The Schrödinger flow in a compact manifold: High-frequency dynamics and dispersion

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Abstract. We discuss different aspects of the dynamics of the Schrödinger flow on a compact Riemannian manifold that are related to the behavior of high-frequency solutions. In particular we show that dispersive (Strichartz) estimates fail on manifolds whose geodesic flow is periodic (thus generalizing the well-known result for spheres proved via zonal spherical harmonics). We also address the issue of the validity of observability estimates. We show that the geometric control condition is necessary in manifolds with periodic geodesic flow and we give a new, geometric, proof of a result of Jaffard on the observability for the Schrödinger flow on the two-torus. All our proofs are based on the study of the structure of semiclassical (Wigner) measures corresponding to solutions to the Schrödinger equation.

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1. Introduction

Let (M, g) be a compact, smooth Riemannian manifold. The Schrödinger flow on (M, g) associates to an initial datum $u_0 \in L^2(M)$ the solution $u(t, \cdot)$ to the Schrödinger equation:

$$\begin{cases} i\partial_t u(t,x) + \Delta_x u(t,x) = 0, & (t,x) \in \mathbb{R} \times M, \\ u|_{t=0} = u_0. \end{cases} \tag{1.1}$$

Above, Δ_x denotes the Laplace-Beltrami operator corresponding to (M, g). Since M is compact, the spectrum of $-\Delta_x$ consists of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 ...$ that tend to infinity. We shall denote by $(\psi_{\lambda_n})_{n\in\mathbb{N}}$ an orthonormal basis of $L^2(M)$

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consisting of eigenfunctions $-\Delta_x \psi_{\lambda_n} = \lambda_n \psi_{\lambda_n}$. One has $u(t,\cdot) = e^{it\Delta_x} u_0$ and the following representation holds:

$$e^{it\Delta_x} u_0 = \sum_{n\in\mathbb{N}} e^{-i\lambda_n t} \widehat{u_0}(\lambda_n)\, \psi_{\lambda_n}, \qquad \text{provided } u_0 = \sum_{n\in\mathbb{N}} \widehat{u_0}(\lambda_n)\, \psi_{\lambda_n}. \tag{1.2}$$

Two direct consequences may be extracted from this formula. First, that the dynamics of the Schrödinger flow are almost-periodic; second, that the $L^2(M)$-norm is conserved by $e^{it\Delta_x}$. Note that both these properties hold regardless of the specific geometry of (M,g).

Another dynamical feature of $e^{it\Delta_x}$ that is not so easily interpreted from (1.2) is its *dispersive* character. The high-frequency modes of a solution to the Schrödinger equation travel at a higher speed than is low-frequency counterparts.[1] This result in a regularizing effect on the singularities of the initial datum which is usually quantified through dispersive estimates (also known as Strichartz estimates) of the type:

$$\left\| e^{it\Delta_x} u_0 \right\|_{L^p([0,1]\times M)} \le C \left\| u_0 \right\|_{H^s(M)}. \tag{1.3}$$

Such an estimate is known to hold for $M = \mathbb{R}^d$ when $p = p_0(d) := 2(2+d)/d$ and $s = 0$. For a general d-dimensional compact manifold M, Burq, Gérard and Tzvetkov [3] have shown that (1.3) also holds for $p = p_0(d)$ but with $s = 1/p$ (which is half the exponent given by the Sobolev embedding theorem).

This value of s is not optimal in general; in fact, the infimum $s(p,M)$ of the values s for which (1.3) holds is a quantity that depends heavily on the specific geometry of the manifold M considered. For instance, when M is the flat torus $\mathbb{T}^d$, Bourgain has shown [2] that $s\left(p_0(d), \mathbb{T}^d\right) = 0$ for $d = 1,2$ (although the estimate is actually false for $s = 0$), and it holds for $d = 1$, $p = 4$, $s = 0$ as shown by Zygmund [32]. When (M,g) has periodic geodesic flow, (1.3) holds for $p = 4$, $s > d/4 - 1/2$ and $d \ge 3$ ($s > 1/8$ if $d = 2$), which is again smaller than $1/p$. Moreover, these values are optimal on standard spheres $\mathbb{S}^d$; these results are proved in [3]. These considerations can be interpreted as the fact that the dispersive effect for the Schrödinger flow is stronger on tori than on spheres.

The validity of dispersive estimates is closely related to the *high-frequency behavior* of the solutions to (1.1). These behavior is tested on highly oscillating sequences of initial data, *i.e.* sequences $\left(u_0^h\right)$ whose $L^2(M)$-norm is concentrated on frequencies localized towards infinity as $h \to 0^+$. Typical examples of such initial data are (strictly) *h-oscillating sequences* $\left(u_0^h\right)$, which are of the form:

$$u_0^h = \sum_{a/h \le \sqrt{\lambda_n} \le b/h} \widehat{u_0}(\lambda_n)\, \psi_{\lambda_n}, \qquad \text{for some } b > a > 0, \tag{1.4}$$

[1] However, this is readily seen when M is the Euclidean space equipped with the standard metric. The solution issued from a plane-wave initial datum $e^{i\xi\cdot x}$ is precisely $e^{i\xi\cdot(x-t\xi)}$, which travels at velocity ξ.

