The effect of Group Velocity in the numerical analysis of control problems for the wave equation*

Fabricio Macià

École Normale Supérieure, D.M.A., 45 rue d'Ulm, 75230 Paris cedex 05, France.

Abstract. In this note we show how the convergence analysis of numerical algorithms for the computation of internal controls for the wave equation depends on the Group Velocity properties of the numerical scheme used to discretize the wave equation. This is done by means of theory of Wigner measures associated to discrete functions developed in [7]. Some results on the convergence of partial controls are also given.

1 An algorithm for the numerical computation of interior controls for the wave equation

The interior control problem for the wave equation with periodic B.C. consists in the following: given an open set $\omega \subset Q := (0,1)^d$ and a time $T > 0$ we would like to know wether or not it is possible, for *any* pair of initial data (u_0, u_1) in the energy space[1] $H^1_{\text{per}}(Q) \times L^2_{\text{per}}(Q)$, to find a **control** $F \in L^2((0,T) \times Q)$, *supported in* $(0,T) \times \omega$, such that the solution of the initial value problem

$$\begin{cases} \rho(x)\,\partial_t^2 u(t,x) - \Delta_x u(t,x) = F(t,x) & \text{in } \mathbf{R}_t \times Q, \\ u|_{t=0} = u_0, \qquad \partial_t u|_{t=0} = u_1, \qquad u(t,\cdot) \text{ is } Q\text{-periodic}, \end{cases} \tag{1}$$

is brought to rest in time $t = T$, i.e. $u|_{t=T} = \partial_t u|_{t=T} = 0$. If this is the case, we say that the system (1) is **exactly controllable** in time T with controls supported in ω. In all the results presented here, the density $\rho(x)$ will be assumed to be a strictly positive, Q-periodic, bounded, twice-differentiable function.

By means of J.-L. Lions' **Hilbert Uniqueness Method, H.U.M.** (see e.g. [6]), exact controllability (for T and ω) is shown to be equivalent to establishing the following **observability inequality** for the homogeneous adjoint problem: there exists $C_{T,\omega} > 0$ such that

$$\int_0^T \int_Q \rho(x)\,|\partial_t \varphi(t,x)|^2\,dxdt \le C_{T,\omega} \int_0^T \int_\omega \rho(x)\,|\partial_t \varphi(t,x)|^2\,dxdt \tag{2}$$

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[1] The subscript *per* accompanying a function space means we are dealing with its variant consisting of periodic functions.

holds for every solution of

$$\begin{cases} \rho(x)\,\partial_t^2\varphi(t,x) - \Delta_x\varphi(t,x) = 0 & \text{in } \mathbf{R}_t \times Q, \\ \varphi|_{t=0} = \varphi_0, \qquad \partial_t\varphi|_{t=0} = \varphi_1, & \varphi(t,\cdot) \text{ is } Q\text{-periodic,} \end{cases} \tag{3}$$

with initial data in $H^1_{\text{per}}(Q) \times L^2_{\text{per}}(Q)$.

A necessary and sufficient condition on (T,ω) for (2) to hold (and hence, for (1) to be exactly controllable) is known (see [12] and [1]):

Geometric Control Condition (G.C.C.): *every* **ray** $x(t)$ *must intersect the control region* ω *in time less than* T.

By a **ray** we intend the x-component of a solution of the **ray equations**:

$$\begin{cases} \dot{x}(t) = \rho(x(t))^{-1}\,\xi(t)/|\xi(t)|, \\ \dot{\xi}(t) = 1/2\rho(x(t))^{-2}\,|\xi(t)|^2\,\nabla_x\rho(x(t)), \end{cases} \tag{4}$$

The reason that the G.C.C. is necessary is closely related to the fact that, given any ray $x(t)$ it is possible to find a sequence of solutions of the wave equation (sometimes called **gaussian beams** solutions) whose energy at time t concentrates around the point $x(t)$.

