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Topological and geometric consequences of curvature and symmetry

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Contents

Introducción y resultados obtenidos

La Geometría Riemanniana estudia cómo la presencia de una métrica Riemanniana en una variedad diferenciable condiciona su topología. Toda variedad diferenciable admite métricas Riemannianas, por lo tanto es necesario imponer condiciones en la métrica para obtener posibles restricciones para su existencia. Estas condiciones suelen ser restricciones sobre alguna de las numerosas cantidades que pueden definirse a partir de una métrica Riemanniana, tradicionalmente en la curvatura, el volumen o el diámetro (ver [58]). La motivación y el objetivo de esta tesis es el estudio de posibles consecuencias tanto topólogicas como geométricas de la existencia de métricas con curvatura seccional positiva y no-negativa.

El primer resultado que conecta la curvatura con la topología es el Teorema de Gauss-Bonnet, que relaciona la integral de la curvatura Gaussiana de una superficie con su característica de Euler. Para una variedad Riemanniana M de dimensión n se pueden definir diferentes nociones de curvatura, y en esta tesis nos vamos a centrar en la curvatura seccional, que tiene fuertes consecuencias cuando se imponen condiciones en ésta. Cuando la curvatura seccional $\sec_M$ de M es constante e igual a K, el cubrimiento universal Riemanniano de M es isométrico a $\mathbb{S}^n$ (si $K = 1$), $\mathbb{R}^n$ (si $K = 0$) ó $\mathbb{H}^n$ (si $K = -1$) con sus métricas canónicas; y se dice que M es una forma espacial esférica (si $K = 1$), Euclídea (si $K = 0$) ó hiperbólica (si $K = -1$). Si asumimos que la curvatura seccional es no-positiva ($\sec_M \leq 0$), el Teorema de Cartan-Hadamard establece que M es difeomorfa a $\mathbb{R}^n$. Por el contrario, en el caso de curvatura seccional no-negativa y positiva no se tiene un grado de conocimiento tan elevado (ver [74]).

Se conocen algunas obstrucciones topológicas para la existencia de métricas de curvatura seccional no-negativa en una variedad compacta M de dimensión n. El Teorema de Gromov establece que existe una constante universal $c(n)$ tal que los números de Betti $b_i(M, \mathbb{F})$ están acotados superiormente por $c(n)$, para cualquier cuerpo de coeficientes $\mathbb{F}$. Además, el grupo fundamental de M tiene un conjunto de generadores con $c(n)$ elementos como máximo. Cheeger y Gromoll demostraron que existe un subgrupo abeliano de $\pi_1(M)$ con índice finito. También ellos determinaron la estructura de variedades abiertas (es decir, no compactas y sin frontera) con curvatura no-negativa en el Soul Theorem (ver más adelante).

Para la existencia de métricas con curvatura seccional positiva sólo se conocen dos obstrucciones adicionales. Sea M una variedad Riemanniana de dimensión n con $\sec_M \geq 1$. El Teorema de Bonnet-Myers establece que el diámetro de M es menor o igual que π, y por lo tanto M es compacta. Su cubrimiento universal Riemanniano $\tilde{M}$ satisface la misma cota $\sec_{\tilde{M}} \geq 1$, por lo que $\tilde{M}$ también es compacta y de ahí se sigue que el grupo fundamental de M es finito. El Teorema de Synge nos dice que $\pi_1(M) = 0$ ó $\mathbb{Z}_2$ cuando n es par, y que M es orientable cuando n es impar. Una consecuencia directa de estos resultados es que ni $\mathbb{S}^n \times \mathbb{S}^1$, ni $\mathbb{RP}^n \times \mathbb{RP}^m$ admiten métricas con curvatura seccional positiva. La clásica conjetura de Hopf plantea si $\mathbb{S}^2 \times \mathbb{S}^2$ admite una métrica con curvatura seccional positiva (recordemos que el producto Riemanniano de dos variedades con curvatura positiva contiene planos tangentes de curvatura seccional cero).

Una gran dificultad presente a la hora de estudiar variedades con curvatura seccional positiva es el escaso número de ejemplos conocidos. Aparte de los espacios simétricos de rango uno compactos (llamados CROSSes), que son las esferas $\mathbb{S}^n$, los espacios proyectivos $\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n$ y el plano de Cayley $\mathrm{Ca}\mathbb{P}^2$, que existen en las dimensiones correspondientes, sólo se conocen ejemplos en dimensiones $6, 7, 12, 13$ y 24. En el estado actual de conocimiento, para la construcción de nuevos ejemplos son necesarias las submersiones Riemannianas: a partir de una variedad con curvatura seccional no-negativa, la idea es construir una submersión Riemanniana sobre otra variedad que pudiera tener curvatura seccional positiva, gracias a la fórmula de O'Neill. En general esta idea es muy difícil de llevar a cabo, lo que sugiere la posibilidad de que haya restricciones para la existencia de submersiones Riemannianas desde una variedad arbitraria de curvatura seccional no-negativa.

El primer capítulo de esta tesis lo dedicaremos a estudiar submersiones Riemannianas $\pi : M^{n+k} \to B^n$ desde una variedad M cerrada (compacta y sin frontera) con curvatura seccional positiva, y examinaremos los posibles valores que puede tomar k, la dimensión de la fibra F^k. Nuestras estimaciones involucran al radio de conjugación de B, denotado por $\mathrm{conj}(B)$; y a la longitud de la geodésica cerrada más corta en B, denotada por $\ell_0(B)$. Recordemos que el radio de conjugación de una variedad Riemanniana es el mínimo sobre las distancias entre puntos conjugados a lo largo de geodésicas; y que en toda variedad compacta (no importa la curvatura) existe al menos una gedésica cerrada. Nuestro primer resultado es el siguiente:

Teorema A. *Sea $\pi : M^{n+k} \to B^n$ una submersión Riemanniana. Si $\sec_M \geq 1$, entonces*

$$k \leq \left(\frac{\pi}{\mathrm{conj}(B)} - 1\right)(n-1), \qquad y \qquad k \leq \left(\frac{2\pi}{\ell_0(B)} - 1\right)(n-1).$$

Obsérvese que, a cambio, el Teorema A nos da una cota superior para $\ell_0(B)$ en términos de n y k. Este hecho sugiere estudiar si alguno de los posibles levantamientos a M de la geodésica cerrada más corta en B (que es geodésica en M) se cierra. En caso afirmativo, tendríamos la desigualdad $\ell_0(M) \leq \ell_0(B)$, y entonces podríamos usar las clásicas cotas inferiores de Heintze-Karcher y Klingenberg para $\ell_0(M)$ dadas en términos de determinados invariantes de M. De esta manera obtendríamos una cota superior para k en términos de invariantes sólo de M. Usaremos Teoría del Punto Fijo de Lefschetz para probar que si la característica de Euler de la fibra no se anula, entonces al menos un levantamiento de una curva diferenciable cerrada en B se cierra en M. De esta manera obtenemos los siguientes resultados:

Teorema B. *Sea* $\pi : M^{n+k} \to B^n$ *una submersión Riemanniana con fibra* F^k*, y supongamos que* $\sec_M \geq 1$.

1. *Si* $\chi(F) \neq 0$*, entonces*

$$k \leq \left(\frac{\mathrm{Vol}(\mathbb{S}^{n+k})}{\mathrm{Vol}\,(M)} - 1 \right) (n-1).$$

2. *Si además* $\chi(M) \neq 0$*, entonces*

$$k \leq (\sqrt{\max \sec_M} - 1)\,(n-1).$$

El método principal para construir submersiones Riemannianas desde una variedad consiste en tomar el cociente bajo una cierta acción por isometrías de un grupo de Lie. De manera más general, se han encontrado numerosas restricciones para la existencia de métricas con curvatura seccional positiva que admitan una cierta acción por isometrías de un grupo de Lie compacto (ver [29]). Por ejemplo, Hsiang y Kleiner demostraron en [41] que $\mathbb{S}^2 \times \mathbb{S}^2$ no admite una métrica de curvatura seccional positiva de manera que el círculo $\mathbb{S}^1$ actúe por isometrías.

La geometría de espacios topológicos X arbitrarios con una acción de un grupo de Lie G que preserva cierta estructura dada tiene interés por sí misma. Vamos a suponer que X y G son compactos y conexos. El caso más restrictivo ocurre cuando X es homogéneo, es decir, la acción del grupo G es transitiva y por lo tanto su espacio de órbitas consta de un solo punto. Si X es una variedad topológica homogénea (respectivamente un orbifold diferenciable homogéneo), entonces es equivariantemente homeomorfa (resp. difeomorfo) a una variedad diferenciable homogénea G/H, donde H denota el grupo de isotropía de la acción. Si X es un espacio de Alexandrov homogéneo (ó en particular un orbifold Riemanniano homogéneo), entonces es equivariantemente isométrico a una variedad Riemanniana homogénea G/H. Recordemos que si la métrica homogénea viene inducida por una métrica bi-invariante en G, entonces G/H tiene curvatura seccional no-negativa y decimos que es una variedad Riemanniana homogénea normal.

La condición de que una acción sea transitiva puede relajarse de diferentes maneras. En este sentido, recordemos que la cohomogeneidad de una acción se define como la dimensión del espacio de órbitas. En el segundo capítulo de esta tesis consideramos acciones diferenciables de cohomogeneidad uno en orbifolds diferenciables cerrados. Los orbifolds son espacios topológicos con una estructura que generaliza la noción de variedad, en el sentido de que son localmente homeomorfos a cocientes de variedades bajo la acción de grupos finitos. Igual que para variedades, un orbifold es cerrado si el espacio topológico subyacente es compacto y no tiene frontera. Recordemos que el cono $C(X)$ sobre un espacio topológico X se define como el espacio cociente $C(X) = (X \times [0,1]) \,/\, (X \times \{0\})$. Obtenemos el siguiente resultado:

Teorema C. *Sea $\mathcal{O}$ un orbifold diferenciable conexo y cerrado con una acción diferenciable y efectiva de un grupo de Lie compacto y conexo G, con grupo de isotropía principal H. Si la acción es de cohomogeneidad uno, entonces el espacio de órbitas $\mathcal{O}/G$ es homeomorfo a un círculo o a un intervalo cerrado y en cada caso se cumple lo siguiente.*

1. *Si el espacio de órbitas es un círculo, entonces $\mathcal{O}$ es equivariantemente difeomorfo a un G/H-fibrado sobre un círculo con grupo de estructura $N(H)/H$, donde $N(H)$ denota el normalizador de H en G. En particular, $\mathcal{O}$ es variedad diferenciable y su grupo fundamental es infinito.*

2. *Si el espacio de órbitas es un intervalo, que podemos suponer que es $[-1, +1]$, entonces:*

   (a) *Hay dos órbitas no-principales, $\pi^{-1}(\pm 1) = G/K_{\pm}$, donde $\pi : \mathcal{O} \to \mathcal{O}/G$ denota la proyección natural y $K_{\pm}$ es el grupo de isotropía de la acción en cualquier punto de la órbita $\pi^{-1}(\pm 1)$.*

   (b) *El conjunto singular del orbifold $\mathcal{O}$ es ó bien vacío, ó bien una de las órbitas no-principales, ó bien ambas órbitas no-principales.*

   (c) *El orbifold $\mathcal{O}$ es equivariantemente difeomorfo al orbifold construido como la unión de dos orbi-fibrados sobre las dos órbitas no-principales y cuyas fibras son conos sobre formas espaciales esféricas, es decir,*

   $$\mathcal{O} \approx G \times_{K_-} C\left(\mathbb{S}_-/\Gamma_-\right) \cup_{G/H} G \times_{K_+} C\left(\mathbb{S}_+/\Gamma_+\right),$$

   donde $\mathbb{S}_{\pm}$ denota la esfera de dimensión $\dim \mathcal{O} - \dim G/K_{\pm} - 1$ y $\Gamma_{\pm}$ es un grupo finito actuando de manera libre y por isometrías en $\mathbb{S}_{\pm}$. La acción queda determinada por el diagrama (G, H, K_-, K_+), donde tenemos las inclusiones de subgrupos $H \leq K_{\pm} \leq G$, y donde $K_{\pm}/H$ son formas espaciales esféricas $\mathbb{S}_{\pm}/\Gamma_{\pm}$.

   (d) *Recíprocamente, un diagrama (G, H, K_-, K_+) con $H \leq K_{\pm} \leq G$ y donde $K_{\pm}/H$ son formas espaciales esféricas, determina un orbifold de cohomogeneidad uno como en el apartado (c).*

Para poner el Teorema C en perspectiva, recordemos que existen teoremas análogos que determinan la estructura de variedades diferenciables, variedades topológicas y espacios de Alexandrov de cohomogeneidad uno (ver [54, 39, 23, 22]). En tales casos, la única diferencia con el Teorema C es que las fibras sobre las dos órbitas no-principales son, respectivamente, conos sobre esferas (es decir, discos), conos sobre esferas o sobre la esfera homológica de Poincaré, y conos sobre variedades Riemannianas homogéneas con curvatura seccional positiva. Como la esfera homológica de Poincaré es una forma espacial esférica, obtenemos el siguiente corolario al Teorema C.

Corolario. *Toda variedad topológica cerrada de cohomogeneidad uno es equivariantemente homeomorfa a un orbifold diferenciable de cohomogeneidad uno.*

En vista del corolario anterior, es natural preguntarse cuándo una variedad topológica cerrada de cohomogeneidad $k \geq 2$ es equivariantemente homeomorfa a un orbifold diferenciable.

El siguiente corolario al Teorema C se sigue del hecho de que la menor dimensión en la que una variedad Riemanniana homogénea con curvatura seccional positiva no es una forma espacial es 4.

Corolario. *Sea X un espacio de Alexandrov cerrado de cohomogeneidad uno. Si la codimensión de ambas órbitas no-principales es como máximo 4, entonces X es equivariantemente homeomorfo a un orbifold diferenciable de cohomogeneidad uno.*

La parte final de esta tesis se centra en variedades abiertas con curvatura seccional no-negativa. Recordemos que el Soul Theorem de Cheeger y Gromoll determina la estructura de dichas variedades: dada M, existe una subvariedad S compacta, sin frontera, totalmente geodésica y totalmente convexa (denominada el "soul" de M) tal que M es difeomorfa al fibrado normal de S.

Una pregunta natural es hasta qué punto se cumple el recíproco del Soul Theorem: dado un fibrado vectorial E sobre una variedad compacta S con curvatura seccional no-negativa, ¿admite E una métrica Riemanniana con curvatura seccional no-negativa con soul S? La respuesta es claramente afirmativa cuando S es una variedad Riemanniana homogénea G/H y E es un fibrado vectorial homogéneo; es decir, un fibrado de la forma $G \times_H \mathbb{R}^m$, donde H actúa en $\mathbb{R}^m$ a través de una representación lineal. Obsérvese que la métrica con curvatura no-negativa en $G \times_H \mathbb{R}^m$ proviene de la submersión Riemanniana $G \times \mathbb{R}^m \to G \times_H \mathbb{R}^m$, gracias a la fórmula de O'Neill. Por el contrario, existen ejemplos de fibrados vectoriales que no admiten métricas con curvatura seccional no-negativa sobre variedades compactas con grupo fundamental no trivial y curvatura seccional no-negativa (ver [57]). La pregunta anterior tiene una respuesta afirmativa para todo fibrado vectorial sobre la esfera $\mathbb{S}^n$ cuando $n \leq 5$ (ver [30]).

Aparte de lo expuesto, sólo se conocen resultados parciales, por lo que se ha considerado una versión más débil de la pregunta inicial: en las mismas condiciones, ¿admite $E \times \mathbb{R}^k$ una métrica Riemanniana con curvatura seccional no-negativa con soul S para algún k? La respuesta en este caso es afirmativa para todo fibrado vectorial sobre todas las esferas $\mathbb{S}^n$ (Rigas, [62]), y sobre las variedades $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ y $\mathbb{CP}^2 \# -\mathbb{CP}^2$ (Grove y Ziller, [31]). En el tercer capítulo probamos que la respuesta a esta última pregunta es afirmativa para todo fibrado vectorial sobre cualquier CROSS. Usaremos resultados previos sobre la K-teoría de dichos espacios para construir un fibrado vectorial homogéneo en cada clase estable de fibrados vectoriales. Recordemos que dos fibrados vectoriales E, F pertenecen a la misma clase estable si existen fibrados triviales k_1, k_2 de manera que $E \oplus k_1$ es isomorfo a $F \oplus k_2$.

Teorema D. *Sea E un fibrado vectorial real arbitrario sobre un espacio simétrico de rango uno compacto S. Denotemos por k el fibrado vectorial trivial de rango k. Entonces, para algún k, la suma de Whitney $E \oplus k = E \times \mathbb{R}^k$ admite una métrica con curvatura seccional no-negativa y soul S.*

Nuestros métodos ofrecen, en el caso de las esferas, una prueba alternativa al Teorema de Rigas, y de hecho, nos permiten acotar superiormente el mínimo entero k que satisface el Teorema D. Para establecer dicho resultado recordemos que el Teorema de Integrabilidad de Bott tiene como consecuencia lo siguiente: si E es un fibrado vectorial real sobre una esfera $\mathbb{S}^n$ de dimensión $n \equiv 0 \pmod 4$, entonces su $(n/4)$-clase de Pontryagin $p_{n/4}(E)$ es de la forma

$$p_{n/4}(E) = ((n/2) - 1)!(\pm \mathsf{p}_E)a$$

para algún numero natural p_E, donde a es un generador de $H^n(\mathbb{S}^n, \mathbb{Z})$.

Teorema E. *Sea E un fibrado vectorial real arbitrario sobre $\mathbb{S}^n$. Sea k_0 el menor entero tal que la suma de Whitney $E \oplus k_0$ admite una métrica con curvatura seccional no-negativa. Entonces se tienen las siguientes desigualdades:*

- $k_0 \leq n + 1$, *si* $n \equiv 3, 5, 6, 7 \ (mod\ 8)$.
- $k_0 \leq 2^n$, *si* $n \equiv 1, 2 \ (mod\ 8)$.
- $k_0 \leq \max\{n + 1, 2^{n-1}\mathsf{p}_E\}$, *si* $n \equiv 0, 4 \ (mod\ 8)$.

Por último también obtenemos resultados para fibrados vectoriales complejos sobre otras variedades:

Teorema F. *Sea E un fibrado vectorial complejo arbitrario sobre una variedad S en alguna de las dos clases de variedades $\mathscr{C}_i$ siguientes:*

- *$\mathscr{C}_1$ es la clase de variedades compactas S con curvatura no-negativa tales que sus números de Betti pares $b_{2i}(S)$ se anulan para $i \geq 1$, y el anillo $H^*(S, \mathbb{Z})$ es libre de torsión.*
- *$\mathscr{C}_2$ es la clase de variedades Riemannianas homogéneas compactas G/H tales que G es un grupo de Lie compacto y conexo, $\pi_1(G)$ es libre de torsión y H es un subgrupo cerrado y conexo de rango máximo.*

Denotemos por k el fibrado vectorial complejo trivial de rango k. Entonces, para algún k, la suma de Whitney $E \oplus k = E \times \mathbb{C}^k$ admite una métrica con curvatura seccional no-negativa y soul S.

La clase $\mathscr{C}_1$ incluye todas las esferas homológicas de dimensión impar que admitan curvatura seccional no-negativa, por ejemplo la esfera exótica de dimensión 7 de Gromoll y Meyer. La clase $\mathscr{C}_2$ incluye variedades como las esferas de dimensión par, variedades Grassmannianas complejas y cuaterniónicas, las variedades de Wallach W^6, W^{12} y W^{24} o el plano de Cayley.

Introduction and statement of results

Every smooth manifold M can be endowed with a Riemannian metric. The question is then whether M admits a metric with certain geometric conditions. The conditions that have been classically studied are lower or upper bounds for the curvature, the volume or the diameter, although many other concepts can be defined from a Riemannian metric (see [58] for a survey by Petersen). The motivation and the goal of this thesis is the study of geometric and topological consequences of positive and nonnegative sectional curvature.

The first result in the study of the topological implications of curvature is the Gauss-Bonnet Theorem, which relates the integral of the Gaussian curvature of a surface with its Euler characteristic. For a Riemannian manifold M of arbitrary dimension n, several notions of curvature can be defined from its curvature tensor. Sectional curvature turns out to be very restrictive and has strong implications. When the sectional curvature $\sec_M$ of M is constant and equal to K, then the Riemannian universal covering of M is isometric to $\mathbb{S}^n$ (if $K = 1$), $\mathbb{R}^n$ (if $K = 0$) or $\mathbb{H}^n$ (if $K = -1$) with their canonical metrics; M is called a spherical (if $K = 1$), Euclidean (if $K = 0$) or hyperbolic (if $K = -1$) space form. If we allow the sectional curvature to be nonpositive ($\sec_M \leq 0$), then Cartan-Hadamard's Theorem states that the universal cover of M is diffeomorphic to $\mathbb{R}^n$ via the exponential map at any point. However, the case of nonnegative and, in particular, positive sectional curvature is not so well understood (see [74] for a survey by Ziller).

For a compact manifold M of dimension n admitting a metric of nonnegative sectional curvature ($\sec_M \geq 0$) one has topological obstructions. Gromov's Theorem states that there exists a universal constant $c(n)$ such that the Betti numbers $b_i(M, \mathbb{F})$ are bounded above by $c(n)$, for any field of coefficients $\mathbb{F}$. Furthermore, the fundamental group of M has a generating set with at most $c(n)$ elements. Cheeger and Gromoll proved that there exists an abelian subgroup of $\pi_1(M)$ with finite index. They also determined the structure of open manifolds (i.e., non-compact and without boundary) with nonnegative sectional curvature in the so-called Soul Theorem (see below).

For the existence of positively curved metrics one has in addition only the two classical obstructions. Let M be a Riemannian manifold of dimension n with $\sec_M \geq 1$. Bonnet-Myers's Theorem states that the diameter of M is at most π, hence M is compact. The Riemannian universal covering $\tilde{M}$ of M satisfies the same curvature bound, so $\tilde{M}$ is compact and therefore the fundamental group of M is finite. Synge's Theorem states that $\pi_1(M) = 0$ or $\mathbb{Z}_2$ if n is even, and that M is orientable if n is odd. A direct consequence of these results is that neither $\mathbb{S}^n \times \mathbb{S}^1$ nor $\mathbb{RP}^n \times \mathbb{RP}^m$ admit a metric with positive curvature. A long-standing conjecture by Hopf asks if $\mathbb{S}^2 \times \mathbb{S}^2$ admits a positively curved metric (observe that the Riemannian product of two positively curved manifolds contains tangent 2-planes of vanishing sectional curvature).