or those of W.K.B. type, $u_0^h(x) := e^{iS_0(x)/h}$ for some $S_0 \in C^\infty(M)$. For small h, the behavior of $e^{it\Delta_x}u_0^h$ turns out to be related to the dynamics of the geodesic flow of (M,g). In particular, up to times t of the order of h, the classical W.K.B. method gives a very precise description of the structure of these solutions in terms of propagation along geodesics of M; however, it fails to describe the global in time evolution. A simpler, although more general, approach consists in understanding the limiting behavior as $h \to 0^+$ of the position densities:

$$n_h(t) := \left|e^{it\Delta_x}u_0^h\right|^2.$$

This object is physically relevant, in the context of the quantum-classical correspondence principle, as it describes the asymptotic behavior of the position probability density of a free quantum particle propagating in M. If $\left(u_0^h\right)$ is bounded in $L^2(M)$, the measures n_h are bounded in $L^\infty(\mathbb{R};\mathcal{M}_+(M))$, where $\mathcal{M}_+(M)$ stands for the set of positive Radon measures on M. Therefore it has at least a weak-$*$ accumulation point $\nu \in L^\infty(\mathbb{R};\mathcal{M}_+(M))$; these are sometimes called *quantum limits* or *defect measures.*

It can be shown (see for instance [24]) that the support of ν is an union of geodesics of M. The precise structure of the set of such accumulation points depends heavily on the particular dynamical properties of the geodesic flow of M. When it is completely integrable, some results have been obtained in [24, 25] by identifying the structure of the set of *semiclassical (or Wigner) measures* corresponding to $\left(e^{it\Delta_x}u_0^h\right)$. These are obtained as limits of some microlocal lifts to T^*M of the densities $n_h(t)$, known as Wigner distributions (a systematic presentation is given in [15, 23, 16, 17, 4], see also Section 2 for precise definitions).[2] In Section 3 we shall present a new approach to the structure result of [25] for semiclassical measures on the flat torus $\mathbb{T}^d$.

The knowledge of the structure of the set of quantum limits in M can be used to show the failure of dispersive estimates (1.3) in the case $s = 0$. This is due to the fact that whenever (1.3) holds one has $n_h \in L^{p/2}([0,1] \times M)$, and the same holds for any quantum limit ν. In particular, since $p/2 \geq 1$, (1.3) implies that any quantum limit must be absolutely continuous with respect to the Riemannian measure in M. If one is able to produce a sequence of initial data $\left(u_0^h\right)$ that gives a quantum limit which has a nontrivial singular component then no dispersive estimate may hold for $e^{it\Delta_x}$ in M. We shall apply this strategy to prove, in Section 4, the following result.

Theorem 1.1. *Let (M,g) be a manifold with periodic geodesic flow. Then the dispersive estimate*

$$\left\|e^{it\Delta_x}u_0\right\|_{L^p([0,1]\times M)} \leq C\left\|u_0\right\|_{L^2(M)} \tag{1.5}$$

fails for every $p > 2$.

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[2]We refer the reader to [7, 31] for a comparison between the semiclassical measure and the W.K.B. approaches.

As was pointed out by the referee, the failure of the dispersive estimate in this setting can also be obtained combining the optimality of the analogous of the Strichartz estimates for spectral projectors proved by Sogge (see [30], Corollary 5.1.2) together with the precise spectral results for the Laplacian on manifolds with periodic geodesic flow by Duistermaat-Guillemin [10] and Colin de Verdière [8]. This strategy allows to show that estimate (1.5) fails even if the L^2-norm is replaced by a Sobolev norm H^s with $s < \delta(p)$, where

$$\delta(p) := \begin{cases} \frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right) & \text{if } 2 < p \leq \frac{2(d+1)}{d-1}, \\ d\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2} & \text{if } p \geq \frac{2(d+1)}{d-1}, \end{cases}$$

denotes Sogge's exponent.

Note that the approach we used to prove Theorem 1.1 cannot be used to disprove the dispersive estimate in the case of the flat torus $\mathbb{T}^2$ and $2 < p < 4 = p_0(2)$ and $s = 0$, since every quantum limit is absolutely continuous with respect to the Lebesgue measure in that case (see [25] for a proof). This suggests that an eventual failure of the dispersive estimate in this case must be realised by a more subtle mechanism.

The third and final aspect of the dynamics of the Schrödinger flow we want to discuss here is related to a quantitative version of the unique continuation property known as *observability.* Take $T > 0$ and an open set $U \subset M$; the Schrödinger flow $e^{it\Delta_x}$ is said to satisfy the observability property for T and U whenever a constant $C = C_{T,U} > 0$ exists such that

$$\|u_0\|_{L^2(M)} \leq C \int_0^T \int_U \left|e^{it\Delta_x} u_0(x)\right|^2 dx dt \tag{1.6}$$

for every initial datum $u_0 \in L^2(M)$. Note that the fact that an estimate like (1.6) holds implies that whenever two solutions to the Schrödinger equation are close to each other in $L^2((0,T)\times U)$-norm they must be globally close. In particular, two solutions that coincide in $(0,T)\times U$ must be identical. The observability property is relevant in Control Theory [22], and Inverse Problems [18].

A sufficient condition for (1.6) to hold was found by Lebeau [20] (see also [9]). It is the following.

$$\begin{array}{c} \text{There exists } L_0 > 0 \text{ such that} \\ \text{every geodesic of } (M,g) \text{ of length smaller than } L_0 \text{ intersects } \overline{U}. \end{array} \tag{1.7}$$

However, this condition is not necessary in general, as follows from the works of Jaffard [19] or Burq and Zworski [6]. Nevertheless, we shall show in Section 4 that (1.7) is equivalent to (1.6) when (M,g) has periodic geodesic flow.