We now bring our attention to the numerical computation of the control F associated to a pair of initial data (u_0,u_1) in energy space. The simplest idea to solve this problem consists in discretizing the initial data on a grid $Q_h := Q \cap h\mathbf{Z}^d$, thus obtaining a couple of discrete functions $\left(U_0^h, U_1^h\right)$, and then computing the (finite-dimensional) control F^h with the properties that $F_n^h = 0$ if $hn \in Q_h \setminus \omega$ and $U^h(T) = \partial_t U^h(T) = 0$, the function $U^h(t)$ being the solution of the semi-discretized wave equation:

$$\begin{cases} \rho(hn)\,\partial_t^2 U_n^h(t) - L_h U_n^h(t) = F_n^h(t) & \text{in } \mathbf{R}_t \times Q_h, \\ U^h(0) = U_0^h, \qquad \partial_t U^h(0) = U_1^h, & U^h(t) \text{ is } Q_h\text{-periodic.} \end{cases} \tag{5}$$

Here we have taken as L_h the usual finite-difference discretization of the Laplacian:

$$L_h U_n^h := \frac{1}{h^2}\sum_{j=1}^{d}\left[U_{n+e_j}^h + U_{n-e_j}^h - 2U_n^h\right],$$

the vectors e_1,..., e_d being those of the canonical basis of $\mathbf{Z}^d$. Now we must ensure that the discrete controls F^h converge[2] to the continuous one F. Again, an application of H.U.M. gives that the convergence of our algorithm is equivalent to proving an **uniform observability estimate**: there exists $C_{T,\omega} > 0$ *independent of* h, such that

$$\int_0^T h^d \sum_{hn\in Q_h} \rho(hn)\left|\partial_t\Phi_n^h(t)\right|^2 dt \le C_{T,\omega}\int_0^T h^d \sum_{hn\in\omega} \rho(hn)\left|\partial_t\Phi_n^h(t)\right|^2 dt, \tag{6}$$

[2] in order to lighten the exposition, we are deliberately vague on this point. For more details see [5] and [11]

for every solution Φ^h of the semi-discrete wave equation:

$$\begin{cases} \rho(hn)\,\partial_t^2\Phi_n^h - L_h\Phi_n^h = 0 & \text{in } \mathbf{R}_t \times Q_h, \\ \Phi^h(0) = \Phi_0^h, \qquad \partial_t\Phi^h(0) = \Phi_1^h, & \Phi^h(t) \text{ is } Q_h\text{-periodic.} \end{cases} \tag{7}$$

2 A non-convergence result: Group Velocity and Wigner Measures

Unfortunately, the algorithm we have just presented does not converge. In the related case of boundary controllability this was numerically observed by R. Glowinski and J.-L. Lions (see e.g. [4]). E. Zuazua proved rigorously that the inequality analog to (6) in that context fails to hold for a constant-coefficient wave equation (see. e.g. [13]). The purpose of this section is to prove this phenomenon takes place in the case of interior control, variable coefficients and, what is more important, to completely describe the convergence failure in terms of Group Velocity. This will be done by means of the theory of Wigner measures associated to discrete functions introduced in [7] (see also [10]).

Our main result states that the semi-discrete wave equation (7) possesses solutions with energy concentrating along **modified rays**, which do not coincide with the ones corresponding to the original continuous problem (4). These new rays are given by the x-component of the solutions to

$$\begin{cases} \dot{x}_j(t) = \rho(x(t))^{-1}\sin\xi_j(t) \Big/ \left(4\sum_{k=1}^d \sin^2(\xi_k(t)/2)\right)^{1/2}, \\ \dot{\xi}_j(t) = 2\rho(x(t))^{-2}\,\partial_{x_j}\rho(x(t))\sum_{k=1}^d \sin^2(\xi_k(t)/2). \end{cases} \tag{8}$$

The above discussion is made precise in the following:

Theorem 1. *Given $x_0, \xi^0 \in \mathbf{R}^d$, $\xi^0 \neq 0$ there exists a sequence of solutions $\left(\Phi^h\right)_{h>0}$ of the semi-discrete wave equation (7) of total energy equal to one and such that its energy density concentrates near the points $x(\pm t)$, the curve $x(t)$ being the (x-component) of the solution of (8) corresponding to the initial data $\left(x_0, \xi^0\right)$. More precisely:*

$$\lim_{h\to 0} h^d \sum_{hn\in Q_h} \rho(hn)\left|\partial_t\Phi_n^h(t)\right|^2 \psi(hn) = \frac{1}{2}\left[\psi(x(t)) + \psi(x(-t))\right] \tag{9}$$

for every $t \in \mathbf{R}$ and $\psi \in C_{per}^\infty(Q)$.

By testing in (9) with functions $\psi \in C_c^\infty(\omega)$ we arrive to the following necessary condition for the uniform observability estimate (6) to hold:

Every modified ray $x(t)$ must intersect the control region ω in a time less than T.