The main difficulty when studying positively curved manifolds is the small number of known examples. Besides the compact rank one symmetric spaces (CROSSes), namely the spheres $\mathbb{S}^n$, the projective spaces $\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n$ and the Cayley plane $\mathrm{Ca}\mathbb{P}^2$, which appear in the corresponding dimensions, there exist examples only in dimensions $6, 7, 12, 13$ and 24. New examples appear in increasing periods of time, and at the present state of knowledge, Riemannian submersions are necessary in their construction: starting with the correct manifold with nonnegative sectional curvature as total space, one searches for some submersion that would guarantee a positively curved base thanks to the well-known O'Neill's formula. However, this is not so easily done, pointing out to the possible presence of restrictions on the existence of such Riemannian submersions from an arbitrary nonnegatively curved manifold.

In the first chapter of this thesis we consider Riemannian submersions $\pi : M^{n+k} \to B^n$ from closed (i.e., compact and without boundary) positively curved manifolds M, and we study the possible values for k, the dimension of the fiber F^k. Our estimates involve the conjugate radius of B, denoted by $\mathrm{conj}(B)$; and the length of the shortest closed geodesic in B, denoted by $\ell_0(B)$. Recall that the conjugate radius of a manifold is the minimum over the distances between conjugate points along geodesics; and that in every compact manifold (without curvature assumptions) there exist at least one closed geodesic. Our first result is:

Theorem A. *Let $\pi : M^{n+k} \to B^n$ be a Riemannian submersion. If $\sec_M \geq 1$, then:*

$$k \leq \left(\frac{\pi}{\mathrm{conj}(B)} - 1 \right)(n-1), \qquad \textit{and} \qquad k \leq \left(\frac{2\pi}{\ell_0(B)} - 1 \right)(n-1).$$

Note that in turn Theorem A gives an upper bound for $\ell_0(B)$ in terms of n and k. This suggest to study if any of the possible horizontal lifts of the shortest closed geodesic in B to M (which is a geodesic in M) is closed. This would give us the inequality $\ell_0(M) \leq \ell_0(B)$, and then one could use the classical lower bounds for $\ell_0(M)$ given in terms of suitable invariants of M by Heintze-Karcher and Klingenberg. That way we would get an upper bound for k in terms of invariants of the total space M. Using Lefschetz Fixed-Point Theory we prove that if the fiber has nonzero Euler characteristic then for a given smooth closed curve in B there is a lift which is closed in M. We get the following results:

Theorem B. *Let $\pi : M^{n+k} \to B^n$ be a Riemannian submersion with fiber F^k. Suppose that* $\sec_M \geq 1$.

1. *If $\chi(F) \neq 0$, then*
$$k \leq \left(\frac{\mathrm{Vol}(\mathbb{S}^{n+k})}{\mathrm{Vol}\,(M)} - 1 \right)(n-1).$$

2. *If in addition $\chi(M) \neq 0$, then*
$$k \leq (\sqrt{\max \sec_M} - 1)\,(n-1).$$

The main source to construct Riemannian submersions from a manifold is taking the quotient under certain isometric actions of a Lie group. More generally, many obstructions to the existence of a positively curved metric in a manifold have been developed under the assumption of the presence of an isometric action on the manifold (see the survey [29] by Grove). For example, Hsiang and Kleiner showed in [41] that $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit a Riemannian metric of positive sectional curvature such that the circle $\mathbb{S}^1$ acts by isometries on it.

The geometry of arbitrary topological spaces X with a certain action of a Lie group G preserving a given structure is of particular interest. We assume that X and G are compact and connected. The most restrictive case occurs when X is homogeneous, i.e., the G-action preserving the given structure is transitive and hence its orbit space is just a point. If X is a homogeneous topological manifold (respectively smooth orbifold), then it is equivariantly homeomorphic (resp. diffeomorphic) to a homogeneous smooth manifold G/H, where H denotes the isotropy group of the action. If X is a homogeneous Alexandrov space (or in particular a Riemannian orbifold), then it is equivariantly isometric to a homogeneous Riemannian manifold G/H. Recall that if the homogeneous metric descends from a bi-invariant metric on G, then G/H has nonnegative sectional curvature and it is called a normal homogeneous Riemannian manifold.

The transitivity condition of the action can be relaxed in different ways. Recall that the cohomogeneity of the action is defined to be the dimension of its orbit space. In the second chapter of this thesis we study cohomogeneity one smooth actions of compact Lie groups on closed, smooth orbifolds. Orbifolds are topological spaces that generalize the notion of manifold in the sense that they are locally homeomorphic to quotients of manifolds under the action of finite groups. Orbifolds were introduced by Satake in the 1950s under the name of V-manifolds, and then Thurston studied these spaces extensively in [66], where he used the terminology orbifold. As for manifolds, a smooth orbifold is closed if its underlying topological space is compact and has no boundary. Recall that the cone $C(X)$ over a topological space X is defined as the quotient space $C(X) = (X \times [0,1])\,/\,(X \times \{0\})$. We obtain the following result:

Theorem C. *Let $\mathcal{O}$ be a closed, connected, smooth orbifold with an (almost) effective smooth action of a compact, connected Lie group G with principal isotropy group H. If the action is of cohomogeneity one, then the orbit space $\mathcal{O}/G$ is homeomorphic to a circle or to a closed interval and the following statements hold.*

1. *If the orbit space is a circle, then $\mathcal{O}$ is equivariantly diffeomorphic to a G/H-bundle over a circle with structure group $N(H)/H$, where $N(H)$ is the normalizer of H in G. In particular, $\mathcal{O}$ is a manifold and its fundamental group is infinite.*

2. *If the orbit space is homeomorphic to an interval, say $[-1, +1]$, then:*

   (a) *There are two non-principal orbits, $\pi^{-1}(\pm 1) = G/K_\pm$, where $\pi : \mathcal{O} \to \mathcal{O}/G$ is the natural projection and $K_\pm$ is the isotropy group of the G-action at a point in $\pi^{-1}(\pm 1)$.*

   (b) *The orbifold singular set of $\mathcal{O}$ is either empty, a non-principal orbit or both non-principal orbits.*

   (c) *The orbifold $\mathcal{O}$ is equivariantly diffeomorphic (as orbifolds) to the union of two orbifiber bundles over the two non-principal orbits whose fibers are cones over spherical space forms, that is,*

   $$\mathcal{O} \approx G \times_{K_-} C\left(\mathbb{S}_-/\Gamma_-\right) \cup_{G/H} G \times_{K_+} C\left(\mathbb{S}_+/\Gamma_+\right),$$

   where $\mathbb{S}_\pm$ denotes the round sphere of dimension $\dim \mathcal{O} - \dim G/K_\pm - 1$ and $\Gamma_\pm$ is a finite group acting freely and by isometries on $\mathbb{S}_\pm$. The action is determined by a group diagram (G, H, K_-, K_+) with group inclusions $H \leq K_\pm \leq G$ and where $K_\pm/H$ are spherical space forms $\mathbb{S}_\pm/\Gamma_\pm$.

   (d) *Conversely, a group diagram (G, H, K_-, K_+) with $H \leq K_\pm \leq G$ and where $K_\pm/H$ are spherical space forms, determines a cohomogeneity one orbifold as in part (c).*

To put Theorem C into perspective, recall that there exist analogous structure results for cohomogeneity one actions on closed smooth manifolds, on closed topological manifolds and on closed Alexandrov spaces (cf. [54, 39, 23, 22]). In these cases, the only difference with Theorem C is that the fibers over the non-principal orbits are, respectively, cones over a round sphere (i.e. balls), cones over a round sphere or the Poincaré homology sphere (i.e. homology balls), and cones over a homogeneous positively curved Riemannian manifold. Since the Poincaré homology sphere is a spherical space form, the following corollary follows from Theorem C.

Corollary. *Every closed cohomogeneity one topological manifold is equivariantly homeomorphic to a smooth cohomogeneity one orbifold.*

It is thus natural to ask when is a cohomogeneity $k \geq 2$ closed topological manifold equivariantly homeomorphic to a smooth orbifold.

The following corollary to Theorem C follows from the fact that the lowest dimension where a homogeneous positively curved manifold is not a space form is 4.

Corollary. *Let X be a closed Alexandrov space of cohomogeneity one. If the codimension of both non-principal orbits is at most* 4*, then X is equivariantly homeomorphic to a smooth cohomogeneity one orbifold.*

The last part of this thesis focuses on open manifolds with nonnegative sectional curvature. Recall that the Soul Theorem by Cheeger and Gromoll determines the structure of such a manifold M: there exists a compact, totally geodesic and totally convex submanifold S (called the soul of M) without boundary such that M is diffeomorphic to the normal bundle of S.

A natural question is then to what extent a converse to the Soul Theorem holds: given a vector bundle E over a compact manifold S with nonnegative sectional curvature, does E admit a complete metric of nonnegative curvature with soul S? The answer is clearly affirmative when S is a homogeneous manifold G/H and E is a homogeneous vector bundle; that is, a bundle of the form $G \times_H \mathbb{R}^m$, where H acts on $\mathbb{R}^m$ by means of a linear representation. Observe that the nonnegatively curved metric on $G \times_H \mathbb{R}^m$ comes from the Riemannian submersion $G \times \mathbb{R}^m \to G \times_H \mathbb{R}^m$, thanks to O'Neill's formula. A negative answer was found for certain bundles over compact nonsimply connected manifolds (see [57]). The question above also has a positive answer for every vector bundle over the round spheres $\mathbb{S}^n$, with $n \leq 5$ (Grove and Ziller, [30]).

Besides that, there are only partial results, and a weaker question has been studied: in the conditions above, does $E \times \mathbb{R}^k$ admit a metric of nonnegative curvature with soul S for some k? The answer in this case is affirmative for every vector bundle over all round spheres $\mathbb{S}^n$ (Rigas, [62]), and over the manifolds $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{CP}^2 \# -\mathbb{CP}^2$ (Grove and Ziller, [31]). In Chapter 3 we give an affirmative answer for every vector bundle over any CROSS. We use previous results on the K-theory of these spaces in order to find a homogeneous vector bundle in every stable class of vector bundles. Recall that two vector bundles E, F are stably equivalent if there exist trivial bundles k_1, k_2 such that $E \oplus k_1$ is isomorphic to $F \oplus k_2$.

Theorem D. *Let E be an arbitrary real vector bundle over a compact rank one symmetric space S. Denote by k the trivial vector bundle of rank k. Then, for some k the Whitney sum $E \oplus k = E \times \mathbb{R}^k$ admits a metric with nonnegative sectional curvature and soul S.*

In the case of the sphere our methods yield an alternative proof of Rigas' Theorem. Moreover, our approach allows us to give an upper bound for the least integer k satisfying Theorem D. In order to state our result we need to recall that, as a consequence of the Bott Integrability Theorem, if E is a real vector bundle over a sphere $\mathbb{S}^n$ of dimension $n \equiv 0 \pmod 4$, then its $(n/4)$-th Pontryagin class $p_{n/4}(E)$ is of the form

$$p_{n/4}(E) = ((n/2) - 1)!(\pm \mathsf{p}_E)a$$

for some natural number p_E, where a is a generator of $H^n(\mathbb{S}^n, \mathbb{Z})$. We obtain the following bounds:

Theorem E. *Let E be an arbitrary real vector bundle over $\mathbb{S}^n$. Let k_0 be the least integer such that the Whitney sum $E \oplus k_0$ admits a metric with nonnegative sectional curvature. The following inequalities hold:*

- $k_0 \leq n+1$, *if* $n \equiv 3, 5, 6, 7$ $(mod\ 8)$.
- $k_0 \leq 2^n$, *if* $n \equiv 1, 2$ $(mod\ 8)$.
- $k_0 \leq \max\{n+1, 2^{n-1}\mathsf{p}_E\}$, *if* $n \equiv 0, 4$ $(mod\ 8)$.

We also obtain results for complex vector bundles over other manifolds:

Theorem F. *Let E be an arbitrary complex vector bundle over a manifold S in one of the two following classes $\mathscr{C}_i$:*

- *$\mathscr{C}_1$ is the class of compact nonnegatively curved manifolds S whose even dimensional Betti numbers $b_{2i}(S)$ vanish for $i \geq 1$, and such that $H^*(S, \mathbb{Z})$ is torsion-free.*
- *$\mathscr{C}_2$ is the class of compact homogeneous spaces G/H such that G is a compact, connected Lie group with $\pi_1(G)$ torsion-free and H a closed, connected subgroup of maximal rank.*

Denote by k the trivial complex vector bundle of rank k. Then, for some k the Whitney sum $E \oplus k = E \times \mathbb{C}^k$ admits a metric with nonnegative sectional curvature and soul S.

Odd-dimensional homology spheres admiting nonnegatively curved metrics belong to class $\mathscr{C}_1$, in particular the 7-dimensional Gromoll-Meyer exotic sphere. The class $\mathscr{C}_2$ includes such manifolds as even-dimensional spheres, complex and quaternionic Grassmannian manifolds, the Wallach flag manifolds W^6, W^{12} and W^{24} or the Cayley plane.

Notation and conventions

In this thesis we assume that the reader is familiar with some background on Differential and Riemannian geometry, see the references [12, 17, 47, 63, 69] for a detailed discussion. By *smooth* we will always mean infinitely differentiable. We will denote a Riemannian metric on a smooth manifold M by $\langle , \rangle$. The *norm* of a vector $v \in T_pM$ in the tangent space of M at p will be denoted by $\|v\| = \langle v, v\rangle^{\frac{1}{2}}$. Denote by ∇ the Levi-Civita connection associated to the Riemannian manifold $(M, \langle , \rangle)$. Let $\alpha : I \to M$ be a curve in M and let $X(t)$ be a vector field along α. We will denote by $\alpha'(t)$ the velocity vector of α, and by $X'(t)$ the covariant derivative $\nabla_{\alpha'(t)}X(t)$. Recall that *geodesics* are curves $\alpha : I \to M$ such that $\alpha''(t) \equiv 0$.

Let X, Y, Z be vector fields on M. For the definition of the *curvature tensor* we adopt the following convention as in [69]:

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where $[X, Y]$ denotes the Lie bracket of X and Y.

For a given point p in M, let $\Pi \subset T_pM$ be a 2-plane with orthonomal basis X, Y. The *sectional curvature* $\sec_M \Pi$ of Π is defined as:

$$\sec_M \Pi = \langle R(X, Y)Y, X\rangle.$$

Recall that the sectional curvature of a 2-plane Π has the following geometric interpretation. Let S_Π be the 2-dimensional submanifold of M consisting of geodesics whose initial tangent vectors lie in Π. Then $\sec_M(\Pi)$ equals the Gaussian curvature of the surface S_Π at the point p.

Given $K \in \mathbb{R}$, we say that $\sec_M \geq K$ (resp. $\sec_M \leq K$) if for every point $p \in M$ and every 2-plane $\Pi \subset T_pM$, the sectional curvature satisfies $\sec_M \Pi \geq K$ (resp. $\sec_M \Pi \leq K$). Note that if $\sec_M \geq K > 0$, we can rescale the metric so that $\sec_M \geq 1$. If $\sec_M \equiv 0$, we say that M is *flat*.

When dealing with groups $*$ such as the fundamental group or the group of stable classes of vector bundles over a compact manifold, we will write $* = 0$ to denote that it only consists of the identity element.

LIST OF SYMBOLS

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integer numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{H}$	Quaternionic numbers
$\mathbb{F}^k$	Cartesian product $\mathbb{F} \times \cdots \times \mathbb{F}$ of the set $\mathbb{F}$ with itself k-times
$\mathbb{Z}_k$	The cyclic subgroup of k elements $\mathbb{Z}/k\mathbb{Z}$
$\mathbb{S}^n$	n-dimensional sphere

Chapter 1

Soft restrictions on positively curved Riemannian submersions

In this chapter we study Riemannian submersions from positively curved manifolds. We assume that all Riemannian manifolds are complete. We will write M^n to denote that the dimension of the manifold M is n. Our motivation is the following conjecture (attributed to F. Wilhelm).

Conjecture. *Let $\pi : M^{n+k} \to B^n$ be a Riemannian submersion between compact positively curved Riemannian manifolds. Then $k \leq n-1$.*

In the very rigid case where the fibers are totally geodesic the conjecture holds by O'Neill's formula (see Section 1.2.2). In the general case, partial progress towards the conjecture appears in the thesis of W. Jiménez [43] where he used results of Kim and Tondeur [44] to obtain that if $\sec_M \geq 1$, then

$$k \leq \frac{1}{3}\left(\max \sec_B - 1\right)(n-1)\,, \tag{1.0.1}$$

where $\max \sec_B$ denotes the maximum of the sectional curvatures in B. It is worth noticing that O'Neill's formula together with [68] guarantees that $\max \sec_B > 1$, and therefore the right hand side in (1.0.1) is positive.

For a different type of restrictions using rational homotopy theory methods, see [4].

In this chapter, we examine the index of Lagrangian subspaces of Jacobi fields (see Section 1.1.1 for the definitions) along horizontal geodesics to prove:

Theorem 1.1. *Let M^{n+k}, B^n be compact Riemannian manifolds with $\sec \geq 1$, and let $\pi : M^{n+k} \to B^n$ be a Riemannian submersion with fiber F^k. Then*

$$k \leq \left(\frac{\pi}{\mathrm{conj}(B)} - 1\right)(n-1),$$

where $\mathrm{conj}(B)$ denotes the conjugate radius of B.

Since the conjugate radius of a positively curved manifold B is at least $\pi/\sqrt{\max \sec_B}$, Theorem 1.1 gives the following improvement of Jimenez's result:

Corollary 1.2. *Under the conditions of Theorem 1.1,*

$$k \leq (\sqrt{\max \sec_B} - 1)\,(n-1)\,.$$

This bound is better than Jimenez's when $\max \sec_B > 4$.

The arguments in the proof of Theorem 1.1 extend to Riemannian foliations, giving the following bound in terms of the focal radius of the foliation (the definition is included at the end of Section 1.2).

Corollary 1.3. *Let $\mathcal{F}$ be a Riemannian foliation with leaves of dimension k in an $n+k$-dimensional compact manifold M with $\sec_M \geq 1$. Then*

$$k \leq \left(\frac{\pi}{\mathrm{foc}(\mathcal{F})} - 1\right)(n-1),$$

where $\mathrm{foc}(\mathcal{F})$ denotes the focal radius of the foliation.

It is also possible to give bounds on the fiber dimension related to the length of the shortest nontrivial closed geodesic in the base (that exists by a theorem of Fet and Lyusternik [19]).

Theorem 1.4. *Let M^{n+k}, B^n be compact Riemannian manifolds with $\sec \geq 1$, and let $\pi : M^{n+k} \to B^n$ be a Riemannian submersion with fiber F^k. Denote by $\ell_0(B)$ the length of the shortest closed geodesic in B. Then*

$$k \leq \left(\frac{2\pi}{\ell_0(B)} - 1\right)(n-1).$$

Observe that in turn Theorem 1.4 gives an upper bound for the length of the shortest closed geodesic in the base manifold B^n of a Riemannian submersion from a manifold M^{n+k} with $\sec_M \geq 1$. Specifically:

$$\ell_0(B) \leq \frac{2\pi(n-1)}{n+k-1}.$$

Motivated by this inequality we study if any of the lifts to M of a closed geodesic (and more generally a picewise smooth curve) c in B closes in the first lap. To do this we apply Lefschetz Fixed-Point Theory to the associated holonomy diffeomorphism $h_c : F \to F$ of the fiber F of the submersion. Denote by $\chi(F)$ the Euler characteristic of the manifold F. We obtain the following result:

Theorem 1.5. *Let $\pi : M \to B$ be a Riemannian submersion with fiber F, and let c be a picewise smooth closed curve in B. If B is simply connected and $\chi(F) \neq 0$, then there is a horizontal lift of c to M that closes in the first lap.*

In the conditions of Theorem 1.5 clearly $\ell_0(M) \leq \ell_0(B)$, where $\ell_0(M)$ denotes the length of the shortest closed geodesic in M. Using this result together with a lower bound for the length of closed geodesics in positively curved manifolds given by Heintze and Karcher, we are able to give an upper bound for the dimension of the fiber in terms of the volume and the dimension of the total space M.

Theorem 1.6. *Let M^{n+k}, B^n be compact Riemannian manifolds with* $\sec \geq 1$*, and let $\pi : M^{n+k} \to B^n$ be a Riemannian submersion with fiber F^k. If $\chi(F) \neq 0$, then*

$$k \leq \left(\frac{\mathrm{Vol}(\mathbb{S}^{n+k})}{\mathrm{Vol}\,(M)} - 1 \right)(n-1),$$

where $\mathbb{S}^{n+k}$ denotes the $n+k$-dimensional sphere of constant curvature equal to 1.

If we require the stronger assumption $\chi(M) \neq 0$, then M is even-dimensional and we can use a result by Klingenberg on a lower bound for the length of closed geodesics in even-dimensional positively curved manifolds to get the following result.

Theorem 1.7. *Let M^{n+k}, B^n be compact Riemannian manifolds with* $\sec \geq 1$*, and let $\pi : M^{n+k} \to B^n$ be a Riemannian submersion. If $\chi(M) \neq 0$, then*

$$k \leq (\sqrt{\max \sec_M} - 1)\,(n-1),$$

where $\max \sec_M$ *denotes the maximum of* $\sec_M$.

The chapter is organized as follows: Sections 1.1 and 1.2 give some preliminaries on the Jacobi equation and on the theory of Riemannian submersions respectively. In Section 1.3 we obtain bounds for the index of Lagrangian subspaces of Jacobi fields in several situations needed for the proofs of Theorems 1.1 and 1.4. Section 1.4 studies the existence of closed lifts of a closed curve and contains the proof of Theorem 1.5. The proofs of the remaining Theorems are contained in Section 1.5.

The results in this chapter are joint work with my thesis advisor, Luis Guijarro. Most of the results are contained in [25]. Theorem 1.4 above is an improved version of Theorem B in [25].

1.1 The Jacobi equation

Given a geodesic α in a Riemannian manifold M, a *Jacobi field* is defined as a vector field $J(t)$ along $\alpha(t)$ satisfying the equation:

$$J''(t) + R(J(t), \alpha'(t))\alpha'(t) = 0.$$

Jacobi fields arise naturally from geodesic variations of the geodesic α. We only consider *normal* Jacobi fields, i.e., those $J(t)$ which remain orthogonal to the velocity vector $\alpha'(t)$.