Theorem 1.2. *Let (M,g) be a compact manifold with periodic geodesic flow. If the observability estimate* (1.6) *holds for some $T > 0$ and some open set $U \subset M$ then U must satisfy* (1.7)*. As a consequence,* (1.6) *and* (1.7) *are equivalent.*

The proof of this result will be again based on the high-frequency properties of the Schrödinger flow; and in particular on the analysis of the set of semiclassical measures on M. Note that the role of semiclassical measures in the context of observability estimates was first noticed by Lebeau [21]. As mentioned before, condition (1.7) is not in general necessary for (1.6) to hold. For instance, when $M = \mathbb{T}^2$, the two-dimensional standard torus equipped with the flat metric, Jaffard [19] proved the following result (see also [5, 27] for related results for eigenfunctions of the Laplacian).

Theorem 1.3. *Let* $(M, g) = \left(\mathbb{T}^2, \text{flat}\right)$. *Given any* $T > 0$ *and any open set* $U \subset \mathbb{T}^2$ *there exist a constant* $C > 0$ *such that the observability estimate (*1.6*) holds.*

The original proof of this result is based on results on pseudo-periodic functions due to Kahane. In Section 4 we shall give a new proof of this result which is completely microlocal and relies on the structure result for semiclassical measures for the Schrödinger flow on the torus presented in [25].

2. Semiclassical measures and the Schrödinger flow

Semiclassical measures are a very convenient tool in the high-frequency analysis of a sequence $\left(u^h\right)$ bounded in $L^2(M)$. These objects are a microlocal version of the well known defect measures, that describe the local concentration of the $L^2(M)$-norm of $\left(u^h\right)$. Assume that $\left(u^h\right)$ is bounded in $L^2(M)$; then the sequence of densities

$$n_h := \left|u^h\right|^2 dm$$

is bounded in $L^1(M)$ (here dm stands for the measure on M induced by the Riemannian metric g). Helly's theorem then ensures that, up to the extraction of a subsequence, that (n_h) weakly converges, as $h \to 0^+$, towards a finite, positive Radon measure $\nu \in \mathcal{M}_+(M)$ which is usually called a *defect measure* for $\left(u^h\right)$. The support of ν describes the regions on which the "energy" of $\left(u^h\right)$ concentrates. For instance, if u^h is supported in some local chart and given by a concentration profile:

$$\frac{1}{h^{d/2}} \rho \left(\frac{x - x_0}{h}\right) \tag{2.1}$$

then one has $\nu(x) = \|\rho\|^2_{L^2(M)} \delta(x - x_0)$. On the other hand, if u^h is oscillating, written in a coordinate chart as:

$$\rho(x) e^{i\xi_0/h \cdot x}, \tag{2.2}$$

then $\nu(x) = |\rho(x)|^2 dm$, whatever the value of ξ_0. The inability of defect measures to distinguish between different directions of oscillation turns out to be a serious difficulty when dealing with solutions to wave-type equations. For instance, suppose $M = \mathbb{R}^d$ equipped with the standard metric, and take u_0^h to be of the form

(2.2). A direct computation gives that the solution $e^{iht\Delta_x}u_0^h$ of the *semiclassical* Schrödinger equation issued from u_0^h satisfies:

$$n_h(ht)(x) := \left|e^{iht\Delta_x}u_0^h(x)\right|^2 = \left|e^{iht\Delta_x}\rho(x-t2\xi_0)\right|^2.$$

Therefore the densities $n_h(ht)$ weakly converge, as $h \to 0^+$, to the defect measure $\nu_t(x) := |\rho(x-t2\xi_0)|^2\, dx$ which does depend on ξ_0. In particular, the defect measure of the initial data $\nu_0 = |\rho|^2\, dx$ does not determine uniquely that corresponding to the evolution, since the latter depends explicitly on ξ_0.

This motivates the introduction of an object that takes into account the nature of the oscillations. The *Wigner distribution* w_h of the function u^h achieves this. Given a test function $a \in C_c^\infty(T^*M)$ on the cotangent bundle of M, we defined the action of w_h against a as:

$$\langle w_h, a\rangle := (\mathrm{op}_h(a)\, u^h | u^h)_{L^2(M)},$$

where $\mathrm{op}_h(a)$ denotes the semiclassical pseudodifferential operator of symbol a obtained by Weyl's quantization rule.[3] When M is the Euclidean space equipped with the standard metric, $\mathrm{op}_h(a)$ is defined by the formula:

$$\mathrm{op}_h(a)\, u(x) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a\left(\frac{x+y}{2}, h\xi\right) u(y)\, e^{i(x-y)\cdot\xi} dy \frac{d\xi}{(2\pi)^d}.$$

This definition extends to a manifold by applying it locally, in a coordinate chart, and then assembling it by means of a partition of unity. This expression for w_h defines it as an element of $\mathcal{D}'(T^*M)$, the set of distributions on T^*M. The Winger distribution is actually a lift of the densities n_h to phase-space T^*M for, if $\varphi \in C^\infty(M)$ one has $\mathrm{op}_h(\varphi) = \varphi$, the operator defined by multiplication by φ, and therefore,

$$\langle w_h, \varphi\rangle = (\varphi u^h | u^h)_{L^2(M)} = \int_M \varphi n_h.$$