This condition is of a completely different nature that the G.C.C.. Notice that (9) has non-trivial stationary solutions $\left(x_0, \xi^0\right)$, corresponding to critical points of the density $\nabla_x \rho\left(x_0\right)=0$ and frequencies $\xi^0 \neq 0$ with components of the form $\pm\pi$ or 0. Formula (9) gives then for this rays the existence of sequences of solutions to (7) with energy density concentrating along the *fixed* point x_0, i.e. for which Group Velocity vanishes.

This implies that, in general, inequality (6) fails to be true. In the constant-coefficient case, due to the simple structure of the ray equations, it is possible to prove that uniform observability fails unless $\omega = Q_h$ up to a null measure set (see [9]). Consequently, the numerical scheme for the computation of the control F proposed in the preceding section does not converge.

The proof of this result is based on showing that the limits of the quadratic quantities:

$$M_t^h\left(x,\xi\right) := \frac{h^d}{\left(2\pi\right)^d} \sum_{m,n\in\mathbf{Z}^d} \overline{\partial_t \Phi_m^h\left(t\right)} \partial_t \Phi_n^h\left(t\right) e^{i(m-n)\cdot\xi} \delta_{hm}\left(x\right), \text{ for } x,\xi\in\mathbf{R}^d.$$

are Radon measures μ_t (the **Wigner measure** of $\partial_t \Phi^h\left(t\right)$) such that

$$\lim_{h\to 0} h^d \sum_{hn\in Q_h} \rho\left(hn\right) \left|\partial_t \Phi_n^h\left(t\right)\right|^2 \psi\left(hn\right) = \int_{Q\times[-\pi,\pi)^d} \psi\left(x\right) d\mu_t\left(x,\xi\right)$$

and then verifying that μ_t is determined from μ_0 by transport along the modified rays (8). This is indeed a property that holds for *every* sequence of solutions of (7). Then it suffices to show the existence of a sequence such that its corresponding μ_0 is precisely $\delta_{x_0}\otimes\delta_{\xi^0}$, typically a gaussian wavepacket concentrating in x_0 and oscillating in the frequency ξ^0 (see [8], [11]). For a general reference on Wigner measures in the continous setting, see [3].

3 Uniform observability for filtered solutions

We finally show that (6) holds provided we restrict ourselves to solutions with filtered high-frequencies. This is not surprising, since the propagation failure results described above are of high-frequency nature. In terms of the convergence of our algorithm this observability inequalities for filtered solutions express the convergence of **partial controls**, i.e. that bring to rest only a finite number of Fourier modes of the solutions (see [5]).

Similarly as done in [13] we consider the spaces

$$J_\gamma^\rho := \left\{ \Phi_n^h\left(t\right) = \sum_{\omega_j^h \le \gamma/\sigma(h)} \left[e^{i\omega_j^h t} \widehat{U_+^h}\left(j\right) + e^{-i\omega_j^h t}\widehat{U_-^h}\left(j\right)\right] E_{j,n}^h \right\};$$

where ω_j^h and E_j^h are respectively the eigenfrequencies and eigenmodes of problem (7), $\gamma > 0$ is the cut-off frequency parameter and σ is positive a function tending to zero as $h \to 0$ such that $h/\sigma(h)$ is bounded.

Two different type of results are obtained, depending on wether $\sigma(h) = h$ or $h << \sigma(h)$: in the second regime we recover the geometric condition of the continuous problem:

Theorem 2. *If $h << \sigma(h)$ then the uniform observability estimate (6) holds for every solution in J_γ^ρ (the cut-off parameter γ being arbitrary) if and only if (T, ω) satisfy the Geometric Control Condition.*

This is no longer true if $\sigma(h) = h$ since, as we explained in the preceding section, the energy propagates following the flow of the modified rays (8). However, the latter rays are close to the original ones (4) for ξ small (simply due to the fact that $\sin\xi \sim \xi$ for $\xi << 1$). This remark leads to the following result:

Theorem 3. *There exists $\gamma > 0$ such that the uniform observability estimate (6) holds in J_γ^h if (T, ω) satisfy the Geometric Control Condition.*

In the proof of both results, the role of Wigner measures is fundamental (see [7] and the proofs of similar results in the continuous setting, e.g. [2]).

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