Two points $\alpha(t_0)$ and $\alpha(t_1)$ are *conjugate points along* α if there exist a nonvanishing Jacobi field J along α such that $J(t_0) = J(t_1) = 0$. The *conjugate radius* $\mathrm{conj}_p(M)$ at $p \in M$ is defined as

$$\mathrm{conj}_p(M) = \inf_{\substack{\alpha \text{ is a geodesic} \\ \text{with } \alpha(0)=p}} \sup \{t : \alpha(t) \text{ is not a conjugate point to } p\}.$$

It is natural to define the conjugate radius of a Riemannian manifold as:

Definition 1.8. The *conjugate radius* $\mathrm{conj}(M)$ of a Riemannian manifold M is

$$\mathrm{conj}(M) = \inf \left\{\mathrm{conj}_p(M) : p \in M\right\}.$$

Jacobi fields play an important role in the study of Riemannian manifolds with prescribed sectional curvature. This is due to Rauch's comparison Theorem:

Theorem 1.9 (Rauch)**.** *Let $M, \tilde{M}$ be Riemannian manifolds, let $\alpha : [0,T] \to M$ and $\tilde{\alpha} : [0,T] \to \tilde{M}$ be unit speed geodesics such that $\tilde{\alpha}(t)$ is not a conjugate point to $\tilde{\alpha}(0)$ along $\tilde{\alpha}$ for any $(0,T]$, and let $J, \tilde{J}$ be Jacobi fields along α and $\tilde{\alpha}$ respectively such that $J(0) = 0 = \tilde{J}(0)$ and $\|J'(0)\| = \|\tilde{J}'(0)\|$. Suppose that the sectional curvatures of 2-planes Π and $\tilde{\Pi}$ containing $\alpha'(t)$ and $\tilde{\alpha}'(t)$ respectively satisfy $\sec_M(\Pi) \leq \sec_{\tilde{M}}(\tilde{\Pi})$. Then*

$$\|J(t)\| \geq \|\tilde{J}(t)\|, \qquad \textit{for all } t \in (0,T].$$

Observe that Rauch's Comparison Theorem implies that $\mathrm{conj}(M) = \infty$ if $\sec_M \leq 0$, and $\mathrm{conj}(M) \geq \pi/\sqrt{\max \sec_M}$ otherwise.

The notion of conjugate point is a special case of the following concept. Consider a submanifold $N \subset M$ and a geodesic α in M with $\alpha(t_0) \in N$ such that $\alpha'(t_0)$ is orthogonal to the tangent space $T_{\alpha(t_0)}N$. A Jacobi field J along α is said to be a *N-Jacobi field* if it satisfies the initial conditions:

$$J(0) \in T_{\alpha(t_0)}N, \quad \text{and} \quad J'(0) + S_{\alpha'(t_0)}J(0) \perp T_{\alpha(t_0)}N.$$

where $S_{\alpha'(t_0)}$ denotes the second fundamental form of N in the orthogonal direction $\alpha'(t_0)$. We say that $\alpha(t_1)$ is a *focal point of N along α* if there exists a nonvanishing N-Jacobi field J along α such that $J(t_1) = 0$.

The *focal radius of* N at $p \in N$ is defined as

$$\mathrm{foc}_p(N) = \inf_{\substack{\alpha \text{ is a geodesic} \\ \text{with } \alpha(0)=p \\ \text{and } \alpha'(t_0) \perp T_{\alpha(t_0)}N}} \sup \{t : \alpha(t) \text{ is not a focal point of } N\}.$$

It is natural to define the focal radius of N:

Definition 1.10. The *focal radius of a submanifold $N \subset M$* in a Riemannian manifold M is

$$\mathrm{foc}(N) = \inf \{\mathrm{foc}_p(N) : p \in N\}.$$

Jacobi fields can be treated in a more general way as the following subsection shows.

1.1.1 Jacobi fields in an abstract setting

This section collects a few facts on the Jacobi equation from [48]. Let $\mathbb{E}$ be a Euclidean vector space of dimension m with positive definite inner product $\langle\,,\,\rangle$. For a smooth one-parameter family of self adjoint linear maps $R : \mathbb{R} \to \mathrm{Sym}(\mathbb{E})$, we consider the equation $J''(t) + R(t)J(t) = 0$ whose solutions we refer to as *R-Jacobi fields* (or just *Jacobi fields* if it is clear from the context to what R we refer). We denote by Jac^R the space of Jacobi fields, a vector space of dimension $2m$; Jac^R is a symplectic vector space with form

$$\omega : \mathrm{Jac}^R \times \mathrm{Jac}^R \to \mathbb{R}, \quad \omega(J_1, J_2) = \langle J_1, J_2'\rangle - \langle J_1', J_2\rangle$$

where the right hand side of ω is independent of the t chosen.

A subspace W is called *isotropic* when ω vanishes in W; a maximal isotropic subspace is called a *Lagrangian subspace*, or simply, a Lagrangian. Since ω is nondegenerate, it is clear that Lagrangian subspaces are just isotropic subspaces of dimension m; in the literature, Lagrangian spaces have often been called *maximal self-adjoint spaces for the Jacobi operator* (see for instance [67] and [71]).

Since the inner product of $\mathbb{E}$ is positive definite, zeros of Jacobi fields are isolated; we should mention that this is not true in the case of nonzero signature, as was noticed in [37] and further studied in [60]. Therefore, if $I \subset \mathbb{R}$ is an interval, we can define the *index* of an isotropic subspace $W \subset \mathrm{Jac}^R$ in I as the number of times (with multiplicity) that fields in W vanish in I; we will denote this index as $\mathrm{ind}_W I$. More precisely,

$$\mathrm{ind}_W I = \sum_{t \in I} \dim \{J \in W \,:\, J(t) = 0\}.$$

As an example, define L_0 to be the subspace of Jacobi fields J along a geodesic α in a Riemannian manifold such that $J(0) = 0$. A direct computation shows that L_0 is a Lagrangian subspace. Observe that $\mathrm{ind}_{L_0}(0, b]$ equals the number of conjugate points (with multiplicities) to $\alpha(0)$ along α in the interval $(0, b]$.

Similarly, given a geodesic α in a Riemannian manifold which is orthogonal to a submanifold $N \subset M$, define L^N to be the subspace of N-Jacobi fields along α. Then L^N is a Lagrangian subspace and $\mathrm{ind}_{L_0}(0, b]$ equals the number of focal points (with multiplicities) to N along α in the interval $(0, b]$.

The indexes of different Lagrangians along the same interval are related by the following inequality in [48].

Proposition 1.11. *Let $\mathbb{E}, R, \mathrm{Jac}^R$ be as previously described. Then for any Lagrangians $L_1, L_2 \subset \mathrm{Jac}^R$ and any interval $I \subset \mathbb{R}$, we have*

$$|\mathrm{ind}_{L_1} I - \mathrm{ind}_{L_2} I| \leq \dim \mathbb{E} - \dim (L_1 \cap L_2). \tag{1.1.1}$$

1.1.2 The transverse Jacobi equation

Let W be an isotropic subspace of Jacobi fields of $(\mathbb{E}, R)$. For a fixed $t \in \mathbb{R}$, we define

$$W(t) = \{J(t) \,:\, J \in W\}, \quad W^t = \{J \in W \,:\, J(t) = 0\}.$$

For each $t \in \mathbb{R}$, the subspace

$$\overline{W}(t) = W(t) \oplus \left\{ J'(t) \,:\, J \in W^t \right\}$$

varies smoothly on t as was shown in [71]; denote by $H(t)$ its orthogonal complement, and by $e = e^h + e^v$ the splitting of a vector under the sum $\mathbb{E} = H(t) \oplus \overline{W}(t)$. We use $\mathcal{H}$ to denote the vector bundle over $\mathbb{R}$ formed by the $H(t)$. There is a covariant derivative on $\mathcal{H}$ induced from $\mathbb{E}$ as follows. If $X : \mathbb{R} \to \mathbb{E}$ is a section of $\mathcal{H}$, we define

$$\frac{D^h X}{dt}(t) = X'(t)^h.$$

The covariant derivative D^h/dt defines parallel sections, and preserves the inner product induced on $\mathcal{H}$ from $\mathbb{E}$. Let $\mathbb{E}_1$ be an inner vector space of dimension the rank of $\mathcal{H}$; using a parallel trivialization of $\mathcal{H}$, we can identify sections of $\mathcal{H}$ with maps $X : \mathbb{R} \to \mathbb{E}_1$, and the covariant derivative D^h/dt with standard derivation.

Modulo these identifications, Wilking's transverse equation reads as

$$X''(t) + R^W(t)X(t) = 0, \quad R^W(t)X(t) = [R(t)X(t)]^h + 3A_t A_t^* X(t),$$

where $A_t : W(t) \to H(t)$ denotes the linear map defined as follows. For a vector $u \in W(t)$, choose a Jacobi field $J_u \in W$ with $J_u(t) = u$, then $A_t(u) = J_u'(t)^h$. By $A_t^* : H(t) \to W(t)$ we denote the adjoint of A_t.

Thus we obtain a new Jacobi setting $(\mathbb{E}_1, R^W)$ with R^W as the new curvature operator used to construct the transverse Jacobi equation; Wilking proved that the projection of any R-Jacobi field onto $\mathcal{H}$ is a solution of the transverse equation, i.e. an R^W-Jacobi field. Moreover, as Lytchak observed, any Lagrangian for $(\mathbb{E}_1, R^W)$ is obtained projecting some Lagrangian that contains W and vice versa.

1.2 Review of Riemannian submersions

In this section we recall briefly some of the main facts about *Riemannian submersions*. The reader can find more information about this topic in [10, 28, 56]; in particular we will use the notation from [28]. First we recall the definition of (smooth) submersion.

Definition 1.12. Let M^{n+k} and B^n be $n+k$ and n-dimensional manifolds respectively and $\pi : M \to B$ a surjective smooth map. We say that π is a *submersion* if its differential π_{*p} at any point $p \in M$ has maximal rank n.

The premiage $\pi^{-1}(b)$ of a point $b \in B$ is a k-dimensional submanifold of M, which we call the *fiber of* π *over* b (even though π need not be a *fibration*, i.e., a surjective map with the homotopy lifting property; on the other hand a fibration is necessarily a submersion). We will denote a generic fiber by F when there is no danger of confusion. We define the *vertical distribution* $\mathcal{V}$ of π to be the kernel of the differential π_*. At a given point $p \in M$, the subspace $\mathcal{V}_p$ equals the tangent space T_pF to the corresponding fiber. If in addition M is a Riemannian manifold, it makes sense to define the *horizontal distribution* $\mathcal{H}$ of π as the orthogonal complement $\mathcal{H} = \mathcal{V}^\perp$ of $\mathcal{V}$. The orthogonal splitting of the tangent bundle of M induces a decomposition $e = e^h + e^v \in \mathcal{H} \oplus \mathcal{V}$ of any vector $e \in TM$.

Definition 1.13. Let $\pi : M \to B$ be a submersion, where M and B are Riemannian manifolds. We say that π is a *Riemannian submersion* if π_* is a linear isometry when restricted to $\mathcal{H}$, i.e., if for every $p \in M$ and $x, y \in \mathcal{H}_p \subset T_pM$,

$$\langle x, y\rangle_M = \langle \pi_{*p}(x), \pi_{*p}(y)\rangle_B,$$

where $\langle , \rangle_M$ and $\langle , \rangle_B$ denote the metrics of M and B respectively.

Let us give some examples of Riemannian submersions:

1. The projection from a product Riemannian manifold $B \times F \to B$ is clearly a Riemannian submersion with fiber F.

2. Let G be a Lie group acting by isometries on a Riemannian manifold M (see Section 2.1.3 for definitions and further details on group actions). Suppose that all orbits have the same type (meaning that any two are equivariantly diffeomorphic). Then there exists a smooth structure and a unique metric on the space of orbits M/G for which the natural projection $\pi : M \to M/G$ is a Riemannian submersion, which we call *homogeneous*. This is the case of the Hopf fibrations. As an example, consider the 3-dimensional unit sphere:

$$\mathbb{S}^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\},$$

then the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ acts on $\mathbb{S}^3$ by the rule

$$z(z_1, z_2) = (zz_1, zz_2).$$

The natural projection onto the orbit space gives us the Riemannian submersion

$$\pi : \mathbb{S}^3(1) \to \mathbb{S}^2(4),$$

where $\mathbb{S}^3(1)$ and $\mathbb{S}^2(4)$ denote the spheres of constant sectional curvature equal to 1 and 4 respectively. The growth of the sectional curvature when descending to the base space is a general fact as O'Neill's formula shows (see Section 1.2.2).

1.2.1 Lifts and holonomy

Let $\pi : M \to B$ be a Riemannian submersion. A *horizontal lift* of a curve $c : [0, l] \to B$ at a point $p \in \pi^{-1}(c(0))$ is a curve $c_p : [0, l] \to M$ such that $\pi \circ c_p = c$, $c_p(0) = p$ and $c_p'(t) \in \mathcal{H}$ for all $t \in [0, l]$. A *basic lift* of a vector field X on B is a horizontal vector field on M projecting to X through π_*; they exist around any point in M. The following properties hold (see [28] and [38] for the proofs):

Proposition 1.14. *Let $\pi : M \to B$ be a Riemannian submersion.*

1. *The basic lift of a smooth vector field on B is smooth.*

2. *The horizontal lift of a curve $c : [0, l] \to B$ at a point $p \in \pi^{-1}(c(0))$ exists and it is unique.*

3. *If $\alpha : I \to M$ is a geodesic with $\alpha'(t_0) \in \mathcal{H}$ for some $t_0 \in I$, then $\alpha'(t) \in \mathcal{H}$ for all $t \in I$, and $\pi \circ \alpha$ is a geodesic in B. Such an α will be called a* ***horizontal geodesic****.*

4. *If M is complete, then*

   (a) *B is complete*

   (b) *the fibers of π are equidistant, i.e., for any two fibers F_0, F_1 and any $p \in F_0$, the distance between p and F_1 equals that between F_0 and F_1;*

   (c) *π is a* locally trivial fiber bundle*, i.e., any point b in B has a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times F$, where $F = \pi^{-1}(b)$. In particular, all the fibers are pairwise diffeomorphic.*

Given a continuous curve $c : [0, l] \to B$, the *holonomy map* $h_c : \pi^{-1}(c(0)) \to \pi^{-1}(c(l))$ is defined as follows: h_c maps a point p in the fiber over the initial point of c to the endpoint of the horizontal lift c_p of c that starts at p. The uniqueness of horizontal lifts of curves implies that h_c is bijective if M is complete.

Proposition 1.15. *Let $\pi : M \to B$ be a Riemannian submersion with M complete. If $c : [0, l] \to B$ is a picewise smooth curve in B, then the holonomy map h_c is smooth and hence a diffeomorphism.*

Proof. It suffices to suppose that

$$\begin{array}{rcl} c : [0, l] & \longrightarrow & B \\ t & \longmapsto & c(t) \end{array}$$

is smooth, since the holonomy map associated to a concatenation of smooth curves is the composition of the holonomy maps associated to each smooth segment of the curve.

The submersion $\pi : M \to B$ is a locally trivial fiber bundle, hence we can consider the pull-back bundle $\pi' : c^*M \to [0, l]$, which is itself a Riemannian submersion with the induced metrics. Since c is smooth, the velocity vector $\partial_t \in T[0, l]$ and hence its basic lift $X \in Tc^*M$ are smooth vector fields. It follows that the flow $\phi_X : [0, l] \times (\pi')^{-1}(0) \to c^*M$ of X is smooth. Finally, observe that $h_c : \pi^{-1}(c(0)) \to \pi^{-1}(c(l))$ equals the restriction of ϕ_X to $\{l\} \times (\pi')^{-1}(0)$. □

1.2.2 Tensors and curvature relations

There are two tensor fields that measure the complexity of a Riemannian submersion $\pi : M \to B$:

1. The *A-tensor* $A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$

$$A_X Y = (\nabla_X Y)^v = \frac{1}{2}[X, Y]^v.$$

   Note that by Frobenius' Theorem, $A \equiv 0$ iff the distribution $\mathcal{H}$ is integrable. Given horizontal vectors x, y and a vertical vector u, denote by $A_x^* : \mathcal{V} \to \mathcal{H}$ the adjoint of $A_x : \mathcal{H} \to \mathcal{V}$:

$$\langle A_x^* u, y \rangle = \langle u, A_x y \rangle.$$

2. The *S-tensor* $S : \mathcal{H} \times \mathcal{V} \to \mathcal{V}$

$$S_X U = -(\nabla_U X)^v.$$

Observe that S_X is just the second fundamental form of a fiber in the horizontal direction X. The Riemannian submersion π has totally geodesic fibers if and only if $S \equiv 0$; in this case all the fibers are pairwise isometric. Moreover, if $S \equiv 0$, then π is a locally trivial fiber bundle where the structure group is a Lie group. Note that the converse of the latter fact does not hold since any homogeneous Riemannian submersion has a Lie group as structure group (see [33]), but it does not have totally geodesic fibers in general.

If both A and S are identically zero then π *locally splits*, i.e., every point $b \in B$ has a neighborhood U such that $\pi^{-1}(U)$ is isometric to the metric product $U \times F$. However, as we pointed out at the beginning of this chapter, A cannot be identically zero if M is a compact manifold with positive sectional curvature (see [68]).

The A and S-tensors also appear in the classical formulas by O'Neill relating the sectional curvature of M with that of B. For a given point $p \in M$, let $x, y \in T_pM$ be orthonormal horizontal vectors. Denote by (x, y) the 2-plane spanned by x and y. Then:

$$\sec_B(\pi_* x, \pi_* y) = \sec_M(x, y) + 3\|A_x y\|^2,$$

where $\|\cdot\|$ denotes the norm of a vector in TM.

For a vertical unit vector $u \in T_pM$,

$$\sec_M(x, u) = \langle (\nabla^v_x S)_x u, u\rangle_M + \|A^*_x u\|^2 - \|S_x u\|^2.$$

1.2.3 Projectable Jacobi fields

Here we describe certain Jacobi fields which occur along horizontal geodesics in the total space of a Riemannian submersion.

Definition 1.16. Let $\alpha : I \to M$ be a horizontal geodesic for the submersion. A Jacobi field J along α is *projectable* if it satisfies

$$J'^v = -S_{\alpha'} J^v - A_{\alpha'} J^h.$$

The interest of projectable Jacobi fields is that they arise from variations by horizontal geodesics. As such, if J is a projectable Jacobi field, $\pi_* J$ is a Jacobi field along the geodesic $\bar\alpha = \pi \circ \alpha$ in the base. Conversely, we have the following

Lemma 1.17. *Let $\bar J$ be a Jacobi field of B along $\bar\alpha$, and v a vertical vector at $\alpha(0)$; then there is a unique projectable Jacobi field J along α such that $\pi_* J = \bar J$ and $J(0)^v = v$.*

A particular case of projectable Jacobi fields arises from taking geodesic variations obtained from lifting a given geodesic in the base; such fields are called *holonomy Jacobi fields*, and they satisfy the stronger condition

$$J' = -A^*_{\alpha'} J - S_{\alpha'} J.$$

It is clear that holonomy fields remain always vertical, i.e., they agree with those projectable Jacobi fields mapping to the zero field under π_*.

1.2.4 Singular Riemannian foliations

Here we recall some basic facts about singular Riemannian foliations. We refer the reader to [53] for further details.

Denote by $(M, \mathcal{F})$, or just $\mathcal{F}$, a partition of a complete smooth manifold M into smooth, complete, connected, injectively immersed submanifolds (called the *leaves* of $\mathcal{F}$). The leaves are allowed to have different dimensions and given a point $p \in M$, we will denote by L_p the leaf through p. Define the *vertical distribution* $T\mathcal{F} = \{T_pL_p : p \in L, L \in \mathcal{F}\}$ and consider the set $\mathfrak{X}_{\mathcal{F}}$ of smooth vector fields X on M such that $X(p) \in T\mathcal{F}_p$ for all $p \in M$. We say that $(M, \mathcal{F})$ is a *singular foliation* if there exits a family $\{X_i\}_i \subset \mathfrak{X}_{\mathcal{F}}$ such that T_pL is spanned by $\{X_i(p)\}_i$ for every $p \in M$. As an example, a smooth submersion $M \to B$ determines a singular foliation on M where the leaves are the fibers of the submersion. When M is a Riemannian manifold it is natural to define a special kind of foliations.

Definition 1.18. A singular foliation on a complete Riemannian manifold is said to be a *singular Riemannian foliation* if each geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

Clearly the fibers of a Riemannian submersion $M \to B$ determine a singular Riemannian foliation on M. Moreover, the leaves of an arbitrary Riemannian foliation are locally given by fibers of Riemannian submersions for adequately defined metrics on the local quotients. Therefore, locally we can define the same concepts as in a Riemannian submersion.

As a second example, if a connected Lie group G acts on a Riemannian manifold M by isometries, then the partition consisting of the G-orbits determines a singular Riemannian foliation on M, which we call *homogeneous*.

Definition 1.19. We define the *focal radius* of a Riemannian foliation $(M, \mathcal{F})$, as the infimum over all the leaves of $\mathcal{F}$ of the focal radius of each leaf. It will be denoted by $\mathrm{foc}(\mathcal{F})$.

Theorem 1.25 will imply that the focal radius of a singular Riemannian foliation $(M, \mathcal{F})$ with $\sec_M \geq 1$ is less than π.

1.3 Bounds on the index

As in Section 1.1.1, let $\mathbb{E}$ be an m-dimensional Euclidean vector space, and consider a one-parameter family of self adjoint linear maps $R : \mathbb{R} \to \mathrm{Sym}(\mathbb{E})$. Recall that given an interval $I \subset \mathbb{R}$ and a Lagrangian $L \subset \mathrm{Jac}^R$, we denote by $\mathrm{ind}_L\, I$ the index of L in I and by $\mathrm{ind}_L(t_0)$ the dimension of the vector subspace of L formed by those Jacobi fields in L that vanish at t_0.