When $M = \mathbb{R}^d$, we may identify $T^*M \equiv \mathbb{R}_x^d \times \mathbb{R}_\xi^d$. If $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$ only depends of ξ then $\mathrm{op}_h(\varphi) = \varphi(hD_x)$ is the Fourier multiplier of symbol φ. Hence,

$$\langle w_h, \varphi\rangle = \int_{\mathbb{R}^d} \varphi(\xi) \left|\widehat{u^h}\left(\frac{\xi}{h}\right)\right|^2 \frac{d\xi}{(2\pi h)^d};$$

this shows that the projection of w_h on the variable ξ measures the concentration of the $L^2\left(\mathbb{R}^d\right)$-norm of the h-rescaled Fourier transform of u^h. The fact that the limits of Wigner distributions are positive measures is non-trivial, and was proved by Gérard [15] and Lions and Paul [23].

Theorem 2.1. *Let $\left(u^h\right)$ be a bounded sequence in $L^2(M)$. Then there exists a subsequence (which we do not relabel) and a finite positive Radon measure $\mu \in \mathcal{M}_+(T^*M)$ such that*

$$w_h \rightharpoonup \mu, \text{ as } h \to 0^+ \text{ in } \mathcal{D}'(T^*M).$$

[3]The books [11, 26] are clear and recent introductions to semiclassical microlocal analysis, we refer the reader to them for background and precise definitions on pseudodifferential operators.

In this situation we say that μ is the *semiclassical measure* of the sequence $\left(u^h\right)$. If in addition, $\left(u^h\right)$ is *h-oscillating*, that is:

$$\limsup_{h\to 0^+} \sum_{\sqrt{\lambda_n}\geq R/h} \left|\widehat{u^h}\left(\lambda_n\right)\right|^2 \to 0, \text{ as } R\to\infty,$$

then the defect measure ν of $\left(u^h\right)$ is obtained by projecting its semiclassical measure μ on the ξ-component:

$$\int_{T_x^*M} \mu\left(x, d\xi\right) = \nu\left(x\right).$$

The additional variable allows to keep track of the directions of oscillation.

A direct computation gives that the semiclassical measure of the oscillating sequence (2.2) is $\left|\rho\left(x\right)\right|^2 dx\delta\left(\xi-\xi_0\right)$, therefore keeping track of the direction of oscillation ξ_0. A particularly interesting example is that of a *wave-packet* or *coherent state*. It is defined as a sequence $\left(u^h\right)$ in $L^2\left(M\right)$, supported in local chart that is written in coordinates as:

$$\frac{1}{h^{d/4}}\rho\left(\frac{x-x_0}{\sqrt{h}}\right)e^{i\xi_0/h\cdot x} \tag{2.3}$$

for some $\rho\in C^\infty\left(M\right)$. The semiclassical measure of this sequence is

$$\|\rho\|^2_{L^2(M)}\,\delta\left(x-x_0\right)\delta\left(\xi-\xi_0\right).$$

For a more detailed account on these issues, we refer the reader to the survey articles [4, 17], and the concise presentation of [16].

Let us now turn to the analysis of semiclassical measures for sequences of solutions to the Schrödinger equation. Let $\left(u_0^h\right)$ be a bounded, h-oscillating sequence in $L^2\left(M\right)$. We define the time-dependent Wigner distributions:

$$\left\langle w_h\left(t\right), a\right\rangle := \left(\mathrm{op}_h\left(a\right)e^{it\Delta_x}u_0^h | e^{it\Delta_x}u_0^h\right)_{L^2(M)}, \qquad a\in C_c^\infty\left(T^*M\right). \tag{2.4}$$

The following result was proved in [24].

Theorem 2.2. *With the above notations and hypotheses, the following holds. There exists a subsequence, which we do not relabel, and a positive measure* $\mu\in L^\infty\left(\mathbb{R};\mathcal{M}_+\left(T^*M\right)\right)$ *such that:*

$$\lim_{h\to 0^+}\int_{\mathbb{R}}\phi\left(t\right)\left\langle w_h\left(t\right), a\right\rangle dt = \int_{\mathbb{R}\times T^*M}\phi\left(t\right)a\left(x,\xi\right)\mu_t\left(dx,d\xi\right)dt, \tag{2.5}$$

for every $\phi\in L^1\left(\mathbb{R}\right)$, $a\in C_c^\infty\left(T^*M\right)$. *Moreover, for all* $\varphi\in C^\infty\left(M\right)$,

$$\lim_{h\to 0^+}\int_{\mathbb{R}\times M}\phi\left(t\right)\varphi\left(x\right)\left|e^{it\Delta_x}u_0^h\left(x\right)\right|^2 dmdt = \int_{\mathbb{R}\times T^*M}\phi\left(t\right)\varphi\left(x\right)\mu_t\left(dx,d\xi\right)dt, \tag{2.6}$$

and for almost every $t\in\mathbb{R}$, *the measure* μ_t *is invariant by the geodesic flow* ϕ_s^g *of* (M,g)*:*

$$\int_{T^*M} a\left(\phi_s^g\left(x,\xi\right)\right)\mu_t\left(dx,d\xi\right) = \int_{T^*M} a\left(x,\xi\right)\mu_t\left(dx,d\xi\right), \quad \text{for every } s\in\mathbb{R}. \tag{2.7}$$

Note that the convergence in (2.5) is precisely the convergence in the weak-$*$ topology in $L^{\infty}(\mathbb{R};\mathcal{D}'(T^*M))$. One cannot expect pointwise convergence of the distributions $w_h(t)$ for every $t\in\mathbb{R}$, since as shown in (2.7) the limit measure becomes instantaneously invariant by the geodesic flow. However, if one considers instead the solutions to the semiclassical Schrödinger equation, which corresponds to taking limits of $(w_h(ht))$, convergence is locally uniform in t, and the limiting measure μ_t is computed through μ_0 by transport along the geodesic flow ϕ_t^g, see [15, 23, 17].

