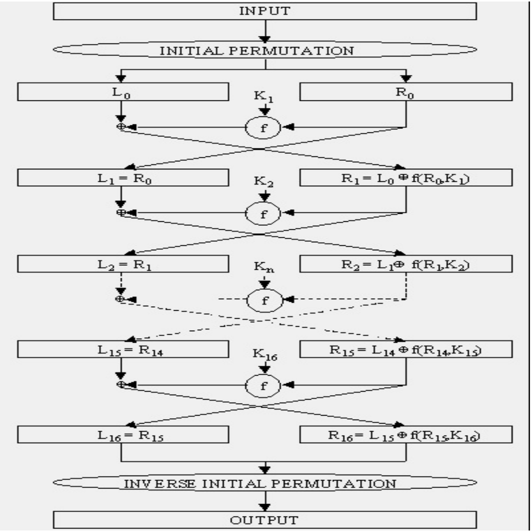

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- Advantages:
- – Encryption is fast and simple.
- – Less computer resources.
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- Disadvantages:

–Secure channel for secret key exchange.

–Ensuring privacy of keys is difficult.

–Origin and authenticity of message cannot be guaranteed.

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It is based on the use of a trapdoor function. A biyective function $f : P \to P$ easy to compute, but very hard to find $f^{-1}$ in any single value, unless an additional information is provided, the deciphering key $K_d$, which is kept secret.

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Public key cryptosystems are often used to send the keys of a symmetric scheme. This is called key exchange. In order to ensure security, probabilistic cryptosystems are used: the same plaintext has many different cipher text, depending on a random parameter.

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Each user $A$ selects two huge primes, $p_A$ and $q_A$ and computes $n_A = p_A q_A$. Then the user selects a random $1 < e_A < \varphi(n_A)$ coprime to $\varphi(n_A)$to be the public key and computes the inverse $e_A^{-1} = d_A \pmod{\varphi(n_A)}$, which will be the private key. $c = m^{e_A}$.
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**Exercise:** How to make the digital signature $f_A^{-1} f_B$ when $n_A$ and $n_B$ are different?