The definition of the Lagrangian L_0 given in Section 1.1.1 can be generalized in an obvious way. Given $a \in \mathbb{R}$, denote by L_a the Lagrangian subspace of Jac^R defined as

$$L_a := \left\{ Y \in \mathrm{Jac}^R \,:\, Y(a) = 0 \right\}. \tag{1.3.1}$$

1.3.1 Upper bounds

All along this subsection we will assume that there is some positive number $C > 0$ such that for any $a \in \mathbb{R}$ and any Jacobi field with $Y(a) = 0$, Y does not vanish again in $(a, a + C]$. Clearly

$$\operatorname{ind}_{L_a}(a, a+C) = \operatorname{ind}_{L_a}(a, a+C] = 0, \quad \operatorname{ind}_{L_a}[a, a+C) = m.$$

Inequality (1.1.1) shows that for an arbitrary Lagrangian L,

$$\operatorname{ind}_L(a, a+C] \leq m. \tag{1.3.2}$$

Our next aim is to extend this to larger intervals:

Proposition 1.20. *For any Lagrangian L and any positive integer r we have*

$$\operatorname{ind}_L[a, a+rC] \leq (r+1)m.$$

Proof. Breaking the interval $[a, a + rC]$ into subintervals of length C and using (1.3.2) repeatedly, we get that

$$\operatorname{ind}_L[a, a+rC] = \operatorname{ind}_L(a) + \sum_{i=0}^{r-1} \operatorname{ind}_L(a+iC, a+(i+1)C] \leq m + rm.$$

□

In the same way we consider the case where there is some positive number ℓ such that for any $a \in \mathbb{R}$,

$$\operatorname{ind}_{L_a}(a, a+\ell) \leq m, \quad \text{i.e.,} \quad \operatorname{ind}_{L_a}[a, a+\ell) \leq 2m.$$

Again we extend this to larger intervals:

Proposition 1.21. *For any Lagrangian L and any positive integer r we have*

$$\operatorname{ind}_L[a, a+r\ell] \leq 2m(r+1).$$

Proof. Breaking the interval $[a, a+r\ell]$ into subintervals of length ℓ and applying inequality (1.1.1) to each subinterval we get

$$\operatorname{ind}_L[a, a+r\ell] = \sum_{i=0}^{r-1} \operatorname{ind}_L[a+i\ell, a+(i+1)\ell) + \operatorname{ind}_L(a+r\ell) \leq 2mr + 2m.$$

□

1.3.2 Curvature-related lower bounds

To get a lower bound on the index of a Lagrangian L, we need to establish the existence of conjugate points for the fields in L; Rauch's Theorem (for the statement of the theorem in an abstract setting see [70]) gives precisely that for a Lagrangian of the form L_a as defined in (1.3.1). We will then use Proposition 1.11 to relate this to the index of an arbitrary Lagrangian.

We will say that the curvature R satisfies $R(t) \geq \delta$ for all $t \in \mathbb{R}$ if $\langle R(t)v, v\rangle \geq \delta\|v\|^2$ for any vector $v \in \mathbb{E}$. Our first result is a quantitative refinement of Corollary 10 in [71].

Proposition 1.22. *Assume that there is some $\delta > 0$ such that the curvature R satisfies $R(t) \geq \delta$ for all $t \in \mathbb{R}$. Then for any $a \in \mathbb{R}$, the set*

$$\mathcal{A} = \left\{ Y \in L_a \,:\, Y(t) = 0 \text{ for some } t \in \left(a, a + \pi/\sqrt{\delta}\right] \right\}$$

generates L_a.

Proof. The proof consists on using of Wilking's transversal equation repeatedly. We describe how to proceed:

1. We compare $R(t)$ to the constant curvature case $\bar{R}(t) = \delta I$; Rauch's Theorem gives us that there is some nonzero Y_1 in $\mathcal{A}$ vanishing for some $t_1 \in (a, a + \pi/\sqrt{\delta}]$.

2. Let $W_1 \subset L_a$ be the vector subspace generated by Y_1; we consider the transverse Jacobi equation induced by W_1 in L_a. In L_a/W_1 there is a Jacobi equation of the form
$$Y'' + R_1 Y = 0, \quad R_1(t) = R(t)^h + 3A_t A_t^*,$$
and therefore $\langle R_1(t)v, v\rangle \geq \langle R(t)v, v\rangle \geq \delta\|v\|^2$ for any $v \in W_1(t)^\perp \subset \mathbb{E}$. Moreover, after taking the W_1-orthogonal component, the fields in L_a give an R_1-Lagrangian L_1. It is clear that every vector field in L_1 vanishes at $t = a$.

3. Once again, we compare $R_1(t)$ to δI to obtain some nonzero $X_2 \in L_a$ such that $X_2^\perp$ vanishes at some time t_2 in $(a, a + \pi/\sqrt{\delta}]$; this merely means that $X_2(t_2) = \lambda Y_1(t_2)$ for some $\lambda \in \mathbb{R}$. It follows that the field $Y_2 = X_2 - \lambda Y_1$ vanishes at $t = t_2$, hence Y_2 lies in $\mathcal{A}$ and it is linearly independent with respect to Y_1.

4. Clearly, the process can be iterated as needed until we obtain a basis of L_a.

□

Proposition 1.22 allows us to obtain good lower bounds for the index of a Lagrangian over long intervals. They can also be obtained using the Morse-Schoenberg lemma [63] and Proposition 1.11.

Proposition 1.23. *Let $a \in \mathbb{R}$; when $R \geq \delta$, the index of any Lagrangian subspace L of Jacobi fields satisfies*

$$\operatorname{ind}_L \left[a, a + r\pi/\sqrt{\delta}\right] \geq rm + \operatorname{ind}_L(a)$$

for any positive integer r.

Proof. Without loss of generality we can assume that $a = 0$ and write the proof for this case. Consider the closed intervals

$$I_j = [j\pi/\sqrt{\delta}, (j+1)\pi/\sqrt{\delta}].$$

Proposition 1.22 says that

$$\mathrm{ind}_{L_{j\pi/\sqrt{\delta}}} I_j \geq 2m;$$

while Proposition 1.11 gives us

$$\mathrm{ind}_L I_j \geq \mathrm{ind}_{L_{j\pi/\sqrt{\delta}}} I_j - m + \dim(L \cap L_{j\pi/\sqrt{\delta}}) \geq m + \mathrm{ind}_L(j\pi/\sqrt{\delta}).$$

Breaking the interval $[0, r\pi/\sqrt{\delta}]$ into the I_j's, we conclude that

$$\begin{aligned}(1.3.3)\quad \mathrm{ind}_L \left[0, r\frac{\pi}{\sqrt{\delta}}\right] &= \sum_{j=0}^{r-1} \mathrm{ind}_L I_j - \sum_{j=1}^{r-1} \mathrm{ind}_L \left(j\frac{\pi}{\sqrt{\delta}}\right) \geq \\ &\geq \sum_{j=0}^{r-1} \left(m + \mathrm{ind}_L \left(j\frac{\pi}{\sqrt{\delta}}\right)\right) - \sum_{j=1}^{r-1} \mathrm{ind}_L \left(j\frac{\pi}{\sqrt{\delta}}\right) = rm + \mathrm{ind}_L(0).\end{aligned}$$

□

A consequence of the last results is the following extension of Proposition 1.22 to arbitrary Lagrangians:

Proposition 1.24. *Let $a \in \mathbb{R}$; when $R \geq \delta$, for every Lagrangian subspace L of* Jac^R *the set*

$$\left\{ Y \in L : \ Y(t) = 0 \text{ for some } t \in \left(a, a + \pi/\sqrt{\delta}\right] \right\}$$

spans L.

Proof. Proposition 1.23 for $r = 1$ gives

$$\mathrm{ind}_L[a, a + \pi/\sqrt{\delta}] \geq m + \mathrm{ind}_L(a).$$

Therefore there exists a $Y_1 \in L$ such that $Y_1(t_1) = 0$ for some $t_1 \in (a, a + \pi/\sqrt{\delta}]$. The proof is then identical to that of Proposition 1.22. □

Applying Proposition 1.24 to the Lagrangian subspace L^N of N-Jacobi fields along α defined in Section 1.1 we get the following geometric application:

Theorem 1.25. *Let M be an n-dimensional manifold with* $\sec \geq 1$ *and $\alpha : \mathbb{R} \to M$ a geodesic orthogonal to a submanifold N at $\alpha(0)$. Then there are at least $n - 1$ focal points of N along α in the interval* $(0, \pi]$.

1.3.3 Index bounds for periodic Jacobi fields

In this section we examine the index of Lagrangians when the solutions of the Jacobi equation are periodic with common period. We will show that such index is always bounded above by some linear function related to multiples of the period.

Proposition 1.26. *Suppose there is some $l > 0$ such that for every Jacobi field J, the field $t \to X(t) = J(t+l)$ is also a Jacobi field. Then for any Lagrangian L in* Jac^R *we have*

$$\mathrm{ind}_L[a, a+rl] \leq r\,(m + \mathrm{ind}_L[a, a+l)) + m$$

for any positive integer r.

Proof. As usual, we will write the proof for $a = 0$. We start by choosing some basis of L, given by $X_1, \ldots, X_m$; for any positive integer r, consider the Jacobi fields defined as

$$X_i^r(t) = X_i(t + rl), \quad i = 1, \ldots, m.$$

Let L^r the subspace generated by $X_1^r, \ldots, X_m^r$; it is Lagrangian, with $L^0 = L$. Clearly

$$\mathrm{ind}_L\,[jl, (j+1)l) = \mathrm{ind}_{L^j}[0, l).$$

Using (1.1.1), we have that

$$\mathrm{ind}_L[0, rl] = \sum_{j=0}^{r-1} \mathrm{ind}_{L^j}[0, l) + \mathrm{ind}_{L^r}(0) \leq \sum_{j=0}^{r-1} (\mathrm{ind}_L\,[0, l) + m) + m,$$

as claimed.

□

1.4 Horizontal closed curves and geodesics

Let $\pi : M \to B$ be a Riemannian submersion (no curvature conditions are assumed in this section) with fiber F and $c : [0, l] \to B$ a picewise smooth closed curve, i.e., $c(0) = c(l)$ but not necessarily $c'(0) = c'(l)$. In this section we study the existence of fixed points for the associated holonomy diffeomorphism $h_c : F \to F$ using Lefschetz Fixed-Point Theory. Note that h_c having a fixed point p means that the horizontal lift $c_p : [0, l] \to M$ of c satisfying $c_p(0) = p$ closes in the first lap, i.e., $c_p(0) = c_p(l) = p$.

Some preliminaries on Lefschetz Fixed-Point Theory are given in Subsection 1.4.1; then we apply these results to different situations in Subsections 1.4.2 and 1.4.3. We finish the section by giving an example of a Riemannian submersion with no horizontal closed geodesics in Subsection 1.4.4.

1.4.1 Lefschetz Fixed-Point Theory

Here we collect some basic facts on Lefschetz Fixed-Point Theory (see [12, 34] for proofs and further details). Let $f : M \to M$ be a smooth map on a compact oriented manifold. The *Lefschetz number* $L(f)$ of f is an integer which measures somehow the cardinality of the fixed-point set of f.

Definition 1.27. Let $f : M \to M$ be a smooth map on a compact orientable manifold. The *Lefschetz number* $L(f)$ of f is

$$L(f) = \sum_i (-1)^i \operatorname{trace} H^i(f)$$

where $H^i(f)$ denotes the linear map induced by f on the cohomology group $H^i(M)$.

As an immediate consequence, we have the following:

Corollary 1.28. *The Lefschetz number $L(Id)$ of the identity map $Id : M \to M$ equals the Euler characteristic $\chi(M)$ of M.*

Observe that The Lefschetz number $L(f)$ of a smooth map $f : M \to M$ is a topological invariant, i.e., $L(f) = L(\bar{f})$ for any smooth map $\bar{f} : M \to M$ homotopic to f. Recall that f is *homotopic* to $\bar{f}$ if there exists a smooth map $H : M \times [0,1] \to M$ such that $H(p,0) = f(p)$ and $H(p,1) = \bar{f}(p)$ for all $p \in M$. The main result is the following:

Theorem 1.29 (Lefschetz Fixed-Point Theorem)**.** *Let $f : M \to M$ be a smooth map on a compact orientable manifold. If $L(f) \neq 0$, then f has a fixed point.*

1.4.2 $\pi_1(B) = 0$ and $\chi(F) \neq 0$

The simply-connectedness of the base space B in a Riemannian submersion $M \to B$ allows us to characterize the holonomy diffeomorphisms topologically.

Proposition 1.30. *Let $\pi : M \to B$ be a Riemannian submersion with fiber F, and let $c : [0,l] \to B$ be a picewise smooth closed curve. If B is simply connected, then h_c is homotopic to the identity map $Id : F \to F$.*

Proof. Since B is simply connected, there exists a smooth homotopy of curves

$$\begin{array}{rcl} H : [0,l] \times [0,1] & \longrightarrow & B \\ (\ t\ ,\ s\ ) & \longmapsto & H(t,s) =: c_s(t) \end{array}$$

with $c_1 \equiv c$, $c_t(0) = c_t(1) = c(0)$ for all t and $c_0 \equiv c(0)$. As in the proof of Proposition 1.15, consider the pull-back bundle $\pi' : H^*M \to [0,l] \times [0,1]$, which is itself a Riemannian submersion with the induced metrics. Since H is smooth, the velocity vector $\partial_t \in T([0,l] \times [0,1])$ and hence its basic lift X to H^*M are smooth vector fields. It follows that the flow $\phi_X : \mathbb{R} \times H^*M \to H^*M$ of X is smooth. Observe that $H_{*(t,s)}(\partial_t) = c_s'(t)$, and that for each $s \in [0,1]$, the restriction of ϕ_X to $\{l\} \times (\pi')^{-1}(0,s)$ equals the holonomy map $h_{c_s} : \pi^{-1}(c(0)) \to \pi^{-1}(c(0))$. Therefore h_{c_s} is a smooth homotopy between $h_{c_1} \equiv h_c$ and $h_{c_0} \equiv Id$. $\square$

Proof of Theorem 1.5. Proposition 1.30 tells us that h_c is homotopic to the identity map and therefore by Corollary 1.28 its Lefschetz number equals the Euler characteristic of the fiber $\chi(F)$. By hypothesis $\chi(F) \neq 0$, thus Theorem 1.29 implies that the map h_c has a fixed point. □

1.4.3 F is a homology sphere

Next we examine the case when the fiber of the Riemannian submersion is a homology sphere, which we denote by $\mathbb{S}^k$. In this case, the Lefschetz number of the holonomy diffeomorphism $h_c : \mathbb{S}^k \to \mathbb{S}^k$ associated to a closed curve c in the base space is:

$$L(h_c) = \operatorname{trace} H^0(h_c) + (-1)^k \operatorname{trace} H^k(h_c) = 1 + (-1)^k \operatorname{trace} H^k(h_c)$$

where

$$\operatorname{trace} H^k(h_c) = \begin{cases} 1 & \text{if } h_c \text{ preserves orientation} \\ -1 & \text{otherwise} \end{cases}$$

The following table shows the Lefschetz number of $L(h_c)$ in all the possible cases:

	k is even	k is odd
h_c preserves orientation	2	0
h_c reverses orientation	0	2

Table 1.1: Lefschetz number $L(h_c)$.

Observe that even if h_c reverses the orientation, $h_c^2 = h_c \circ h_c$ preserves the orientation. Thus, if k is even, $L(h_c^2) = 2$.

Proposition 1.31. *Let $\pi : M \to B$ be a Riemannian submersion with fiber a homology sphere of dimension k, and $c : [0, l] \to B$ a picewise smooth closed curve. Then,*

1. *If k is even and h_c preserves orientation, then there is a horizontal lift of c that closes in the first lap.*
2. *If k is even and h_c reverses orientation, then there is a horizontal lift of c that closes in the second lap.*
3. *If k is odd and h_c reverses orientation, then there is a horizontal lift of c that closes in the first lap.*

Note that in contrast to Theorem 1.5, in Proposition 1.31 we do not require simply-connectedness of the base space.

Observe that in the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ with fiber the circle $\mathbb{S}^1$, every geodesic in $\mathbb{S}^2$ is closed of length π, and every geodesic in $\mathbb{S}^3$ is closed of length 2π. This implies that any lift of a closed geodesic always closes in the second lap but never in the first one. By part (3) of Proposition 1.31, it follows that the holonomy map associated to a closed geodesic in $\mathbb{S}^2$ preserves orientation.

One can obtain similar results to Proposition 1.31 considering other spaces as fibers of the Riemannian submersion. Natural spaces to examine are orientable manifolds with simple cohomology rings, such as (homology) odd-dimensional real or complex projective spaces.

1.4.4 An example with no horizontal closed geodesics

Here we show how to produce examples of Riemannian submersions where the horizontal lifts of a closed curve never closes. In these examples, the fibers are circles and hence the associated holonomy diffeomorphisms preserve orientation.

For each $\alpha \in \mathbb{R}$, we have the following action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$:

$$\begin{array}{rcl} (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (a,b),(x,y) & \longmapsto & (x+a, y+b+a\alpha) \end{array}$$

This action is smooth, free and properly discontinuous (see the definition in Section 2.1.1) for every $\alpha \in \mathbb{R}$. It follows that the quotient space is a manifold, which will be denoted by T_α. Clearly, T_0 is the standard 2-dimensional torus, and T_α is diffeomorphic to T_0 for every α as the following well-defined map shows:

$$\begin{array}{rcl} T_\alpha & \longrightarrow & T_0 \\ [x,y] & \longmapsto & [x, y-\alpha x] \end{array}$$

The action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$ is clearly by isometries, so the Euclidean metric on $\mathbb{R}^2$ descends to a flat metric on T_α. Now we define the following Riemannian submersion

$$\begin{array}{rcl} \pi_\alpha : T_\alpha & \longrightarrow & \mathbb{R}/\mathbb{Z} = \mathbb{S}^1 \\ [x,y] & \longmapsto & [x] \end{array}$$

where the circle has the obvious metric of length 1. The curve $c(t) = [t]$ in the circle is a geodesic, which is closed as $c(j) = c(0)$ for every integer j. A horizontal lift of c to the point $[0,y]$ in the fiber over $c(0)$ is of the form $c_y(t) = [t,y]$. At integer times,

$$c_y(j) = [j,y] = [0, y - j\alpha],$$

so $c_y(j) = c_y(0)$ if and only if $j\alpha$ is an integer.

Proposition 1.32. *For the Riemannian submersion $T_\alpha \to \mathbb{S}^1$ the following holds:*

1. *If $\alpha = 1/m$ with $m \in \mathbb{N}$, the lift of a closed geodesic closes exactly in the m^{th} lap.*
2. *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there are no closed horizontal geodesics.*

Observe that the horizontal distribution of the submersion is integrable for all $\alpha \in \mathbb{R}$. Moreover, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, each of the integral submanifolds is dense and it is called a *irrational winding of the torus.*

1.5 Proofs

Let $\pi : M \to B$ be such a submersion where M and B have dimensions $n+k$ and n respectively, with fiber F^k. We will usually overline the notation for objects in the base, to distinguish them from those in M.

1.5.1 Proof of Theorem 1.1

Let $\alpha : \mathbb{R} \to M$ be a horizontal geodesic and $\bar{\alpha} = \pi \circ \alpha$ its projection by the submersion. The Lagrangian subspace $\bar{L}_o$ can be lifted to α by considering the subspace spanned by projectable Jacobi fields Y that vanish at $t = 0$ (and will therefore have horizontal $Y'(0)$), and by holonomy Jacobi fields along α. We use L to denote such Lagrangian, and W to denote the subspace generated by the holonomy fields. It is interesting to observe that L agrees with the L^N from the proof of Theorem 1.25 when N is the fiber through $\alpha(0)$. By Lemma 3.1 in [48],

$$\mathrm{ind}_{L/W} + \mathrm{ind}_W = \mathrm{ind}_L \tag{1.5.1}$$

along any interval, where in L/W we are using the transverse Jacobi equation induced by $W \subset L$. Observe that since holonomy Jacobi fields never vanish, $\mathrm{ind}_W = 0$ over any interval. We claim that $\mathrm{ind}_{L/W} = \mathrm{ind}_{\bar{L}_0}$. To prove it, we use that, as stated in [48, Section 3.2], the transverse Jacobi equation corresponding to W along α agrees with the usual Jacobi equation along $\bar{\alpha}$. Since Lagrangians for the Jacobi equation project to Lagrangians for the transverse Jacobi equation, and every field Y in L satisfies $Y(0) \in W(0)$, we have the mentioned equivalence of indices. Thus we have $\mathrm{ind}_{\bar{L}_0} = \mathrm{ind}_L$.

We will estimate this common value over the intervals $[0, r\pi]$ using some of the previous inequalities on the index; choose an arbitrary $C < \mathrm{conj}(B)$:

$$\mathrm{ind}_{\bar{L}_0}[0, r\pi] \;=\; \mathrm{ind}_{\bar{L}_0}\left[0, \frac{r\pi}{C} \cdot C\right] \leq \left(\left[\frac{r\pi}{C}\right] + 1\right)(n-1) \tag{1.5.2}$$

by Proposition 1.20, and

$$\mathrm{ind}_L[0, r\pi] \;\geq\; r(n-1+k) + \mathrm{ind}_L(0) = r(n-1+k) + (n-1) \tag{1.5.3}$$

by Proposition 1.23.

To finish the proof, divide both inequalities by r and make it tend to infinity to conclude that

$$k \leq \left(\frac{\pi}{C} - 1\right)(n-1).$$

Letting C tend to $\mathrm{conj}(B)$ gives us Theorem 1.1. □

The above proof can be easily extended to metric foliations:

Proof of Corollary 1.3. Let F be a leaf of $\mathcal{F}$ and $\alpha : \mathbb{R} \to M$ a geodesic orthogonal to F with $\alpha(0) \in F$. Denote by W the set of holonomy Jacobi fields along α, and by

L the Lagrangian spanned by W and those Jacobi fields along α with $J(0) = 0$ and $J'(0) \perp T_{\alpha(0)}F$,i.e., the F-Jacobi fields. Since $\operatorname{ind}_W I = 0$, equation 1.5.1 gives

$$\operatorname{ind}_L I = \operatorname{ind}_{L/W} I$$

for any interval I. Observe that L/W corresponds to the Lagrangian $\bar{L}_0 = \{J : J(0) = 0\}$ of Jacobi fields for Wilking's transverse equation for the isotropic W.

From the definition of the focal radius of $\mathcal{F}$ (Definition 1.19) it follows that for every $C < \operatorname{foc}(\mathcal{F})$,

$$\operatorname{ind}_{\bar{L}_0}(0, C] = 0,$$

and therefore we are in the situation of Proposition 1.24, thus

$$r(n-1+k) \leq \operatorname{ind}_L[0, r\pi] = \operatorname{ind}_{\bar{L}_0}[0, r\pi] \leq \left(\left[\frac{r\pi}{C}\right] + 1\right)(n-1)$$

for any integer $r > 0$. As before, divide both sides by r and let it tend to zero to obtain the inequality claimed in the corollary. □

1.5.2 Proof of Theorem 1.4

Let m be the smallest positive integer such that $\pi_i B = 0$ when $i = 1, \ldots, m-1$ and $\pi_m B \neq 0$. Hurewicz's Theorem implies that $m \leq n$. If ΛB denotes the free loop space of B, then $\pi_{m-1}\Lambda B = \pi_m B$, and Lyusternik-Schnirelmann theory implies that there is a closed geodesic $\bar{\alpha} : [0, \ell] \to B$ such that the number of conjugate points to $\bar{\alpha}(0)$ along $\bar{\alpha}$ in the interval $(0, \ell)$ does not exceed $m - 1$ (see [7, Theorem 1.3]). If we iterate $\bar{\alpha}$ and consider it as a geodesic $\bar{\alpha} : \mathbb{R} \to B$, we have that $\operatorname{ind}_{\bar{L}_a}(a, a+\ell) \leq m - 1 \leq n - 1$ for all $a \in \mathbb{R}$.