3. Manifolds with completely integrable geodesic flow

In order to gain further insight on the structure of the set of semiclassical measures obtained as a limit (2.5) we must make additional hypotheses on the dynamics of the geodesic flow ϕ_t^g of the manifold under consideration. Here we shall deal with manifolds with *completely integrable* geodesic flow; in particular, we shall focus on two particular classes of geometries: manifolds with periodic geodesic flow (also known as Zoll manifolds, see the book [1] for a comprehensive discussion on this dynamical hypothesis) and the flat torus (which is a model case for completely integrable geodesic flows).

In the first case we have an explicit formula for the semiclassical measure μ_t in terms of that of the initial data μ_0. In [24], the following is proved.

Theorem 3.1. *Let (M,g) be a manifold with periodic geodesic flow. Let $\left(u_0^h\right)$ be as in Theorem 2.2; suppose that (2.5) holds and that $w_h(0)$ converges to a semiclassical measure μ_0. If $\mu_0(\{\xi=0\})=0$ then, for a.e. $t\in\mathbb{R}$ and $a\in C_c^{\infty}(T^*M)$ we have:*

$$\int_{T^*M} a(x,\xi)\,\mu_t(dx,d\xi)=\int_{T^*M}\langle a\rangle(x,\xi)\,\mu_0(dx,d\xi), \tag{3.1}$$

where $\langle a\rangle$ denotes the average of a along the geodesic flow:

$$\langle a\rangle(x,\xi):=\lim_{T\to\infty}\frac{1}{T}\int_0^T a\left(\phi_t^g(x,\xi)\right)dt.$$

Note that, in particular, μ_t is constant for a.e. $t\in\mathbb{R}$. When $(M,g)=\left(\mathbb{T}^d,\text{flat}\right)$ the situation is rather different, the structure of μ_t is rather more involved. In order to get some insight on the form of the limits of $w_h(t)$ start noticing that Egorov's theorem (see [11, 26]) is an identity when dealing with the Weyl quantization rule on the torus:

$$e^{-it\Delta_x}\operatorname{op}_h(a)\,e^{it\Delta_x}=\operatorname{op}_h\left(a\circ\phi_{t/h}^{\text{flat}}\right).$$

Hence, in view of (2.4), for $\varphi\in L^1(\mathbb{R})$ and $a\in C_c^{\infty}\left(T^*\mathbb{T}^d\right)$ one has:

$$\int_{\mathbb{R}}\varphi(t)\langle w_h(t),a\rangle\,dt=\left\langle w_h(0),\langle a\rangle_{\varphi}^h\right\rangle,$$

where

$$\langle a\rangle_{\varphi}^h(x,\xi):=\int_{\mathbb{R}}\varphi(t)\,a\left(x+\frac{t}{h}\xi,\xi\right)dt. \tag{3.2}$$

Let us introduce some notation. Denote by $\mathbb{W}$ the set whose elements are straight lines in $\mathbb{Z}^d \setminus \{0\}$ passing through the origin. We have a disjoint union

$$\mathbb{Z}^d = \bigsqcup_{\omega \in \mathbb{W}} \omega \sqcup \{0\}.$$

Given $a \in C_c^\infty\left(T^*\mathbb{T}^d\right)$ we have a Fourier series decomposition:

$$a(x,\xi) = \sum_{k \in \mathbb{Z}^d} \widehat{a}(k,\xi)\,\psi_k(x), \qquad \psi_k(x) := \frac{e^{ik\cdot x}}{(2\pi)^{d/2}}.$$

Now, denote by a_ω the orthogonal projection of a into the set of functions in $L^2\left(\mathbb{T}^d\right)$ whose Fourier modes lie in ω, *i.e.*,

$$a_\omega := \sum_{k \in \omega} \widehat{a}(k,\cdot)\,\psi_k.$$

Taking now (3.2) into account we find that:

$$\langle a_\omega \rangle_\varphi^h (x,\xi) = b_{a,\varphi}^\omega \left(x, \xi, \frac{\xi \cdot \nu_\omega}{h}\right),$$

with

$$b_{a,\varphi}^\omega (x,\xi,\sigma) := \int_{\mathbb{R}} \varphi(t)\, a_\omega (x + t\sigma\nu_\omega, \xi)\, dt,$$

where ν_ω denotes a unit vector in the direction ω. Therefore, testing $w_h(0)$ against $\langle a_\omega \rangle_\varphi^h$ amounts to perform a blow-up of $w_h(0)$ in the direction ν_ω. This type of object has been already studied in the literature (in the context of Euclidean space) under the name of two-microlocal semiclassical measures. We refer the reader to the works of Fermanian-Kammerer [13, 12], Fermanian-Kammerer and Gérard [14], Miller [28], and Nier [29]. Following [25] one shows that, given $\omega \in \mathbb{W}$ there exists a positive measure $\mu_{\mathcal{R}}^0(\omega,\cdot)$ on