Denote by $\alpha : \mathbb{R} \to M$ some horizontal lift of $\bar{\alpha}$ to M. Choose along α the Lagrangian L of Jacobi fields spanned by the vertical holonomy Jacobi fields and projectable Jacobi fields that vanish at $t = 0$. As in the proof of Theorem 1.1, we have

$$\operatorname{ind}_L I = \operatorname{ind}_{\bar{L}_0} I. \tag{1.5.4}$$

We are going to use this equality in intervals of the form $[0, r\pi]$ for r a positive integer; the left hand side in (1.5.4) can be bound with the help of Proposition 1.23, giving

$$r(n-1+k) + \operatorname{ind}_L(0) \leq \operatorname{ind}_L[0, r\pi];$$

on the other hand the right hand side can be bound with Proposition 1.21 to get

$$\operatorname{ind}_{\bar{L}_0}[0, r\pi] \leq \operatorname{ind}_{\bar{L}_0}\left[0, \left(\left[\frac{r\pi}{\ell}\right] + 1\right)\ell\right] \leq 2\left(\left[\frac{r\pi}{\ell}\right] + 2\right)(n-1).$$

Dividing by r and letting it tend to infinity gives

$$n - 1 + k \leq \frac{2\pi}{\ell}(n-1) \leq \frac{2\pi}{\ell_0(B)}(n-1).$$

□

1.5.3 Proof of Theorems 1.6 and 1.7

Denote by $\tilde{M}$ and $\tilde{B}$ the Riemannian universal coverings of M and B respectively. Observe that $\tilde{M}$ and $\tilde{B}$ satisfy the same curvature bounds of M and B respectively and hence are compact. The composition of the covering map $\tilde{M} \to M$ with $\pi : M \to B$ is a Riemannian submersion $\tilde{M} \to B$, which can be lifted to a Riemannian submersion $\tilde{\pi} : \tilde{M}^{n+k} \to \tilde{B}^n$ using basic covering space theory. The fiber of $\tilde{\pi}$ is a manifold F' which is a covering space of F by construction (note that F' need not be simply connected).

Applying Theorem 1.4 to the Riemannian submersion $\tilde{\pi} : \tilde{M}^{n+k} \to \tilde{B}^n$ we get

$$\ell_0(\tilde{B}) \leq \frac{2\pi(n-1)}{n+k-1}, \tag{1.5.5}$$

where $\ell_0(\tilde{B})$ denotes the length of the shortest closed geodesic in $\tilde{B}$.

The coverings $\tilde{M}$ and $\tilde{B}$ are compact and simply connected, hence orientable. We claim that the fiber F' is also orientable. To prove the this fact, observe that by part (4)-(c) of Proposition 1.14, any point in $\tilde{B}$ has a neighborhood U such that $\tilde{\pi}^{-1}(U)$ is diffeomorphic to $U \times F'$. On the other hand, $\tilde{\pi}^{-1}(U)$ is an open submanifold N of the same dimension as $\tilde{M}$, and hence orientable. It follows that $U \times F'$ and therefore F' are orientable.

For the proof of Theorem 1.6 observe that since F' is a covering space of F, it follows that $\chi(F') = d\chi(F)$, where d denotes the degree of the covering map $F' \to F$. Then $\chi(F') \neq 0$, and therefore the Riemannian submersion $\tilde{\pi} : \tilde{M}^{n+k} \to \tilde{B}^n$ satisfies the conditions of Theorem 1.5. It follows that a lift to $\tilde{M}$ of the shortest closed geodesic in $\tilde{B}$ closes in the first lap, and hence $\ell_0(\tilde{M}) \leq \ell_0(\tilde{B})$.

Recall that from the work of Heintze and Karcher in [36] on the length of a closed geodesic in a positively curved manifold we have that:

$$\ell_0(\tilde{M}) \geq \frac{2\pi \operatorname{Vol}(\tilde{M})}{\operatorname{Vol}(\mathbb{S}^{n+k})}, \tag{1.5.6}$$

where $\operatorname{Vol}(\mathbb{S}^{n+k})$ denotes the volume of the $n+k$-dimensional sphere of constant sectional curvature equal to 1. Put together (1.5.5) with (1.5.6) to get

$$k \leq \left(\frac{\operatorname{Vol}(\mathbb{S}^{n+k})}{\operatorname{Vol}(\tilde{M})} - 1\right)(n-1).$$

Now the obvious inequality $\operatorname{Vol}(\tilde{M}) \geq \operatorname{Vol}(M)$ gives Theorem 1.6.

For the proof of Theorem 1.7 observe that since $\tilde{M}$ is a covering space of M, it follows that $\chi(\tilde{M}) = d\chi(M)$, where d denotes the degree of the covering map $\tilde{M} \to M$. Thus $\chi(\tilde{M}) \neq 0$. In particular, Poincaré duality implies that $\tilde{M}$ is even-dimensional.

The submersion $\tilde{M} \to \tilde{B}$ is a fibration with fiber F' and then $\chi(\tilde{M}) = \chi(\tilde{B})\chi(F')$. It follows that $\chi(F') \neq 0$, so we can apply Theorem 1.5 to the Riemannian submersion $\tilde{\pi} : \tilde{M}^{n+k} \to \tilde{B}^n$. We get the inequality $\ell_0(\tilde{M}) \leq \ell_0(\tilde{B})$.

Now, by work of Klingenberg in [46], we have that the length $\ell_0(\tilde{M})$ of the shortest closed geodesic in a simply connected even-dimensional positively curved manifold $\tilde{M}$ satisfies:

$$\ell_0(\tilde{M}) \geq \frac{2\pi}{\sqrt{\max \sec_{\tilde{M}}}}. \tag{1.5.7}$$

Put together the inequalities (1.5.5) and (1.5.7) to obtain:

$$k \leq \left(\sqrt{\max \sec_{\tilde{M}}} - 1\right)(n-1).$$

Clearly $\sqrt{\max \sec_{\tilde{M}}} = \sqrt{\max \sec_M}$, and we get Theorem 1.7.

Chapter 2

Cohomogeneity one orbifolds

Let G be a compact Lie group acting on a topological space X. The *cohomogeneity* of the action is, by definition, the dimension of the orbit space X/G. In this chapter we study cohomogeneity one smooth actions of compact Lie groups on closed, smooth orbifolds. As for manifolds, a smooth orbifold is *closed* if its underlying topological space is compact and has no boundary. Throughout this chapter, we will work in the category of orbifolds. Therefore, smooth maps, diffeomorphisms, bundles, etc. will be understood to be morphisms and objects in this category.

We generalize the well-known structure theorem for closed cohomogeneity one smooth manifolds. Recall that the *cone* $C(X)$ over a topological space X is defined as the quotient space $C(X) = (X \times [0,1]) \,/\, (X \times \{0\})$; as an example, observe that the cone over the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the unit ball in $\mathbb{R}^{n+1}$.

Theorem 2.1. *Let $\mathcal{O}$ be a closed, connected, smooth orbifold with an (almost) effective smooth action of a compact, connected Lie group G with principal isotropy group H. If the action is of cohomogeneity one, then the orbit space $\mathcal{O}/G$ is homeomorphic to a circle or to a closed interval and the following statements hold.*

1. *If the orbit space is a circle, then $\mathcal{O}$ is equivariantly diffeomorphic to a G/H-bundle over a circle with structure group $N(H)/H$, where $N(H)$ is the normalizer of H in G. In particular, $\mathcal{O}$ is a manifold and its fundamental group is infinite.*

2. *If the orbit space is homeomorphic to an interval, say $[-1,+1]$, then:*
   - *(a) There are two non-principal orbits, $\pi^{-1}(\pm 1) = G/K_{\pm}$, where $\pi : \mathcal{O} \to \mathcal{O}/G$ is the natural projection and $K_{\pm}$ is the isotropy group of the G-action at a point in $\pi^{-1}(\pm 1)$.*
   - *(b) The orbifold singular set of $\mathcal{O}$ is either empty, a non-principal orbit or both non-principal orbits.*
   - *(c) The orbifold $\mathcal{O}$ is equivariantly diffeomorphic to the union of two orbifiber bundles over the two non-principal orbits whose fibers are cones over spherical space*

forms, that is,

$$\mathcal{O} \approx G \times_{K_-} C\left(\mathbb{S}_-/\Gamma_-\right) \cup_{G/H} G \times_{K_+} C\left(\mathbb{S}_+/\Gamma_+\right),$$

where $\mathbb{S}_\pm$ *denotes the round sphere of dimension* $\dim \mathcal{O} - \dim G/K_\pm - 1$ *and* $\Gamma_\pm$ *is a finite group acting freely and by isometries on* $\mathbb{S}_\pm$*. The action is determined by a group diagram* (G, H, K_-, K_+) *with group inclusions* $H \leq K_\pm \leq G$ *and where* $K_\pm/H$ *are spherical space forms* $\mathbb{S}_\pm/\Gamma_\pm$*.*

(d) *Conversely, a group diagram* (G, H, K_-, K_+) *with* $H \leq K_\pm \leq G$ *and where* $K_\pm/H$ *are spherical space forms, determines a cohomogeneity one orbifold as in part (c).*

Note that, although not explicitly contained in the group diagram (G, H, K_-, K_+), the inclusions $H \hookrightarrow K_\pm \hookrightarrow G$ are an important part of the group action information. Indeed, the same 4-tuple (G, H, K_-, K_+) may give raise to different cohomogeneity one manifolds, depending on the inclusion maps. For example, both $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$ admit cohomogeneity one actions of the torus T^2 with associated group diagram $(T^2, 1, \mathbb{S}^1, \mathbb{S}^1)$, where 1 denotes the trivial group, but different inclusion maps (see, for example, [55]).

Observe that the free and isometric actions of the finite groups $\Gamma_\pm$ on the round spheres $\mathbb{S}_\pm$ are important as well to obtain the orbifold structure of the points in the non-principal orbits. This is particularly important when $\dim \mathbb{S}_\pm = 1$, since $\mathbb{S}_\pm/\Gamma_\pm$ is again diffeomorphic to $\mathbb{S}_\pm$. For example, consider the standard $\mathbb{S}^1$-action on the topological 2-sphere X. Endow X with the usual smooth structure, the tear drop structure, and the rugby ball structure respectively. Since the topological action is the same, the group diagram is $(\mathbb{S}^1, 1, \mathbb{S}^1, \mathbb{S}^1)$ in all cases, and hence $K_\pm/H = \mathbb{S}^1$. In order to distinguish their orbifold structures, it is important to explicitly consider the $\mathbb{Z}_n$-action on the singular point (resp. the two singular points) of the tear drop (resp. rugby ball).

Remark. In the context of the present chapter, the word "singular" may refer to two different properties. It may refer to the *singular orbits* of a compact Lie group action (i.e. orbits whose dimension is less than the dimension of a principal orbit) or to the singular set of an orbifold. A priori these are not related. We will be careful in making clear the conditions in which we use the term. In the cohomogeneity one literature, the non-principal orbits corresponding to the endpoints of the orbit space of a closed cohomogeneity one manifold are sometimes referred to as "singular orbits" (although in principle they could be exceptional orbits). To avoid confusion, we will always refer to these orbits as the *non-principal orbits* of the action.

To put Theorem 2.1 into perspective, recall that there exist analogous structure results for cohomogeneity one actions on closed smooth manifolds, on closed topological manifolds and on closed Alexandrov spaces (cf. [54, 39, 23, 22]). In all these cases, the only differences with Theorem C appear when the orbit space is homeomorphic to an interval. If X is a such a cohomogeneity one smooth manifold (respectively smooth orbifold, topological manifold, Alexandrov space), then X is equivariantly diffeomorphic (resp. diffeomorphic, homeomorphic, homeomorphic) to the smooth manifold (resp. smooth orbifold, topological manifold, Alexandrov space) constructed as the union of two fiber bundles over the non-principal orbits whose fibers are cones over certain spaces $K_{\pm}/H$ specified in the following table:

X	$K_{\pm}/H$
Smooth manifold	A round sphere
Topological manifold	A round sphere or the Poincaré homology sphere
Riemannian orbifold	A spherical space form
Alexandrov space	A homogeneous positively curved Riemannian manifold

The Poincaré homology sphere is a 4-dimensional spherical space form, thus the following corollary follows from Theorem 2.1.

Corollary 2.2. *Let X be a closed cohomogeneity one topological manifold with a cohomogeneity one action by a Lie group G. Then X admits a smooth orbifold structure $\mathcal{O}_X$ such that the G-action is smooth.*

Clearly, every topological manifold with a transitive action of a compact Lie group is homeomorphic to a smooth manifold. In view of Corollary 2.2, it is thus natural to ask the following

Question 2.3. *Given $k \geq 2$, when is a cohomogeneity k closed topological manifold equivariantly homeomorphic to a smooth orbifold?*

Recall that Alexandrov spaces are inner metric spaces with a lower curvature bound (in the triangle comparison sense); they are synthetic generalizations of Riemannian manifolds with (sectional) curvature bounded below and, more generally, of Riemannian orbifolds with a lower curvature bound (see [15, 16]). The following corollary to Theorem 2.1 follows from the fact that the lowest dimension where a homogeneous positively curved manifold is not a space form is 4.

Corollary 2.4. *Let (X, d) be a closed cohomogeneity one Alexandrov space with a cohomogeneity one action by a Lie group G such that the codimension of both non-principal orbits is at most 4. Then X admits a smooth orbifold structure $\mathcal{O}_X$ such that the G-action is smooth.*

Since Riemannian orbifolds are Alexandrov spaces, results for cohomogeneity-one Alexandrov spaces hold for cohomogeneity-one orbifolds. This is the case, for example, for results relating to the group diagram (see [22, Section 2]).

The chapter is organized as follows. In Section 2.1, we fix notation and review some basic facts about orbifolds and smooth actions. We prove Theorem 2.1 in Section 2.2.

The results in this chapter are joint work with Fernando Galaz-García (KIT) and they will be contained in the forthcoming paper [20].

2.1 Preliminaries

In this section we collect the basic definitions and facts about orbifolds that we will use in the proof of Theorem 2.1. We have based our discussion on [13, 21, 45].

2.1.1 Smooth orbifolds

Definition 2.5. An *n-dimensional (differentiable) orbifold atlas* on a second-countable, Hausdorff topological space Q is given by the following data:

1. An open cover $\{V_i\}_{i\in I}$ of Q indexed by a set I.

2. For each $i \in I$, a finite subgroup Γ_i of the group of diffeomorphisms of a simply connected n-manifold X_i and a continuous map $\mathsf{q}_i : X_i \to V_i$ such that q_i induces a homeomorphism from X_i/Γ_i onto V_i. The collection $(V_i, X_i, \Gamma_i, \mathsf{q}_i)$ is called a *local (uniformizing) chart.*

3. For all $z_i \in X_i$ and $z_j \in X_j$ such that $\mathsf{q}_i(z_i) = \mathsf{q}_j(z_j)$, there is a diffeomorphism h from an open connected neighborhood W of z_i to a neighborhood of z_j such that $\mathsf{q}_j \circ h = \mathsf{q}_i|_W$. Such a map h is called a *change of chart*; it is well defined up to composition with an element of Γ_j. In particular, if $i = j$, then h is the restriction of an element of Γ_i.

The family $\{(V_i, X_i, \Gamma_i, \mathsf{q}_i)\}_{i\in I}$ is called an *orbifold atlas* on Q.

The sources X_i can be thought to be open balls in $\mathbb{R}^n$.

Definition 2.6. Let $\{(V_i, X_i, \Gamma_i, \mathsf{q}_i)\}_{i\in I_1}$ and $\{(V_i, X_i, \Gamma_i, \mathsf{q}_i)\}_{i\in I_2}$ be orbifold atlases over a given topological space Q. We say that they define the same orbifold structure on Q if the union atlas $\{(V_i, X_i, \Gamma_i, \mathsf{q}_i)\}_{i\in I_1\cup I_2}$ satisfies the compatibility condition (3) in Definition 2.5.

Definition 2.6 determines a equivalence relation on the set of orbifold atlases over a given topological space Q.

Definition 2.7. An n-dimensional *smooth orbifold*, denoted by $\mathcal{O}$, is a second-countable, Hausdorff topological space $|\mathcal{O}|$, called the *underlying topological space* of $\mathcal{O}$, together with an equivalence class of orbifold atlases on $\mathcal{O}$.

Let $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ be a uniformizing chart with $p \in V_p$. If $\mathsf{q}_p^{-1}(p)$ consists only of one point, then $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ is called a *good local chart* around $p \in V_p$. In particular, $\mathsf{q}_p^{-1}(p)$ is fixed by the action of Γ_p on X_p.

We will write $p \in \mathcal{O}$ to denote a point p in the topological space $|\mathcal{O}|$. Given $p \in \mathcal{O}$, one can always find a good local chart $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ around p. Moreover, the corresponding group Γ_p does not depend on the choice of good local chart around p and it is referred to as the *local (orbifold) group at p.* From now on we will consider only good local charts.

Definition 2.8. The *singular set* $\Sigma_{\mathcal{O}}$ of an orbifold $\mathcal{O}$ consists of those points $p \in \mathcal{O}$ whose local group Γ_p is non-trivial. The regular part $\mathcal{O} \setminus \Sigma_{\mathcal{O}}$ will be denoted by $\mathcal{O}_0$ and it is a (possibly non-complete) manifold.

Proposition 2.9 (Newmann, Thurston)**.** *The singular set $\Sigma_{\mathcal{O}}$ of an orbifold $\mathcal{O}$ is a closed set with empty interior.*

Let us give some examples:

1. A manifold is a particular case of orbifold whose singular set is empty.
2. A manifold M with boundary can be given an orbifold structure in which its boundary becomes a "mirror". Any point on the boundary has a neighborhood modelled on $\mathbb{R}^n/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts by reflection in a hyperplane.
3. Let M be a manifold equipped with a properly discontinuous action of a discrete group Γ. Recall that the action of Γ is said to be *properly discontinuous* if for every compact set $K \subset M$, there are only finitely many $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$. The quotient space M/Γ inherits an orbifold structure (see [66, Proposition 13.2.1]), which we will denote simply by M/Γ. An orbifold that arise in this way is called *good orbifold*. Observe that if in addition the Γ-action is free then M/Γ is a manifold.
4. The topological 2-dimensional sphere can be endowed with different orbifold structures.
   (a) If the singular set consists of one point, modelled on $\mathbb{R}^2/\mathbb{Z}_k$, where $\mathbb{Z}_k$ acts by rotations on $\mathbb{R}^2$, the orbifold structure is known as *the tear drop* and provides an example of an orbifold which is not good.
   (b) If the singular set consists of two points, modelled on $\mathbb{R}^2/\mathbb{Z}_{k_1}$ and $\mathbb{R}^2/\mathbb{Z}_{k_2}$ respectively, the orbifold structure is not good unless $k_1 = k_2$. In the latter case, the orbifold structure is known as *the rugby ball*.

Definition 2.10. A *smooth map* $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds is given by a continuous map $|\varphi| : |\mathcal{O}_1| \to |\mathcal{O}_2|$ such that, if $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ and $(V_{|\varphi|(p)}, X_{|\varphi|(p)}, \Gamma_{|\varphi|(p)}, \mathsf{q}_{|\varphi|(p)})$ are good local charts around $p \in \mathcal{O}_1$ and $|\varphi|(p) \in \mathcal{O}_2$ respectively, then there is a (possibly non-unique) smooth map $\tilde{\varphi}_p : X_p \to X_{|\varphi|(p)}$ so that the diagram

$$\begin{tikzcd} X_p \arrow[r, "\tilde{\varphi}_p"] \arrow[d, "\mathsf{q}_p"'] & X_{|\varphi|(p)} \arrow[d, "\mathsf{q}_{|\varphi|(p)}"] \\ V_p \arrow[r, "|\varphi|"'] & V_{|\varphi|(p)} \end{tikzcd}$$

commutes. The map $\tilde{\varphi}_p$ is called a *lift* of φ around p. Given two such lifts $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ around p, there exists $\gamma \in \Gamma_{|\varphi|(p)}$ such that $\tilde{\varphi}_1 = \gamma\tilde{\varphi}_2$.

Definition 2.11. A smooth map $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ is a *diffeomorphism* if it is bijective and has a smooth inverse.

We say that a map $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds satisfies certain topological condition (e.g. surjectivity, continuity, etc.) if the map $|\varphi| : |\mathcal{O}_1| \to |\mathcal{O}_2|$ between the underlying topological spaces does.

Definition 2.12. An *orbifiber bundle* $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ is a surjective smooth map together with a third orbifold $\mathcal{O}_3$ such that

1. for each $p \in |\mathcal{O}_2|$, there is a uniformizing chart $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ around p, along with an action of Γ_p on $\mathcal{O}_3$ and a diffeomorphism

$$(\mathcal{O}_3 \times X_p)\,/\Gamma_p \longrightarrow \mathcal{O}_1|_{|\varphi|^{-1}(V_p)},$$

   where $\mathcal{O}_1|_{|\varphi|^{-1}(V_p)}$ denotes the induced orbifold structure on the topological space $|\varphi|^{-1}(V_p) \subset |\mathcal{O}_1|$ and

2. the following diagram commutes:

$$\begin{tikzcd} (\mathcal{O}_3 \times X_p)\,/\Gamma_p \arrow[r] \arrow[d] & \mathcal{O}_1 \arrow[d] \\ X_p/\Gamma_p \arrow[r] & \mathcal{O}_2 \end{tikzcd}$$

Definition 2.13. 1. An *orbivector space* is a triple $(\mathbb{E}, \Gamma, \rho)$ where $\mathbb{E}$ is a vector space, Γ is a finite group and ρ is a linear representation of Γ in $\mathbb{E}$.

2. A *linear map* between orbivector spaces $(\mathbb{E}, \Gamma, \rho)$ and $(\mathbb{E}', \Gamma', \rho')$ consists of a linear map $T : \mathbb{E} \to \mathbb{E}'$ in the usual sense and a homomorphism $H : \Gamma \to \Gamma'$ such that $T \circ (\rho(\gamma)) = \rho'(H(\gamma)) \circ T$ for all $\gamma \in \Gamma$.

3. An *orbivector bundle* is an orbifiber bundle $E \to \mathcal{O}$ which is locally isomorphic to $(\mathbb{E} \times X_p)/\Gamma_p$, where $(\mathbb{E}, \Gamma, \rho)$ is an orbivector space on which Γ_p acts linearly.