$$I_\omega := \left\{\xi \in \mathbb{R}^d : k \cdot \xi = 0 \text{ for } k \in \omega\right\}$$

taking values in the set of trace-class operators $\mathcal{L}^1\left(L^2(\gamma_\omega)\right)$ on the space of square-summable functions defined on any geodesic γ_ω in the direction ω such that:

$$\lim_{h \to 0^+} \left\langle w_h(0), b\left(x, \xi, \frac{\xi \cdot \nu_\omega}{h}\right)\right\rangle = \operatorname{tr} \int_{I_\omega} \tilde{b}(s, \xi, D_s)\, \mu_{\mathcal{R}}^0(\omega, d\xi)$$

where $b \in C_c^\infty\left(T^*\mathbb{T}^d \times \mathbb{R}\right)$ is a functions whose non-zero Fourier modes in x corresponds to frequencies in ω. Note that in this case,

$$b(x,\xi,\sigma) = \tilde{b}(x \cdot \nu_\omega, \xi, \sigma)$$

where $\tilde{b} \in C_c^\infty\left(\gamma_\omega \times \mathbb{R}^d \times \mathbb{R}\right)$ is the restriction of $b(\cdot,\xi,\sigma)$ to γ_ω. For every $\xi \in I_\omega$, the pseudodifferential operator $\tilde{b}(s,\xi,D_s)$ is a compact operator in $L^2(\gamma_\omega)$. A straightforward computation then gives:

$$\lim_{h \to 0^+} \left\langle w_h(0), \langle a_\omega \rangle_\varphi^h \right\rangle = \int_{\mathbb{R}} \varphi(t) \operatorname{tr} \int_{I_\omega} \tilde{a}_\omega(\cdot,\xi)\, \mu_{\mathcal{R}}^t(\omega, d\xi)\, dt, \tag{3.3}$$

where $\tilde{a}_\omega(\cdot,\xi)$ denotes the operator of multiplication in $L^2(\gamma_\omega)$ by the restriction of $a_\omega(\cdot,\xi)$ to γ_ω, and the trace-class operator valued measures $\mu^t_{\mathcal{R}}(\omega,\cdot)$ are defined as the solutions to the initial-value problem for a density-matrix Schrödinger equation on $L^2(\gamma_\omega)$:

$$\begin{cases} i\partial_t \mu^t_{\mathcal{R}}(\omega,\xi) = \left[-\partial_s^2, \mu^t_{\mathcal{R}}(\omega,\xi)\right], \\ \mu^t_{\mathcal{R}}(\omega,\xi)\,|_{t=0} = \mu^0_{\mathcal{R}}(\omega,\xi). \end{cases} \tag{3.4}$$

The right-hand side of (3.3) can be written as

$$\int_{\mathbb{R}} \varphi(t) \int_{\mathbb{T}^d\times I_\omega} a_\omega(x,\xi)\,\rho^t_\omega(dx,d\xi),$$

where ρ_ω is a signed measure on I_ω whose projection on x is absolutely continuous with respect to the Lebesgue measure and whose non-zero Fourier modes lie in ω. The measure ρ^t_ω is obtained as the extension to $\mathbb{T}^d\times I_\omega$ of the density defined on $\gamma_\omega\times I_\omega$ by formula (3.3), see [25] (the sum in ω of these two-microlocal measures was called there the *resonant semiclassical measure* of $\left(u_0^h\right)$). Therefore, we recover the main result [25].

Theorem 3.2. *Let $(M,g) = \left(\mathbb{T}^d, \text{flat}\right)$, suppose $\left(u_0^h\right)$ satisfies the hypotheses of Theorem 2.2 and that $w_h(0) \rightharpoonup \mu_0$ as $h\to 0^+$. Then, for a.e. $t\in\mathbb{R}$ we have:*

$$\mu_t = \sum_{\omega\in\mathbb{W}} \rho^t_\omega + dx\otimes\overline{\mu_0},$$

where

$$\overline{\mu_0}(\xi) := (2\pi)^{-d}\int_{\mathbb{T}^d} \mu_0(dy,\xi),$$

and the ρ^t_ω are defined by the above construction. In particular, they are signed measures concentrated on $\mathbb{T}^d\times I_\omega$, their non-zero Fourier modes in x are in the line ω and its projection on the x-component is absolutely continuous with respect to the Lebesgue measure. Moreover, each of the measures

$$\rho^t_\omega + dx\otimes\overline{\mu_0}\rceil_{I_\omega}$$

is non-negative.

Let us stress that the measures ρ^t_ω are *not determined* by the semiclassical measures of the initial data μ_0. In [24, 25] examples of sequences $\left(u_0^h\right)$ and $\left(v_0^h\right)$ are given having the same semiclassical measure μ_0 but such that their respective time-dependent measures ρ^t_ω differ. In fact, a sufficient condition to have $\rho^t_\omega = 0$ is that

$$\lim_{h\to 0^+} \left\|\chi(\nu_\omega\cdot D_x)\,u_0^h\right\|_{L^2(\mathbb{T}^d)} = 0,$$

for every $\chi\in C_c^\infty(\mathbb{R})$ (see [25]).

Note also that the term $\sum_{\omega\in\mathbb{W}}\rho^t_\omega$ is concentrated on the set

$$\Omega := \left\{\xi\in\mathbb{R}^d : \xi\cdot k = 0 \text{ for some } k\in\mathbb{Z}^d\setminus\{0\}\right\}$$

of *resonant frequencies*. When the measure μ_0 of the initial data does not charge this set then the measure μ_t equals $dx \otimes \overline{\mu_0}$. This is the analogue in this context of the averaging formula (3.1).

When $\mu_0(\{\xi = 0\}) = 0$ and $d = 2$, it is proved in [25] that in fact the whole measure $\int_{\mathbb{R}^d} \mu_t(\cdot, d\xi)$ is absolutely continuous with respect to the Lebesgue measure. This is in great contrast with the situation on Zoll manifolds, where the semiclassical measures μ_t may be singular with respect to the Riemannian measure. This again can be interpreted as the fact that the dispersive effect is much stronger on the torus than on manifolds with periodic geodesic flow.

4. Dispersion and observability for the Schrödinger flow

Let us now turn to the proof of the main results of this article.

Proof of Theorem 1.1. Take $(x_0, \xi_0) \in T^*M$ with $\xi_0 \neq 0$ and let $\left(u_0^h\right)$ be a wave-packet type sequence of initial data, as defined in (2.3) with $\left\|u_0^h\right\|_{L^2(M)} = 1$. Then we have that $w_h(0) \rightharpoonup \delta(x - x_0)\,\delta(\xi, -\xi_0)$ as $h \to 0^+$. The averaging formula (3.1) in Theorem 3.1 then gives, for every $a \in C_c^\infty(T^*M)$:

$$\lim_{h\to 0^+} \int_0^1 \langle w_h(t), a\rangle\, dt = \int_{T^*M} a(x,\xi)\,\delta_\gamma(dx, d\xi),$$

where δ_γ is the Dirac mass supported on γ, the geodesic in T^*M issued from (x_0, ξ_0). Identity (2.6) then gives:

$$\lim_{h\to 0^+} \int_0^1 \int_M \varphi(x) \left|e^{it\Delta_x} u_0^h\right|^2 dtdx = \int_M \varphi(x)\,\delta_{\gamma_M}(dx), \tag{4.1}$$

where γ_M stands for the projection of γ onto M. Since δ_{γ_M} is singular with respect to the Riemannian measure, we conclude that no dispersive estimate may hold for $p > 2$. □

Proof of Theorem 1.2. Suppose that the open set $U \subset M$ does not satisfy the geometric condition (1.7). Therefore, there exists a geodesic γ_M in M that does not intersect $\overline{U}$. Let γ denote the lift of γ_M to T^*M. Let $(x_0, \xi_0) \in \gamma$ and consider the wave-packet sequence $\left(u_0^h\right)$ centered at that point and satisfying $\left\|u_0^h\right\|_{L^2(M)} = 1$. Reasoning as in the preceding proof, we find that (4.1) holds. In particular, if $\varphi \in C^\infty(M)$ is supported in a neighborhood of $\overline{U}$ that does not intersect γ_M we have:

$$\lim_{h\to 0^+} \int_0^1 \int_M \varphi(x) \left|e^{it\Delta_x} u_0^h\right|^2 dtdx = 0.$$

Since $\left\|u_0^h\right\|_{L^2(M)} = 1$ we conclude that no constant $C > 0$ exists such that estimate (1.6) holds. □

Proof of Theorem 1.3. Before proving Jaffard's result Theorem 1.3, we recall that the semiclassical reduction argument in [20] (which combines a Littlewood-Payley decomposition with a unique continuation results for eigenfunctions of the Laplacian) reduces the proof of an observability estimate (1.6) for any function in $L^2(M)$ to establishing it for strictly oscillating sequences of initial data. This, in turn, is equivalent to establishing the following fact.

Let $\left(u_0^h\right)$ *be a strictly* h*-oscillating sequence (*i.e. *verifying* (1.4)*) such that*

$$\lim_{h\to0^+}\int_0^T\int_U\left|e^{it\Delta_x}u_0^h(x)\right|^2dxdt=0. \tag{4.2}$$

Then

$$\lim_{h\to0^+}\left\|u_0^h\right\|_{L^2(\mathbb{T}^2)}=0.$$

This equivalence is a straightforward consequence of the closed graph theorem. Let $\mu\in L^\infty\left(\mathbb{R};\mathcal{M}_+\left(T^*\mathbb{T}^2\right)\right)$ denote the semiclassical measure (in the sense of (2.5)) associated to (possibly a subsequence of) $\left(e^{it\Delta_x}u_0^h\right)$. Suppose moreover that $\left(u_0^h\right)$ has a semiclassical measure μ_0. Our goal is to show that, assuming (1.4)), we can conclude that (4.2) implies that $\mu_0=0$. Start noticing that (4.2) implies that for every $\varphi\in C\left(\mathbb{T}^2\right)$ supported in U we have:

$$\int_0^T\int_U\varphi(x)\,\mu_t(dx,d\xi)\,dt=0.$$

As shown in Theorem 3.2 the measure μ can be written as:

$$\mu_t=\sum_{\omega\in\mathbb{W}}\rho_\omega^t+dx\otimes\overline{\mu_0}$$

and $\rho_\omega^t+dx\otimes\overline{\mu_0}\rceil_{I_\omega}\geq0$. Moreover, the Fourier coefficients of ρ_ω^t lie in ω.