The *tangent bundle* $T\mathcal{O}$ of an orbifold $\mathcal{O}$ is an orbivector bundle which is locally isomorphic to TX_p/Γ_p. We call the orbivector space TX_p/Γ_p the *tangent space* to $\mathcal{O}$ at p and denote it by $T_p\mathcal{O}$. A smooth map $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ induces a smooth map $\varphi_* : T\mathcal{O}_1 \to T\mathcal{O}_2$ in terms of the differential of local lifts $\tilde{\varphi}_p$ of φ around points $p \in \mathcal{O}_1$. The map φ_* is called the *differential* of φ.

We now recall the definitions of orbifold covering space and universal covering space (see [66, Chapter 13]).

Definition 2.14. Let $\mathcal{O}_1, \mathcal{O}_2$ be smooth orbifolds. An *orbifold covering map* is a continuous map $|\varphi| : |\mathcal{O}_1| \to |\mathcal{O}_2|$ such that each point $p \in |\mathcal{O}_2|$ has a good local chart $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ for which each connected component U_i of $|\varphi|^{-1}(V_p)$ is homeomorphic to

X_p/Γ_i, where Γ_i is a subgroup of Γ_p. These homeomorphisms $U_i \to X_p/\Gamma_i$ give rise to the following commutative diagram:

$$\begin{tikzcd} U_i \arrow[r] \arrow[dr, "|\varphi|"'] & X_p/\Gamma_i \arrow[d] \\ & X_p/\Gamma_p = V_p \end{tikzcd}$$

where the map $X_p/\Gamma_i \to X_p/\Gamma_p$ is the obvious quotient map.

Observe that, in general, the map $|\varphi| : |\mathcal{O}_1| \to |\mathcal{O}_2|$ is not a covering map in the topological sense. Note that an orbifiber bundle with a zero-dimensional fiber is an orbifold covering map.

Definition 2.15. Let $|\varphi| : |\tilde{\mathcal{O}}| \to |\mathcal{O}|$ be an orbifold covering map and choose a point $\tilde{p} \in \tilde{\mathcal{O}}$ in the regular part of $\tilde{\mathcal{O}}$. We say that $\tilde{\mathcal{O}}$ is the *universal orbifold covering space* of $\mathcal{O}$ if for any other orbifold covering map $|\varphi'| : |\mathcal{O}'| \to |\mathcal{O}|$ and any election of a regular point $p' \in \mathcal{O}'$ such that $|\varphi|(\tilde{p}) = |\varphi'|(p')$, there exists a lift $|\phi| : |\tilde{\mathcal{O}}| \to |\mathcal{O}'|$ of $|\varphi|$. In other words, the diagram

$$\begin{tikzcd} |\tilde{\mathcal{O}}| \arrow[dd, "|\varphi|"'] \arrow[dr, "|\phi|"] & \\ & |\mathcal{O}'| \arrow[dl, "|\varphi'|"] \\ |\mathcal{O}| & \end{tikzcd}$$

commutes and $|\phi|$ is an orbifold covering map.

Proposition 2.16 ([66, Proposition 13.2.4])**.** *A smooth orbifold has a universal orbifold covering space.*

2.1.2 Riemannian orbifolds

Definition 2.17. Let $\{(V_i, X_i, \Gamma_i, \mathsf{q}_i)\}_{i\in I}$ be an orbifold atlas defining a smooth orbifold structure $\mathcal{O}$ on a given topological space. A *Riemannian metric* on $\mathcal{O}$ is a family of Γ_i-invariant Riemannian metrics $\langle , \rangle_i$ on the manifolds X_i such that each change of charts is an isometry.

We say that a Riemannian orbifold $\mathcal{O}$ has sectional curvature *bounded below by* $K \in \mathbb{R}$ if the Riemannian metric on each good local chart has sectional curvature bounded below by K. The underlying topological space $|\mathcal{O}|$ of a Riemannian orbifold $\mathcal{O}$ inherits a metric space structure as follows. A *smooth curve* in an orbifold $\mathcal{O}$ is a smooth map $\beta : [0, l] \to \mathcal{O}$ from an interval $[0, l]$ to $\mathcal{O}$; its *length* is defined as $L(\beta) = \int_0^l \|\beta'(t)\| dt$, where $\|\beta'(t)\|$ denotes the norm of the vector $\beta'(t)$ given by a local lifting of β to a good local chart. This induces a length structure on $|\mathcal{O}|$ with corresponding metric d. If $\mathcal{O}$ has sectional curvature bounded below by K, then $(|\mathcal{O}|, d)$ is an Alexandrov space with curvature bounded below by K (see [15, Proposition 10.2.4]). This follows from the fact that the curvature bound descends to quotients by a finite isometric group action. Observe that the tangent space $T_p\mathcal{O}$ corresponds to the tangent cone of the Alexandrov space $(|\mathcal{O}|, d)$ at p. We say that $\mathcal{O}$ is *complete* if $(|\mathcal{O}|, d)$ is a complete metric space. As pointed out in [45, Section 2.5], one can think of Riemannian orbifolds as Alexandrov spaces equipped with an additional structure that allows one to make sense of smooth functions. A *minimal geodesic* is a curve $\beta : [0, l] \to |\mathcal{O}|$ that realizes the distance between its endpoints. The lifts of minimal geodesic to local charts satisfy the geodesic equation.

Definition 2.18. A *local isometry* $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ between n-dimensional Riemannian orbifolds is a smooth map such that each lift is a local isometry. A *(Riemannian) isometry* $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds is a diffeomorphism which is a local isometry.

More generally, one can also define isometries of metric spaces.

Definition 2.19. Let (Y_1, d_1) and (Y_2, d_2) be two metric spaces and let $f : Y_1 \to Y_2$ be a bijective map.

1. The map f is a *local radial isometry* if for every point $p \in Y_1$ there exists a neighborhood $U_p \subset Y_1$ such that $d_1(p, p') = d_2(f(p), f(p'))$ for all $p' \in U_p$.

2. The map f is a *(metric) isometry* if for any $p, p' \in Y_1$,
$$d_1(p, p') = d_2(f(p), f(p')).$$

Lemma 2.20. *Let Y_1 and Y_2 be convex Riemannian manifolds (i.e., any two points can be joined by a minimal geodesic) with distance functions d_1 and d_2, respectively. Let $f : Y_1 \to Y_2$ be a bijective map that is a local radial isometry. Then f is a metric isometry.*

Proof. Let $p, p' \in Y_1$ be arbitrary points and let $\alpha : [0, l] \to Y_1$ be a minimal geodesic joining p to p'. By definition, for each $t \in [0, l]$ there exists a neighborhood $U_{\alpha(t)}$ of the point $\alpha(t)$ such that $d_1(\alpha(t), p) = d_2(f(\alpha(t)), f(p))$ for all $p \in U_{\alpha(t)}$.

The collection of the sets $U_{\alpha(t)}$ for $t \in [0, l]$ clearly covers the image of the curve α. Since $\alpha([0, l])$ is compact, there exists a finite subcovering that covers $\alpha([0, l])$. In other words, there exists a sequence of points $p_i = \alpha(t_i)$ for $0 \leq i \leq k$, with $t_0 < t_1 < \dots t_{k-1} < t_k$, such that the finite collection of the sets U_{p_i} covers $\alpha([0, l])$. We may suppose that the intersections $U_{p_i} \cap U_{p_{i+1}}$ are non-empty and that $p \in U_1$ and $p' \in U_k$. Let q_i be a point in $U_{p_i} \cap U_{p_{i+1}}$ for $1 \leq i \leq k-1$.

Using that α is a minimal geodesic and that f is a local radial isometry we get

$$\begin{aligned} d_1(p, p') &= d_1(p, p_1) + d_1(p_1, q_1) + d_1(q_1, p_2) + \dots d_1(q_{k-1}, p_k) + d_1(p_k, p') \\ &= d_2(f(p), f(p_1)) + d_2(f(p_1), f(q_1)) + d_2(f(q_1), f(p_2)) + \dots \\ &\quad + d_2(f(q_{k-1}), f(p_k)) + d_2(f(p_k), f(p')). \end{aligned}$$

On the other hand observe that f maps minimal geodesics to minimal geodesics, so

$$d_2(f(p), f(p')) = d_2(f(p), f(p_1)) + d_2(f(p_1), f(q_1)) + \dots d_2(f(p_k), f(p')),$$

it then follows that $d_1(p, p') = d_2(f(p), f(p'))$. □

It is well-known that for Riemannian manifolds metric and Riemannian isometries are the same [59, Theorem 18, p. 147]. Therefore, we can simply speak of *isometries* of a Riemannian manifold. For Riemannian orbifolds, since all the local lifts of a Riemannian isometry preserve the norm of tangent vectors, it is clear that a Riemannian isometry must be a metric isometry. As in the manifold case, the converse is true for Riemannian orbifolds.

Proposition 2.21. *Let $\mathcal{O}_1, \mathcal{O}_2$ be Riemannian orbifolds with induced distances d_1 and d_2. Then a metric isometry $|\varphi| : (|\mathcal{O}_1|, d_1) \to (|\mathcal{O}_2|, d_2)$ is a Riemannian isometry.*

Proof. For each $p \in \mathcal{O}_1$, let $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ and $(V_{|\varphi|(p)}, X_{|\varphi|(p)}, \Gamma_{|\varphi|(p)}, \mathsf{q}_{|\varphi|(p)})$ be good local charts around p and $|\varphi|(p)$ respectively such that $V_{|\varphi|(p)} = |\varphi|(V_p)$. The first step is to construct continuous local lifts $\tilde{\varphi}_p : X_p \to X_{|\varphi|(p)}$ of $|\varphi|$.

Note that $|\varphi|$ restricted to V_p is a homeomorphism onto its image $V_{|\varphi|(p)}$. It follows that $(V_{|\varphi|(p)}, X_p, \Gamma_p, |\varphi| \circ \mathsf{q}_p)$ is a good local chart. Observe that the maps $|\varphi| \circ \mathsf{q}_p : X_p \to V_{|\varphi|(p)}$ and $\mathsf{q}_{|\varphi|(p)} : X_{|\varphi|(p)} \to V_{|\varphi|(p)}$ are orbifold covering maps. We may assume that both X_p and $X_{|\varphi|(p)}$ are universal orbifold covering spaces of $V_{|\varphi|(p)}$. By Proposition 2.16, there exist a lift $\tilde{\varphi}_p$ of $|\varphi| \circ \mathsf{q}_p$, i.e., a continuous map $\tilde{\varphi}_p : X_p \to X_{|\varphi|(p)}$ such that the following diagram commutes:

$$\begin{tikzcd} X_p \arrow[r, "\tilde{\varphi}_p"] \arrow[d, "\mathsf{q}_p"'] & X_{|\varphi|(p)} \arrow[d, "\mathsf{q}_{|\varphi|(p)}"] \\ V_p \arrow[r, "|\varphi|"'] & V_{|\varphi|(p)} \end{tikzcd}$$

Observe that $\tilde{\varphi}_p$ is in particular a lift of $|\varphi| : V_p \to V_{|\varphi|(p)}$. The second step is to prove that $\tilde{\varphi}_p$ is a local radial isometry.

Let $\tilde{d}_1$ and $\tilde{d}_2$ be the induced distance functions on the Riemannian manifolds X_p and $X_{|\varphi|(p)}$ respectively. Denote by $\tilde{p} \in X_p$ the preimage of $p \in V_p$ by q_p. We claim that

$$\tilde{d}_1(\tilde{p}, z) = \tilde{d}_2(\tilde{\varphi}_p(\tilde{p}), \tilde{\varphi}_p(z)), \quad \text{for all } z \in X_p. \tag{2.1.1}$$

To prove the latter, suppose that

$$\tilde{d}_1(\tilde{p}, z) \neq \tilde{d}_2(\tilde{\varphi}_p(\tilde{p}), \tilde{\varphi}_p(z)), \quad \text{for some } z \in X_p.$$

Since Γ_p and $\Gamma_{|\varphi|(p)}$ act by isometries on X_p and $X_{|\varphi|(p)}$ respectively, it follows that

$$\tilde{d}_1(\gamma_1(\tilde{p}), \gamma_1(z)) \neq \tilde{d}_2(\gamma_2(\tilde{\varphi}_p(\tilde{p})), \gamma_2(\tilde{\varphi}_p(z))), \quad \text{for all } \gamma_1 \in \Gamma_p,\ \gamma_2 \in \Gamma_{|\varphi|(p)}.$$

Now observe that $\tilde{\varphi}_p(\tilde{p}) \in X_{|\varphi|(p)}$ is the preimage of $|\varphi|(p)$ by $\mathsf{q}_{|\varphi|(p)}$, so the elements in Γ_p and $\Gamma_{|\varphi|(p)}$ fix $\tilde{p}$ and $\tilde{\varphi}_p(\tilde{p})$ respectively. Therefore

$$\tilde{d}_1(\tilde{p}, \gamma_1(z)) \neq \tilde{d}_2(\tilde{\varphi}_p(\tilde{p}), \gamma_2(\tilde{\varphi}_p(z))), \quad \text{for all } \gamma_1 \in \Gamma_p,\ \gamma_2 \in \Gamma_{|\varphi|(p)},$$

From the definition of the distance functions $d_1, d_2, \tilde{d}_1, \tilde{d}_2$, it follows that:

$$\begin{aligned} \tilde{d}_1(\tilde{p}, \gamma_1(z)) &= d_1(p, \mathsf{q}_p(z)), \text{ for all } \gamma_1 \in \Gamma_p, \\ \tilde{d}_2(\tilde{\varphi}_p(\tilde{p}), \gamma_2(\tilde{\varphi}_p(z))) &= d_2(|\varphi|(p), \mathsf{q}_{|\varphi|(p)}(\tilde{\varphi}_p(z)), \text{ for all } \gamma_2 \in \Gamma_{|\varphi|(p)}. \end{aligned}$$

Denote the point $\mathsf{q}_p(z) \in V_p$ by p', and note that $\mathsf{q}_{|\varphi|(p)}(\tilde{\varphi}_p(z)) = |\varphi|(p')$, thus we get that

$$d_1(p, p') \neq d_2(|\varphi|(p), |\varphi|(p')),$$

which is a contradiction to the fact that $|\varphi|$ is a metric isometry.

Now let p' be a point in V_p different than p. Choose good local charts $(V_{p'}, X_{p'}, \Gamma_{p'}, \mathsf{q}_{p'})$ and $(V_{|\varphi|(p')}, X_{|\varphi|(p')}, \Gamma_{|\varphi|(p')}, \mathsf{q}_{|\varphi|(p')})$ around p' and $|\varphi|(p')$ respectively such that $V_{|\varphi|(p')} = |\varphi|(V_{p'})$. We can repeat the same argument to show that there exist a lift $\tilde{\varphi}_{p'} : X_{p'} \to X_{|\varphi|(p')}$ such that

$$\tilde{d}_1(\tilde{p'}, z) = \tilde{d}_2(\tilde{\varphi}_{p'}(\tilde{p'}), \tilde{\varphi}_{p'}(z)), \quad \text{for all } z \in X_{p'},$$

where $\tilde{d}_1$ and $\tilde{d}_2$ denote the distance functions in the Riemannian manifolds $X_{p'}$ and $X_{|\varphi|(p')}$ respectively, and $\tilde{p'} \in X_{p'}$ the preimage of $p' \in V_{p'}$ by $\mathsf{q}_{p'}$.

The intersections $V_p \cap V_{p'}$ and $V_{|\varphi|(p)} \cap V_{|\varphi|(p')}$ are non-empty by construction. Since $\mathcal{O}$ is a Riemannian orbifold, the associated changes of charts $X_p \to X_{p'}$ and $X_{|\varphi|(p)} \to X_{|\varphi|(p')}$ are Riemannian isometries and hence preserve distances. Denote $\tilde{p'}$ by y and $V_p \cap V_{p'}$ by U_y. We have proved that for each $y \in X_p$ there exists a neighborhood U_y such that

$$\tilde{d}_1(y, z) = \tilde{d}_2(\tilde{\varphi}_p(y), \tilde{\varphi}_p(z)), \quad \text{for all } z \in U_y \subset X_p,$$

i.e., $\tilde{\varphi}_p$ is a local radial isometry.

We may assume that both X_p and $X_{|\varphi|(p)}$ are convex, so Lemma 2.20 implies that $\tilde{\varphi}_p$ is a metric isometry, and hence a Riemannian isometry. This fact holds for every point $p \in \mathcal{O}$, thus the metric isometry $|\varphi| : (|\mathcal{O}_1|, d_1) \to (|\mathcal{O}_2|, d_2)$ induces a Riemannian isometry $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$. □

2.1.3 Smooth and isometric actions on orbifolds

A smooth action of a Lie group G on an orbifold $\mathcal{O}$ is a smooth map

$$\begin{aligned} \varphi : G \times \mathcal{O} &\to \mathcal{O} \\ (g, p) &\mapsto gp := |\varphi|(g, p) \end{aligned}$$

such that

1. $g_1(g_2p) = g_1g_2p$ for any $g_1, g_2 \in G$ and $p \in \mathcal{O}$,

2. $ep = p$, where e denotes the neutral element of G.

The *orbit* of $p \in \mathcal{O}$ is defined as the set $G(p) = \{gp \mid g \in G\}$. The *isotropy group* G_p at $p \in \mathcal{O}$ is the subgroup of G consisting of those elements that fix p. The orbit space will be denoted by $\mathcal{O}/G$. The *ineffective kernel* of the action is the subgroup $\mathrm{Ker} = \cap_{p\in\mathcal{O}} G_p$. The action is *(almost) effective* if the ineffective kernel is (discrete) trivial. Observe that the group $G' = G/\mathrm{Ker}$ always acts effectively on $\mathcal{O}$. Therefore we focus out attention on effective actions.

By definition, an orbifold diffeomorphism induces an isomorphism between the local groups of the corresponding points. Then every point in an orbit has the same local group. It follows that G also acts on both its regular part $\mathcal{O}_0$ and singular part $\Sigma_{\mathcal{O}}$.

Let G be a compact Lie group acting continuously on a topological space X. Then $G(p)$ is a *principal orbit* if there exists a neighborhood V of $p \in X$ such that for each $q \in V$, we have that $G_p < G_{gq}$ for some $g \in G$. The set of principal orbits is open and dense in X. Let k be the dimension of a principal orbit. Non-principal orbits are classified in *exceptional orbits* (if the dimension equals k) or *singular orbits* (if the dimension is less than k).

Definition 2.22. Let $\mathcal{O}_1, \mathcal{O}_2$ be orbifolds with a smooth action of a Lie group G. A smooth map $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ is *equivariant* if $|\varphi|(g(p)) = g(|\varphi|(p))$ for all $p \in \mathcal{O}_1$ and $g \in G$.

Let us give some examples of group actions:

1. Suppose that the action of a compact Lie group G on an orbifold $\mathcal{O}$ is *transitive*, i.e., for every $p, p' \in \mathcal{O}$, there exists $g \in G$ such that $gp = p'$. It follows that the singular set is empty and hence $\mathcal{O}$ is a manifold, which is said to be *homogeneous*.

2. Consider the tear drop orbifold structure on the unit 2-sphere, and let the north pole be the singular point. Consider the $\mathbb{S}^1$-rotation of the sphere around the axis joining the north and south poles. The north pole is both a singular orbit for the action and a singular point of the tear drop, while the south pole is a singular orbit for the action but a regular point of the orbifold structure.

Lemma 2.23 (Kleiner's Isotropy Lemma, cf. [21])**.** *Let $\mathcal{O}$ be a complete Riemannian orbifold with an isometric and effective action of a compact Lie group. Let $c : [0, l] \to \mathcal{O}$ be a minimal geodesic between the orbits $G(c(0))$ and $G(c(l))$. Then the isotropy group $G_{c(t)}$ is constant for $t \in (0, l)$; and it is a subgroup of the isotropy groups $G_{c(0)}$ and $G_{c(l)}$.*

Proposition 2.24. *[21, Proposition 2.12] Let $\mathcal{O}$ be an orbifold with a smooth effective action by a Lie group G. Let $p \in \mathcal{O}$ have isotropy subgroup $G_p \leq G$ and let $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ be a G_p-invariant good local chart around p. Then there exists a Lie group $\tilde{G}_p$ such that:*

1. $\tilde{G}_p$ acts on X_p and $X_p/\tilde{G}_p = V_p/G_p$;

2. $\tilde{G}_p$ is an extension of G_p by Γ_p, i.e. there exists a short exact sequence

$$\{e\} \longrightarrow \Gamma_p \longrightarrow \tilde{G}_p \xrightarrow{\rho} G_p \longrightarrow \{e\}$$

where e denotes the identity element in Γ_p and ρ denotes the obvious projection map.

Theorem 2.25 (Slice Theorem, cf. [72, Proposition 2.3.7])**.** *Suppose that a compact Lie group G acts on an orbifold $\mathcal{O}$ equipped with a G-invariant metric. Let $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ be a good local chart around p, let $\tilde{p} = \mathsf{q}_p^{-1}(p)$, and let $\tilde{T}_p G(p) \subset T_{\tilde{p}} X_p$ be the tangent space of $\mathsf{q}_p^{-1}(G(p) \cap V_p)$ at $\tilde{p}$. Then a G-invariant neighborhood of the orbit is equivariantly diffeomorphic to*

$$G \times_{G_p} \left(\tilde{T}_p G(p)^{\perp} / \Gamma_p \right);$$

and by Proposition 2.24 this is equivariantly diffeomorphic to

$$G \times_{\tilde{G}_p} \tilde{T}_p G(p)^{\perp},$$

where $\tilde{G}_p$ acts on G via the projection ρ.

Lemma 2.26. *If a compact Lie group G acts smoothly on a smooth orbifold $\mathcal{O}$, then $\mathcal{O}$ admits a Riemannian metric such that the action of G is by isometries.*

Proof. The proof of this lemma goes as in the manifold setting: given a Riemannian metric on $\mathcal{O}$, use the Haar measure $d\mu$ on G to average it and obtain a new metric constructed as follows. For a point p in $\mathcal{O}$, a good local chart $(V_p, X_p, \Gamma_p, \mathsf{q}_p)$ around p with a Riemannian metric $\langle , \rangle_p$ on X_p, and vectors $x, y \in T_p X_p$, define the new metric $\langle , \rangle'_p$ on X_p as

$$\langle x, y \rangle'_p = \int_G \langle \tilde{g}_{p*} x, \tilde{g}_{p*} y \rangle_{gp} d\mu_g,$$

where $\tilde{g}_{p*}$ denotes the differential of $\tilde{g}_p$, which is the lift of the smooth map given by the element $g \in G$ around p. The new Riemannian metric on $\mathcal{O}$ is G-invariant by construction. □

2.2 Proof of Theorem 2.1

By Lemma 2.26 we may assume that the smooth G-action is isometric with respect to some invariant Riemannian metric on $\mathcal{O}$.