Since $\left(u_0^h\right)$ is strictly oscillating we have $\mu_0(\{\xi=0\})=0$. Therefore, setting $\Omega:=\bigcup_{\omega\in\mathbb{W}}I_\omega$ we have

$$\overline{\mu_0}:=\sum_{\omega\in\mathbb{W}}\overline{\mu_0}\rceil_{I_\omega}+\overline{\mu_0}\rceil_{\Omega^c}.$$

Since all the measures $\mu_\omega^t:=\rho_\omega^t+dx\otimes\overline{\mu_0}\rceil_{I_\omega}$ are positive, we can write, for a.e. $t\in\mathbb{R}$,

$$\mu_t=\sum_{\omega\in\mathbb{W}}\mu_\omega^t+dx\otimes\overline{\mu_0}\rceil_{\Omega^c},$$

in the sense of weak convergence of measures. Now, if $\varphi\in C\left(\mathbb{T}^2\right)$ is supported in U, the above remarks imply:

$$0=\sum_{\omega\in\mathbb{W}}\int_0^T\int_{U\times I_\omega}\varphi(x)\,\mu_\omega^t(dx,d\xi)\,dt+T\overline{\mu_0}(\Omega^c)\int_U\varphi(x)\,dx.$$

Since φ is arbitrary we conclude, since μ is positive:

$$\overline{\mu_0}(\Omega^c)=\frac{1}{(2\pi)^2}\mu_0\left(\mathbb{T}^2\times\Omega^c\right)=0, \tag{4.3}$$

and, for every $t \in [0,T]$,

$$\mu_\omega^t (U \times I_\omega) = 0.$$

To conclude that $\mu_0 = 0$ it remains to show that μ_0 does not charge the set Ω of resonant frequencies. By construction, $\int_{I_\omega} \mu_\omega^t$ is invariant by translations along directions in I_ω. Therefore, $\mu_\omega^t (U_\omega \times I_\omega) = 0$, where $U_\omega := \{x + s\xi : x \in U, \xi \in I_\omega\}$. Let $\mu_{\mathcal{R}}^0$ denote a resonant Wigner measure corresponding to $\left(u_0^h\right)$ as defined by (3.3). Let γ_ω be the geodesic in $\mathbb{T}^2$ through the origin in the direction ω. Define $m_\omega^t \in \mathcal{L}^1\left(L^2(\gamma_\omega)\right)$ as the Hermitian, positive operators that solve the density-matrix Schrödinger equation:

$$i\partial_t m_\omega^t = \left[-\partial_s^2, m_\omega^t\right], \qquad m_\omega^t|_{t=0} = \mu_{\mathcal{R}}^0(\omega, I_\omega). \tag{4.4}$$

With our preceding notations, we have $m_\omega^t = \mu_{\mathcal{R}}^t(\omega, I_\omega)$. Let $J_\omega := U_\omega \cap \gamma_\omega$, denote by $\mathbf{1}_{J_\omega}$ the characteristic function of J_ω in γ_ω; note that $\mathbf{1}_{U_\omega}(x) = \mathbf{1}_{J_\omega}(x \cdot \nu_\omega)$, where ν_ω is a unit vector in ω. Let $\lambda_{\mathbf{1}_{J_\omega}}$ denote the operator on $L^2(\gamma_\omega)$ acting by multiplication by $\mathbf{1}_{J_\omega}$. Then, Theorem 3.2 shows that

$$\begin{aligned}\operatorname{tr}\left(\lambda_{\mathbf{1}_{J_\omega}} m_\omega^t\right) &= \int_{U_\omega \times I_\omega} \rho_\omega^t(dx, d\xi) + \frac{|U_\omega|}{(2\pi)^2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega) \\ &= \int_{U_\omega \times I_\omega} \mu_\omega^t(dx, d\xi) - |U_\omega| \left[\overline{\mu_0}(I_\omega) - (2\pi)^{-2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega)\right].\end{aligned}$$

Therefore $\operatorname{tr}\left(\lambda_{\mathbf{1}_{J_\omega}} m_\omega^t\right) + |U_\omega| \left[\overline{\mu_0}(I_\omega) - (2\pi)^{-2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega)\right] = 0$ for $t \in [0,T]$; unique continuation for (4.4) then implies

$$\operatorname{tr} m_\omega^t + |U_\omega| \left[\overline{\mu_0}(I_\omega) - (2\pi)^{-2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega)\right] = 0,$$

for every $t \in \mathbb{R}$. Finally, notice that $\overline{\mu_0}(I_\omega) \geq (2\pi)^{-2} \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega)$ ([25], Proposition 8). We conclude that $\operatorname{tr} m_\omega^t = 0$ and, consequently, $\mu_\omega^0\left(\mathbb{T}^2 \times I_\omega\right) = \operatorname{tr} \mu_{\mathcal{R}}^0(I_\omega) = \operatorname{tr} m_\omega^0 = 0$ as well. Therefore, we have shown that $\mu_0(\mathbb{T}^2 \times \Omega) = 0$, combining this with (4.3) we conclude that $\mu_0 = 0$ as we wanted to prove. □

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