To prove part (1), it suffices to show that $\mathcal{O}$ is a smooth manifold, then the conclusion follows from the structure theorem for cohomogeneity one smooth manifolds (cf. [39]). Assume that $\mathcal{O}/G$ is a circle. Then Kleiner's Isotropy Lemma implies that every orbit is principal. Since the local orbifold group in points of the same orbits must be constant, there cannot be points in $\mathcal{O}$ with non-trivial local group. Otherwise, the orbifold singular set would have non-empty interior, which is a contradiction.

Suppose now that the orbit space is homeomorphic to a closed interval. We may assume, after rescaling the metric in $\mathcal{O}$, that the orbit space is isometric to the closed interval $[-1,+1]$. Denote by $\pi : \mathcal{O} \to \mathcal{O}/G$ the projection map. Let $c : [-1,1] \to \mathcal{O}$ be a minimal geodesic between the orbits $\pi^{-1}(\pm 1)$. From Kleiner's Isotropy Lemma it follows that $G_{c(t)} = G_{c(0)}$ for all $t \in (-1,1)$ and $G_{c(t)}$ is a subgroup of $G_{c(\pm 1)}$. We let $H := G_{c(0)}$ and $K_\pm := G_{c(\pm 1)}$. By the Slice Theorem, a principal orbit must be in the interior of $\mathcal{O}/G = [-1,+1]$ and hence H must be a proper subgroup of $K_\pm$. This yields part (a).

Denote by $\Sigma_{\mathcal{O}}$ the orbifold singular set of $\mathcal{O}$. By [11, Theorem 3], one of the following occurs:

- $c \subset \Sigma_{\mathcal{O}}$ or
- $c \cap \Sigma_{\mathcal{O}} = \emptyset, c(-1), c(+1)$ or $c(\pm 1)$.

The first case can not happen: since every point in an orbit has the same orbifold isotropy (local) group, $\Sigma_{\mathcal{O}}$ would be the whole orbifold, contradicting Proposition 2.9. The second case yields part (b).

Denote by $p_\pm$ the point $c(\pm 1)$. Let $(V_\pm, X_\pm, \Gamma_\pm, \mathsf{q}_\pm)$ be a good local chart around $p_\pm$. Let $\tilde{T}_{p_\pm} G(p_\pm) \subset T_{\tilde{p}_\pm} X_\pm$ be the tangent space of $(\mathsf{q}_\pm)^{-1}(G(p_\pm) \cap V_\pm)$ at $\tilde{p}_\pm = (\mathsf{q}_\pm)^{-1}(p_\pm)$. Let $D_\pm$ be the unit disk in the orthogonal subspace $\tilde{T}_{p_\pm} G(p_\pm)^\perp$.

Since the isotropy group $K_\pm$ fixes $p_\pm$, we may suppose that the neighborhood $V_\pm$ is $K_\pm$-invariant, so that $K_\pm$ acts on $V_\pm$ by isometries. Let $\tilde{K}_\pm$ be the extension of $K_\pm$ given by Proposition 2.24 acting on $X_\pm$. The induced action of $\tilde{K}_\pm$ on $T_{\tilde{p}_\pm} X_\pm$ leaves $\tilde{T}_{p_\pm} G(p_\pm)$ invariant, therefore $\tilde{K}_\pm$ acts on $D_\pm$.

Since every vector in $D_\pm$ descends to the one-dimensional space $\mathcal{O}/G$, it follows that $\tilde{K}_\pm$ acts transitively on the boundary of $D_\pm$ with isotropy H. Observe that the boundary of the unit disk $D_\pm$ is the unit sphere of the corresponding dimension, which we denote by $\mathbb{S}_\pm$, and that $\mathbb{S}_\pm = \tilde{K}_\pm / H$.

The actions of $\Gamma_\pm$ and $\tilde{K}_\pm$ on $X_\pm$ commute, hence $\Gamma_\pm$ acts on $\mathbb{S}_\pm$ as well. It follows that $K_\pm$ acts transitively on (the a priori orbifold) $\mathbb{S}_\pm/\Gamma_\pm$ with isotropy H, therefore $\mathbb{S}_\pm/\Gamma_\pm = K_\pm/H$ is a homogeneous manifold, and in particular a spherical space form.

By the Slice Theorem for orbifolds, the following G-equivariant tubular neighborhoods of the non-principal orbits are equivariantly diffeomorphic to orbifiber bundles of the form:

$$\pi^{-1}[-1,0] = G \times_{K_-} (D_-/\Gamma_-) \qquad \pi^{-1}[0,1] = G \times_{K_+} (D_+/\Gamma_+) \,. \tag{2.2.1}$$

Then we have the following decomposition of our cohomogeneity one orbifold into two orbifiber bundles $G \times_{K_\pm} (D_\pm/\Gamma_\pm)$ glued along their common boundary $\pi^{-1}(0) \approx G/H$:

$$\mathcal{O} \approx G \times_{K_-} (D_-/\Gamma_-) \cup_{G/H} G \times_{K_+} (D_+/\Gamma_+) \,.$$

The action of $\Gamma_\pm$ on $D_\pm$ is by isometries so $\Gamma_\pm$ acts on $\mathbb{S}_\pm$. It follows that $D_\pm/\Gamma_\pm$ is isometric to the cone $C(\mathbb{S}_\pm/\Gamma_\pm)$ over $\mathbb{S}_\pm$ equipped with the so-called spherical cone metric (see [15]). This proves part (c).

To prove part (d), suppose we have group inclusions $H \le K_\pm \le G$ such that $K_\pm/H$ are spherical space forms $\mathbb{S}_\pm/\Gamma_\pm$. As in the manifold case (cf. [39, Section 1.1]), one can construct smooth orbifiber bundles as in (2.2.1) and glue them via an equivariant diffeomorphism. $\square$

Chapter 3

Nonnegative curvature on stable bundles over compact rank one symmetric spaces

In 1972, Cheeger and Gromoll proved the fundamental structure theorem for open (i.e., noncompact and without boundary) nonnegatively curved Riemannian manifolds:

Theorem 3.1 (The Soul Theorem [18])**.** *Let M be an open Riemannian manifold with nonnegative sectional curvature. There exists a compact, totally geodesic and totally convex submanifold S without boundary such that M is diffeomorphic to the normal bundle of S.*

Such a submanifold is called a *soul* of M. As an example, every point of $\mathbb{R}^2$ with the canonical flat metric is a soul. In contrast, if we endow $\mathbb{R}^2$ with the paraboloid metric, only the focal point is a soul. In the cylinder $\mathbb{S}^1 \times \mathbb{R}$, every circle $\mathbb{S}^1 \times \{a\}$ is a soul. More generally, any compact manifold S with nonnegative sectional curvature can be realized as the soul of some open nonnegatively curved manifold, the simplest one being $S \times \mathbb{R}^k$ with the product metric. It is natural to ask to what extent a converse to the Soul Theorem holds.

Question 3.2. *Let E be a vector bundle over a compact manifold S with nonnegative sectional curvature. Does E admit a complete metric of nonnegative sectional curvature with soul S?*

The answer is clearly affirmative when S is a homogeneous Riemannian manifold G/H of a compact Lie group G and E is a *homogeneous vector bundle*; that is, a vector bundle of the form $(G \times \mathbb{F}^m)/H$, where $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$ and H acts on $\mathbb{F}^m$ by means of a linear representation. The quotient $(G \times \mathbb{F}^m)/H$ is usually denoted by $G \times_H \mathbb{F}^m$, and its nonnegatively curved metric comes from the Riemannian submersion $G \times \mathbb{R}^m \to G \times_H \mathbb{R}^m$, thanks to O'Neill's formula.

The first obstructions to the above question were found by Özaydin and Walschap in [57]: a plane bundle over a torus admits a nonnegatively curved metric if and only if its rational Euler class vanishes. Later, Guijarro in his thesis [32] and Belegradek and Kapovitch in the series of papers [8] and [9], extended these results to a larger class of bundles over some other nonsimply connected souls.

However, in all these examples the obstructions are always due to the existence of a nontrivial fundamental group. So it is still important to see whether nonnegatively curved metrics exist when the base of the bundle is simply connected.

Even the case of the sphere $\mathbb{S}^n$ is still open, except for dimensions $n \leq 5$; see the article [30] by Grove and Ziller. So it is rather welcome to see that for any sphere there is a positive answer after passing to the stable realm.

Theorem 3.3 (Rigas [62]). *Let E be a real vector bundle over a sphere $\mathbb{S}^n$. Denote by k the trivial real vector bundle of rank k. Then, for some k the Whitney sum $E \oplus k = E \times \mathbb{R}^k$ admits a metric with nonnegative sectional curvature.*

The starting point in Rigas' proof is the isomorphism between *stable classes* of real vector bundles over $\mathbb{S}^n$ and the homotopy group $\pi_n(\mathrm{BO})$, where BO is the classifying space of the infinite orthogonal group O. He shows that the generators of $\pi_n(\mathrm{BO})$ can be realized by isometric embeddings of standard Euclidean spheres as totally geodesic submanifolds of Grassmannian manifolds. Using this fact he is able to prove the existence of homogeneous bundles in every stable class. Recall that two vector bundles E, F over a compact space are stably equivalent if there exist trivial bundles k_1, k_2 such that $E \oplus k_1$ is isomorphic to $F \oplus k_2$.

The statement of Rigas' Theorem was shown over $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{CP}^2 \# -\mathbb{CP}^2$ using *cohomogeneity one methods* (see [31]). Our goal is to extend these results to some other nonnegatively curved compact spaces. Natural candidates are the remaining compact rank one symmetric spaces (CROSSes).

Recall that a *symmetric space* is a homogeneous Riemannian manifold G/H such that for each point $p \in M$ there exist an isometry $\varphi : G/H \to G/H$ fixing p and such that its differential φ_{*p} equals the antipodal map $-Id$. On the other hand, the *rank of a geodesic* α in an arbitrary Riemannian manifold M is simply the dimension of the subspace of parallel fields $X(t)$ along α such that $R(X(t), \alpha'(t))\alpha'(t) = 0$ for all t. This subspace always includes the vector $\alpha'(t)$ (therefore the rank of a geodesic is always ≥ 1) and the subspace of parallel normal Jacobi fields along α. The *rank* of M is now defined as the minimum rank over all of the geodesics in M. For a symmetric space, the rank can be computed in terms of the Lie algebras of G and H. The only existing compact rank one symmetric spaces are the spheres $\mathbb{S}^n$, the projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$ and the Cayley plane $\mathrm{Ca}\mathbb{P}^2$.

In order to obtain Rigas' Theorem for all the CROSSes, the main tool will be the isomorphism between stable classes and reduced K-theory. K-theory of complex vector bundles over a topological space X was introduced around 1960 by Grothendieck, Atiyah and Hirzebruch (see [5]); in [6] the last two studied more closely the particular case when X is a compact homogeneous space. K-theory concerning real vector bundles has been also studied (see for example [40], [64]), although it is not so well understood as in the complex case. The following is the main result in this chapter.

Theorem 3.4. *Let E be an arbitrary real (resp. complex) vector bundle over a compact rank one symmetric space S. Denote by k the trivial real (resp. complex) vector bundle of rank k. Then, for some k the Whitney sum $E \oplus k = E \times \mathbb{R}^k$ (resp. $E \times \mathbb{C}^k$) is a homogeneous real (resp. complex) vector bundle and hence it admits a metric with nonnegative sectional curvature and soul S.*

In the case of the sphere our methods yield an alternative proof of Rigas' Theorem. Moreover, our approach allows us to give an upper bound for the least integer k satisfying Theorem 3.4. In order to state our result we need to recall that, as a consequence of the Bott Integrability Theorem (see [42], Chapter 20), if E is a real vector bundle over a sphere $\mathbb{S}^n$ of dimension $n \equiv 0 \pmod 4$, then its $(n/4)$-th Pontryagin class $p_{n/4}(E)$ is of the form

$$p_{n/4}(E) = ((n/2) - 1)!(\pm \mathsf{p}_E)a$$

for some natural number p_E, where a is a generator of $H^n(\mathbb{S}^n, \mathbb{Z})$.

Theorem 3.5. *Let E be an arbitrary real vector bundle over $\mathbb{S}^n$. Let k_0 be the least integer such that the Whitney sum $E \oplus k_0$ admits a metric with nonnegative sectional curvature. The following inequalities hold:*

- $k_0 \leq n + 1$, *if* $n \equiv 3, 5, 6, 7 \ (mod\ 8)$.
- $k_0 \leq 2^n$, *if* $n \equiv 1, 2 \ (mod\ 8)$.
- $k_0 \leq \max\{n + 1, 2^{n-1}\mathsf{p}_E\}$, *if* $n \equiv 0, 4 \ (mod\ 8)$.

The results by Atiyah and Hirzebruch on K-theory of complex vector bundles over homogeneous spaces were extended by several authors (see for example [1], [35], [51], [52], [61]). The following theorem will be a consequence of some of these results.

Theorem 3.6. *Let E be an arbitrary complex vector bundle over a manifold S in one of the two following classes $\mathscr{C}_i$:*

- *$\mathscr{C}_1$ is the class of compact nonnegatively curved manifolds S whose even dimensional Betti numbers $b_{2i}(S)$ vanish for $i \geq 1$, and such that $H^*(S, \mathbb{Z})$ is torsion-free.*
- *$\mathscr{C}_2$ is the class of compact normal homogeneous Riemannian manifolds G/H such that G is a compact, connected Lie group with $\pi_1(G)$ torsion-free and H a closed, connected subgroup of maximal rank.*

Denote by k the trivial complex vector bundle of rank k. Then, for some k the Whitney sum $E \oplus k = E \times \mathbb{C}^k$ admits a metric with nonnegative sectional curvature and soul S.

Odd-dimensional homology spheres admiting nonnegatively curved metrics belong to class $\mathscr{C}_1$, in particular the 7-dimensional exotic sphere which was shown to admit nonnegative curvature by Gromoll and Meyer in [27]. The class $\mathscr{C}_2$ includes such manifolds as even-dimensional spheres, complex and quaternionic Grassmannian manifolds, the Wallach flag manifolds W^6, W^{12} and W^{24} or the Cayley plane. Recall that manifolds in the class $\mathscr{C}_2$ inherit a nonnegatively curved metric from a biinvariant metric on G.

Remark. In the context of this chapter, the word "rank" may refer to three different notions. The *rank of a Riemannian manifold* is defined above. The *rank of a vector bundle* is just the dimension of its fibers. Finally, the *rank of a Lie group* G is defined as the dimension of the maximal torus in G, meaning the maximal compact, connected, abelian Lie subgroup of G (and therefore isomorphic to the standard torus). We say that a subgroup $H < G$ is of *maximal rank* if the rank of H equals the rank of G.

The chapter is organized as follows. Section 3.1 recalls basic definitions and facts about K-theory, stable classes and characteristic classes of vector bundles, and relates them in the homogeneous setting. Section 3.2 contains the proof of Theorem 3.6. Section 3.3 contains the proofs of Theorem 3.4 for the spheres and of Theorem 3.5. The proofs of Theorem 3.4 for projective spaces and the Cayley plane are given in Sections 3.4 and 3.5 respectively.

The contents of this chapter are in the article [24].

3.1 Stable classes and homogeneous bundles

Throughout this section $\mathbb{F}$ will denote either one of the fields $\mathbb{R}$ or $\mathbb{C}$.

3.1.1 Stable classes of vector bundles and $K_{\mathbb{F}}$-theory

We will denote by $\mathrm{Vect}_{\mathbb{F}}(M)$ the set of isomorphism classes of $\mathbb{F}$-vector bundles over a manifold M. The Whitney sum $\oplus$ and the tensor product of bundles $\otimes_{\mathbb{F}}$ endow $\mathrm{Vect}_{\mathbb{F}}(M)$ with a semiring structure. Let

$$\mathsf{c} : \mathrm{Vect}_{\mathbb{R}}(M) \to \mathrm{Vect}_{\mathbb{C}}(M) \quad \text{and} \quad \mathsf{r} : \mathrm{Vect}_{\mathbb{C}}(M) \to \mathrm{Vect}_{\mathbb{R}}(M)$$

be the complexification and the real restriction maps of vector bundles respectively. We will write $m_{\mathbb{F}}$ or just m (when there is no danger of confusion) for the trivial $\mathbb{F}$-vector bundle of rank m, which is isomorphic to $M \times \mathbb{F}^m$; and mE for the Whitney sum of E with itself m times.

If the manifold M is compact we have the following well-known result (see e.g. Lemma 9.3.5 in [3]).

Lemma 3.7. *Let $E \in \mathrm{Vect}_{\mathbb{F}}(M)$ with M compact. Then there exists $F \in \mathrm{Vect}_{\mathbb{F}}(M)$ such that $E \oplus F$ is isomorphic to a trivial bundle.*

From now on we assume that M is compact. We say that $E, F \in \mathrm{Vect}_{\mathbb{F}}(M)$ are *stably equivalent* if there exist trivial bundles m_1, m_2 such that $E \oplus m_1$ is isomorphic to $F \oplus m_2$. We will denote by $S_{\mathbb{F}}(M)$ the set of *stable classes* of bundles over M and by $\{E\}_{\mathbb{F}}$ the stable class of E. The Whitney sum gives $S_{\mathbb{F}}(M)$ the structure of an abelian semigroup. Furthermore, by Lemma 3.7, every element $\{E\}_{\mathbb{F}}$ has an inverse, so $S_{\mathbb{F}}(M)$ is an abelian group. Later on we will use the following theorem (see e.g. [42], Chapter 9).

Theorem 3.8. *Let E and F be real vector bundles of the same rank k over a compact n-dimensional manifold M such that $E \oplus m$ is isomorphic to $F \oplus m$ for some integer m. If $k \geq n + 1$, then E and F are isomorphic.*

We write $K_{\mathbb{F}}(M)$ for the *K-theory* ring of $\mathbb{F}$-vector bundles over M. This is the ring completion of the semiring $\mathrm{Vect}_{\mathbb{F}}(M)$. Its elements, called *virtual bundles*, are usually written in the form $[E]-[F]$, where $[E_1]-[F_1]$ equals $[E_2]-[F_2]$ if there exists another bundle E_3 such that $E_1 \oplus F_2 \oplus E_3$ and $E_2 \oplus F_1 \oplus E_3$ are isomorphic. Observe that $K_{\mathbb{F}}(M)$ is a commutative ring with unity.

When M is compact, every element in $K_{\mathbb{F}}(M)$ can be written in the form $[E]-[m]$. To prove this, choose a virtual bundle $[E_1]-[F_1]$. By Lemma 3.7 there exists a vector bundle $F_1^{\perp}$ such that $F_1 \oplus F_1^{\perp} = m$. Then clearly $[E_1]-[F_1]$ equals $[E_1 \oplus F_1^{\perp}]-[m]$.

Consider the ring homomorphism $d : K_{\mathbb{F}}(M) \to \mathbb{Z}$ given by $d([E]-[F]) = \mathrm{rank}(E) - \mathrm{rank}(F)$. The kernel of d is called the *reduced* K-theory ring and we will denote it by $\tilde{K}_{\mathbb{F}}(M)$. It is an ideal of $K_{\mathbb{F}}(M)$ and thus a ring without unity. There is a natural splitting $K_{\mathbb{F}}(M) = \tilde{K}_{\mathbb{F}}(M) \oplus \mathbb{Z}$. We recall the following well-known theorem that relates the two latter constructions (see e.g. Theorem 9.3.8 in [3]).

Theorem 3.9. *Let M be a compact manifold. Then $\tilde{K}_{\mathbb{F}}(M) \approx S_{\mathbb{F}}(M)$ as abelian groups. An isomorphism is given by:*

$$\begin{array}{rcl} \Phi_{\mathbb{F}} : \tilde{K}_{\mathbb{F}}(M) & \to & S_{\mathbb{F}}(M) \\ {[E]-[m]} & \mapsto & \{E\}_{\mathbb{F}} \end{array}$$

To simplify notation, from now on $E-F$ will denote the virtual bundle $[E]-[F]$. More details about these concepts can be found in [3], [5] and [42]. In the literature, the rings $K_{\mathbb{R}}(M), K_{\mathbb{C}}(M), \tilde{K}_{\mathbb{R}}(M)$ and $\tilde{K}_{\mathbb{C}}(M)$ are frequently denoted by $K(M), KO(M), \tilde{K}(M)$ and $\widetilde{KO}(M)$ respectively.

3.1.2 Characteristic classes

Roughly speaking, a *characteristic class* is a way of assigning to each $E \in \mathrm{Vect}_{\mathbb{F}}(M)$ a *cohomology class* of M which measures somehow the complexity of the bundle $E \to M$. We refer the reader to the classical reference [49] for all the definitions and details. In this chapter we use certain characteristic classes; let us recall some basic facts.

Let E be a complex vector bundle over M. The *k-th Chern class* of E, denoted by $c_k(E)$, is an element in the $2k$*-th integral cohomology group* $H^{2k}(M,\mathbb{Z})$. Here we list some of their properties:

- For any complex vector bundle E we have that $c_0(E) = 1$, and that $c_k(E) = 0$ for $k > \mathrm{rank}_{\mathbb{C}}(E)$. If in addition M is compact, then clearly $c_k(E) = 0$ for $k > n/2$, where n is the dimension of M, since the corresponding cohomology groups vanish.

- The Chern classes $c_k(E)$ of a trivial bundle $E = M \times \mathbb{C}^m$ are zero for all $k \geq 1$.

- The top Chern class of E (meaning $c_{\mathrm{rank}_{\mathbb{C}}(E)}(E)$) is always equal to the *Euler class* of its real restriction $\mathsf{r}(E)$.

- One can define the *total Chern class* of E as

$$c_T(E) = c_0(E) + c_1(E) + \cdots + c_{\mathrm{rank}_{\mathbb{C}}(E)}(E) \in H^*(M, \mathbb{Z}),$$

where $H^*(M, \mathbb{Z})$ denotes the graded *integral cohomology ring* of M.

- For $E, F \in \mathrm{Vect}_{\mathbb{C}}(M)$, we have the so-called *Whitney Product Formula*:

$$c_T(E \oplus F) = c_T(E)c_T(F),$$

where $c_T(E)c_T(F)$ denotes the product of $c_T(E)$ with $c_T(F)$ in $H^*(M, \mathbb{Z})$. It follows that the total Chern class is stable in the sense that $C_T(E \oplus m_{\mathbb{C}}) = c_T(E)$.

- The *Chern character* of E, denoted by $ch(E)$, is defined as

$$\begin{aligned} ch(E) = \mathrm{rank}_{\mathbb{C}}(E) + c_1(E) + \frac{1}{2}\left(c_1^2(E) - 2c_2(E)\right) \\ + \frac{1}{6}\left(c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E)\right) + \dots \end{aligned}$$

and it induces a ring homomorphism $ch : K_{\mathbb{C}}(M) \to H^*(M, \mathbb{Q})$ from the complex K-theory of M to its *rational cohomology ring*.

Finally, let E be a real vector bundle over a manifold M. The *k-th Pontryagin class*, denoted by $p_k(E)$, is defined as

$$p_k(E) = (-1)^k c_{2k}(\mathsf{c}(E)) \in H^{4k}(M, \mathbb{Z}).$$

3.1.3 Homogeneous vector bundles

For a Lie group G, denote by $\mathrm{Rep}_{\mathbb{F}}(G)$ the set of isomorphism classes of $\mathbb{F}$-representations of G. The direct sum $\oplus$ and the tensor product $\otimes_{\mathbb{F}}$ of representations endow $\mathrm{Rep}_{\mathbb{F}}(G)$ with a semiring structure. Let

$$\mathsf{c} : \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Rep}_{\mathbb{C}}(G) \quad \text{and} \quad \mathsf{r} : \mathrm{Rep}_{\mathbb{C}}(G) \to \mathrm{Rep}_{\mathbb{R}}(G)$$

stand for complexification and real restriction of representations. We will write $m_{\mathbb{F}}$ or simply m for the trivial representation of G on $\mathbb{F}^m$; and $m\rho$ for the sum of $\rho \in \mathrm{Rep}_{\mathbb{F}}(G)$ with itself m times.

Let ρ be a representation of G in the vector space $\mathbb{F}^m$. Recall that the *k-th exterior product* of ρ, denoted by $\Lambda^k(\rho)$, is the representation in $\Lambda^k(\mathbb{F}^m)$ induced in the obvious way. As a vector space, $\Lambda^k(\mathbb{F}^m)$ is isomorphic to $\mathbb{F}^{\binom{m}{k}}$. We set $\Lambda^0(\rho) = 1$ and $\Lambda^1(\rho) = \rho$. Observe that $\Lambda^k(\rho) = 0$ for $k > m$.

If $i : H \to G$ is the inclusion of a closed subgroup H, we denote by

$$i_{\mathbb{F}}^* : \mathrm{Rep}_{\mathbb{F}}(G) \to \mathrm{Rep}_{\mathbb{F}}(H)$$

the semiring homomorphism defined by restricting representations of G to H.

For each $\rho \in \mathrm{Rep}_{\mathbb{F}}(H)$ we have the diagonal action of H on $G \times \mathbb{F}^m$ from the right given by

$$\begin{array}{rcc} (G \times \mathbb{F}^m) \times H & \longrightarrow & G \times \mathbb{F}^m \\ ((g,v),h) & \longmapsto & (gh, \rho(h)^{-1}v) \end{array}$$

where m is the dimension of the representation ρ. The quotient space $E_\rho := (G \times \mathbb{F}^m)/H$ is the total space of an associated $\mathbb{F}$-vector bundle $\pi_\rho : E_\rho \to G/H$ over the homogeneous manifold G/H, where π_ρ is the obvious projection map. Vector bundles arising in this way are called *homogeneous.*

We have an analogue result to Lemma 3.7 in the homogeneous setting. More precisely (see [65]), we have the following:

Lemma 3.10. *Let G be a compact Lie group, H a closed subgroup and $E_\rho \in \mathrm{Vect}_{\mathbb{F}}(G/H)$ a homogeneous bundle. Then there exists a representation $\rho^\perp \in \mathrm{Rep}_{\mathbb{F}}(H)$ such that $E_\rho \oplus E_{\rho^\perp}$ is isomorphic to a trivial bundle.*

Recall that E_ρ is isomorphic to a trivial bundle if and only if ρ is the restriction to H of a representation of G (see [26], page 131), i.e., if $\rho = i_{\mathbb{F}}^*(\tau)$ for some $\tau \in \mathrm{Rep}_{\mathbb{F}}(G)$.

It is straightforward to check that the following map is a morphism of semirings

$$\begin{array}{rcc} \alpha_{\mathbb{F}} : \mathrm{Rep}_{\mathbb{F}}(H) & \to & \mathrm{Vect}_{\mathbb{F}}(G/H) \\ \rho & \mapsto & E_\rho \end{array}$$

Composing $\alpha_{\mathbb{F}}$ with the map $\{\}_{\mathbb{F}} : \mathrm{Vect}_{\mathbb{F}}(G/H) \to S_{\mathbb{F}}(G/H)$ that assigns stable classes to vector bundles we get the induced morphism of semigroups

$$\begin{array}{rcc} \{\alpha\}_{\mathbb{F}} : \mathrm{Rep}_{\mathbb{F}}(H) & \to & S_{\mathbb{F}}(G/H) \\ \rho & \mapsto & \{E_\rho\}_{\mathbb{F}} \end{array}$$

The ring completion $\mathfrak{R}_{\mathbb{F}}(G)$ of the semiring $\mathrm{Rep}_{\mathbb{F}}(G)$ is defined in the same manner as the ring completion $K_{\mathbb{F}}(M)$ of the semiring $\mathrm{Vect}_{\mathbb{F}}(M)$. The semiring morphisms $\mathsf{r}, \mathsf{c}, i_{\mathbb{F}}^*$ and $\alpha_{\mathbb{F}}$ extend to ring morphisms of the corresponding ring completions, which we denote in the same way. We will write $\rho_1\rho_2$ and $\rho_1 + \rho_2$ (resp. E_1E_2 and $E_1 + E_2$) to denote the multiplication and the sum laws in $\mathfrak{R}_{\mathbb{F}}(G)$ (resp. $K_{\mathbb{F}}(M)$) induced from tensor product and direct sum of representations (resp. vector bundles). The following diagrams commute:

$$\begin{tikzcd} \mathfrak{R}_{\mathbb{R}}(G) \arrow[r, "i_{\mathbb{R}}^*"] \arrow[d, "\mathsf{c}"] & \mathfrak{R}_{\mathbb{R}}(H) \arrow[d, "\mathsf{c}"] \\ \mathfrak{R}_{\mathbb{C}}(G) \arrow[r, "i_{\mathbb{C}}^*"] & \mathfrak{R}_{\mathbb{C}}(H) \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{R}_{\mathbb{R}}(H) \arrow[r, "\alpha_{\mathbb{R}}"] \arrow[d, "\mathsf{c}"] & K_{\mathbb{R}}(G/H) \arrow[d, "\mathsf{c}"] \\ \mathfrak{R}_{\mathbb{C}}(H) \arrow[r, "\alpha_{\mathbb{C}}"] & K_{\mathbb{C}}(G/H) \end{tikzcd}$$

The maps $\{\alpha\}_{\mathbb{F}} : \mathrm{Rep}_{\mathbb{F}}(H) \to S_{\mathbb{F}}(G/H)$ and $\alpha_{\mathbb{F}} : \mathfrak{R}_{\mathbb{F}}(H) \to K_{\mathbb{F}}(G/H)$ are related, as shown in the lemma below. Denote by $\tilde{\mathfrak{R}}_{\mathbb{F}}(H)$ the kernel of the map $d : \mathfrak{R}_{\mathbb{F}}(H) \to \mathbb{Z}$ defined by $d(\rho_1 - \rho_2) = \dim \rho_1 - \dim \rho_2$. It is an ideal of $\mathfrak{R}_{\mathbb{F}}(H)$.

Lemma 3.11. *Let G be a compact Lie group and H a closed subgroup. Then, with the notations above,*

1. *$\alpha_{\mathbb{F}}(\tilde{\mathfrak{R}}_{\mathbb{F}}(H)) \subset \tilde{K}_{\mathbb{F}}(G/H)$, and if the map $\alpha_{\mathbb{F}} : \mathfrak{R}_{\mathbb{F}}(H) \to K_{\mathbb{F}}(G/H)$ is surjective, then the restriction $\alpha_{\mathbb{F}} : \tilde{\mathfrak{R}}_{\mathbb{F}}(H) \to \tilde{K}_{\mathbb{F}}(G/H)$ is also surjective.*

2. *The following equality holds:*

$$\Phi_{\mathbb{F}} \circ \alpha_{\mathbb{F}}(\tilde{\mathfrak{R}}_{\mathbb{F}}(H)) = \{\alpha\}_{\mathbb{F}}(\mathrm{Rep}_{\mathbb{F}}(H)),$$

where $\Phi_{\mathbb{F}}$ is the map from Theorem 3.9. In particular, if $\alpha_{\mathbb{F}}$ is surjective, then $\{\alpha\}_{\mathbb{F}}$ is also surjective.

Proof. The first statement follows immediately from the definition of $\alpha_{\mathbb{F}}$.

As for the second part, the inclusion $\{\alpha\}_{\mathbb{F}}(\mathrm{Rep}_{\mathbb{F}}(H)) \subset \Phi_{\mathbb{F}} \circ \alpha_{\mathbb{F}}(\tilde{\mathfrak{R}}_{\mathbb{F}}(H))$ is obvious.

Now, every element in $\alpha_{\mathbb{F}}(\tilde{\mathfrak{R}}_{\mathbb{F}}(H))$ is of the form $E_{\rho_1} - E_{\rho_2}$, for some $\rho_1, \rho_2 \in \mathrm{Rep}_{\mathbb{F}}(H)$ satisfying $\dim \rho_1 = \dim \rho_2$. By Lemma 3.10 there exists $\rho_2^{\perp} \in \mathrm{Rep}_{\mathbb{F}}(H)$ such that $E_{\rho_2} \oplus E_{\rho_2^{\perp}} = m$. Thus

$$E_{\rho_1} - E_{\rho_2} = E_{\rho_1} - E_{\rho_2} + E_{\rho_2^{\perp}} - E_{\rho_2^{\perp}} = E_{\rho_1 \oplus \rho_2^{\perp}} - m,$$

hence we have

$$\Phi_{\mathbb{F}}\left(E_{\rho_1} - E_{\rho_2}\right) = \Phi_{\mathbb{F}}\left(E_{\rho_1 \oplus \rho_2^{\perp}} - m\right) = \left\{E_{\rho_1 \oplus \rho_2^{\perp}}\right\}_{\mathbb{F}}$$

□

We recall the following theorem by Pittie which relates the complex representation and K-theory rings of a certain class of homogeneous spaces.

Theorem 3.12 (Pittie, [61]). *Let G be a compact, connected Lie group such that $\pi_1(G)$ is torsion free. Let H be a closed, connected subgroup of maximal rank. Then the homomorphism*

$$\alpha_{\mathbb{C}} : \mathfrak{R}_{\mathbb{C}}(H) \to K_{\mathbb{C}}(G/H)$$

is surjective.

3.1.4 Nonnegative sectional curvature

Let G/H be a homogeneous manifold. If H is compact, then for every $\rho \in \mathrm{Rep}_{\mathbb{F}}(H)$ we can assume that $\mathsf{r}(\rho(H))$ lies in some orthogonal group $\mathrm{O}(n)$. Suppose that G admits a metric $\langle , \rangle_G$ of nonnegative sectional curvature which is invariant under the action of H from the right (for instance a biinvariant metric in the case of compact G), hence inducing a nonnegatively metric on the quotient manifold G/H by O'Neill's Theorem on Riemannian submersions (see Section 1.2.2). Endow $G \times \mathbb{F}^n$ with the product metric of $\langle , \rangle_G$ and the flat Euclidean metric. Then, again by O'Neill's Theorem on Riemannian submersions, E_ρ inherits a quotient metric of nonnegative curvature of which $G \times_H \{0\} = G/H$ is a soul.

Now suppose that there is a homogeneous bundle in every stable class $S_{\mathbb{F}}(G/H)$. Then, for an arbitrary $\mathbb{F}$-vector bundle E over G/H there exist $\rho \in \mathrm{Rep}_{\mathbb{F}}(H)$ and $n, m \in \mathbb{N}$ such that

$$E \oplus n = E_\rho \oplus m = E_{\rho \oplus m}$$

Therefore $E \oplus n$ is a homogeneous vector bundle and it admits a metric with nonnegative sectional curvature. We have proved:

Lemma 3.13. *Let G be a compact Lie group and H a closed subgroup. Suppose that there is a homogeneous $\mathbb{F}$-vector bundle in every stable class $S_{\mathbb{F}}(G/H)$. Then for every $\mathbb{F}$-vector bundle E there exists $k \in \mathbb{N}$ such that $E \oplus k_{\mathbb{F}} = E \times \mathbb{F}^k$ admits a metric with nonnegative sectional curvature.*

3.2 Proof of Theorem 3.6

The Chern character induces a ring homomorphism $ch : K_{\mathbb{C}}(M) \to H^*(M, \mathbb{Q})$. Atiyah and Hirzebruch studied extensively this homomorphism in [6]. A consequence of their results is the following

Theorem 3.14 ([6])**.** *Let M be a compact manifold. Then $K_{\mathbb{C}}(M)$ is additively a finitely generated abelian group, and its rank equals the sum of the even-dimensional Betti numbers of M. Moreover, if $H^*(M, \mathbb{Z})$ is torsion-free, then $K_{\mathbb{C}}(M)$ is free abelian, i.e.,*

$$K_{\mathbb{C}}(M) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \ times},$$

where n is the sum of the even-dimensional Betti numbers.

Theorem 3.14 implies that manifolds M in the class $\mathscr{C}_1$ satisfy that $K_{\mathbb{C}}(M) = \mathbb{Z}$, and therefore $\tilde{K}_{\mathbb{C}}(M) = 0$. Thus every complex vector bundle E is stably trivial, i.e., for some integer k the Whitney sum $E \oplus k_{\mathbb{C}}$ is isomorphic to a trivial bundle $M \times \mathbb{C}^{k'}$, and hence the product metric has nonnegative sectional curvature.

Theorem 3.12 applies directly to manifolds in the class $\mathscr{C}_2$, and then Lemma 3.11 together with Lemma 3.13 completes the proof.

3.3 The spheres

As a homogeneous space, the sphere can be viewed as

$$\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n) = \mathrm{Spin}(n+1)/\mathrm{Spin}(n).$$

Recall that the *spin group* $\mathrm{Spin}(n)$ is the double cover of the special orthogonal group $\mathrm{SO}(n)$. For $n > 2$, the group $\mathrm{Spin}(n)$ is simply connected and so coincides with the universal cover of $\mathrm{SO}(n)$.

3.3.1 Representation rings of $\mathrm{Spin}(n)$

Denote by Λ the canonical representation of $\mathrm{SO}(n)$ in $\mathbb{R}^n$ and by Λ^k the k-th exterior product of Λ. As usual, we set $\Lambda^0 = 1$ and $\Lambda^1 = \Lambda$. Abusing notation, denote also by Λ^k its complexification $\mathsf{c}(\Lambda^k)$. These representations induce representations of $\mathrm{Spin}(n)$ via the double covering map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ which are usually denoted in the same way.

The representation rings of $\mathrm{Spin}(n)$ are known (see [2] and [14], chapter VI). In the odd case, $\mathfrak{R}_{\mathbb{C}}(\mathrm{Spin}(2n+1))$ equals the polynomial ring:

$$\mathfrak{R}_{\mathbb{C}}(\mathrm{Spin}(2n+1)) = \mathbb{Z}[\Lambda^1, \dots, \Lambda^{n-1}, \Delta].$$

The special 2^n-dimensional representation Δ satisfies:

$$\Delta\Delta = 1 + \Lambda^1 + \cdots + \Lambda^{n-1} + \Lambda^n.$$

In the even case, $\mathfrak{R}_{\mathbb{C}}(\mathrm{Spin}(2n))$ is also a polynomial ring, namely

$$\mathfrak{R}_{\mathbb{C}}(\mathrm{Spin}(2n)) = \mathbb{Z}[\Lambda^1, \dots, \Lambda^{n-2}, \Delta_+, \Delta_-].$$

The special 2^{n-1}-dimensional representations Δ_+, Δ_- satisfy:

$$\Delta_+\Delta_+ = \Lambda^n_+ + \Lambda^{n-2} + \Lambda^{n-4} + \dots \tag{3.3.1}$$

$$\Delta_+\Delta_- = \Lambda^{n-1} + \Lambda^{n-3} + \Lambda^{n-5} + \dots \tag{3.3.2}$$

$$\Delta_-\Delta_- = \Lambda^n_- + \Lambda^{n-2} + \Lambda^{n-4} + \dots \tag{3.3.3}$$

where Λ^n_+ and Λ^n_- are irreducible representations such that $\Lambda^n_+ + \Lambda^n_- = \Lambda^n$. The sums end in $\Lambda^2 + 1$ or $\Lambda^3 + \Lambda^1$ depending on the parity of n.

The irreducible representations Λ^k with $k \leq n-1$ (resp. $k \leq n-2$) of $\mathrm{Spin}(2n+1)$ (resp. $\mathrm{Spin}(2n)$) are real, meaning that they are the complexification of a real representation. Moreover (see [14], chapter VI), we have the following:

Proposition 3.15. *For $n \equiv m$ (mod 8), the special representations Δ, Δ_+ and Δ_- of* $\mathrm{Spin}(n)$ *have the following type:*

m	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Type	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$

In the case when Δ_+, Δ_- or Δ are of real type we denote both the underlying real representation (not to be mistaken for the real restriction) and its complexification in the same way.

Consider the standard inclusion

$$\begin{array}{ccc} \mathrm{SO}(n) & \to & \mathrm{SO}(n+1) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

and its covering group homomorphism $i_n : \mathrm{Spin}(n) \to \mathrm{Spin}(n+1)$. The following relations hold (see [14], chapter VI):

$$i^*_{2n,\mathbb{C}}(\Lambda^k) = \Lambda^k + \Lambda^{k-1} \quad \text{for } 1 \le k \le n \tag{3.3.4}$$
$$i^*_{2n,\mathbb{C}}(\Delta) = \Delta_+ + \Delta_- \tag{3.3.5}$$
$$i^*_{2n-1,\mathbb{C}}(\Lambda^k) = \Lambda^k + \Lambda^{k-1} \quad \text{for } 1 \le k \le n-1 \tag{3.3.6}$$
$$i^*_{2n-1,\mathbb{C}}(\Lambda^n_\pm) = \Lambda^{n-1} \tag{3.3.7}$$
$$i^*_{2n-1,\mathbb{C}}(\Delta_\pm) = \Delta \tag{3.3.8}$$

Thus we get identities on the corresponding stable classes of complex vector bundles over the sphere:

Corollary 3.16. *The following relations hold:*

- *Over* $\mathbb{S}^{2n} = \mathrm{Spin}(2n+1)/\mathrm{Spin}(2n)$,

$$\{E_{\Lambda^k}\}_\mathbb{C} = \{1\}_\mathbb{C} \quad \textit{for } 1 \le k \le n \tag{3.3.9}$$
$$\{E_{\Delta_+}\}_\mathbb{C} + \{E_{\Delta_-}\}_\mathbb{C} = \{1\}_\mathbb{C} \tag{3.3.10}$$
$$\{E_{\Delta_+ \otimes_\mathbb{C} \Delta_-}\}_\mathbb{C} = \{1\}_\mathbb{C} \tag{3.3.11}$$
$$\{E_{\Delta_+ \otimes_\mathbb{C} \Delta_+}\}_\mathbb{C} = 2^n \{E_{\Delta_+}\}_\mathbb{C} \tag{3.3.12}$$

- *Over* $\mathbb{S}^{2n-1} = \mathrm{Spin}(2n)/\mathrm{Spin}(2n-1)$,

$$\{E_{\Lambda^k}\}_\mathbb{C} = \{1\}_\mathbb{C} \quad \textit{for } 1 \le k \le n-1 \tag{3.3.13}$$
$$\{E_\Delta\}_\mathbb{C} = \{1\}_\mathbb{C} \tag{3.3.14}$$

Proof. The relations (3.3.10) and (3.3.14) follow immediately from (3.3.5) and (3.3.8). The relations (3.3.9) and (3.3.13) follow recursively from (3.3.4) and (3.3.6) respectively, since $\Lambda^0 = 1 = i^*_{2n,\mathbb{C}}(1)$. The latter, together with (3.3.2), gives us (3.3.11). Finally, observe that

$$\begin{aligned} \Delta_+\Delta_+ + \Delta_-\Delta_- &= \Lambda^n_+ + \Lambda^n_- + 2\Lambda^{n-2} + 2\Lambda^{n-4} + \dots \\ &= \Lambda^n + 2\Lambda^{n-2} + 2\Lambda^{n-4} + \dots \\ &= i^*_{2n,\mathbb{C}}(\rho) \end{aligned} \tag{3.3.15}$$

for some representation $\rho \in \mathrm{Rep}_\mathbb{C}(\mathrm{Spin}(2n+1))$. On the other hand, by (3.3.5) we have that $\Delta_- = i^*_{2n,\mathbb{C}}(\Delta) - \Delta_+$, hence:

$$\begin{aligned} \Delta_+\Delta_+ + \Delta_-\Delta_- &= \Delta_+\Delta_+ + (i^*_{2n,\mathbb{C}}(\Delta) - \Delta_+)(i^*_{2n,\mathbb{C}}(\Delta) - \Delta_+) \\ &= 2\Delta_+\Delta_+ + i^*_{2n,\mathbb{C}}(\Delta\Delta) - 2i^*_{2n,\mathbb{C}}(\Delta)\Delta_+ \end{aligned} \tag{3.3.16}$$

Combining (3.3.15) and (3.3.16) we get

$$2\Delta_+\Delta_+ + i^*_{2n,\mathbb{C}}(\Delta\Delta) = i^*_{2n,\mathbb{C}}(\rho) + 2i^*_{2n,\mathbb{C}}(\Delta)\Delta_+,$$

which proves (3.3.12) since $E_{i^*_{2n,\mathbb{C}}(\Delta)} = E_{\dim \Delta} = 2^n$. $\square$

3.3.2 The K-theory of the sphere

The rings $K_{\mathbb{F}}(\mathbb{S}^n)$ are well known (see [50], chapter IV). In the complex case:

$$\tilde{K}_{\mathbb{C}}(\mathbb{S}^{2n+1}) = 0, \qquad \tilde{K}_{\mathbb{C}}(\mathbb{S}^{2n}) = \mathbb{Z}.$$

In the real case:

$$\begin{array}{ll} \tilde{K}_{\mathbb{R}}(\mathbb{S}^{8n}) = \mathbb{Z}, & \tilde{K}_{\mathbb{R}}(\mathbb{S}^{8n+4}) = \mathbb{Z}, \\ \tilde{K}_{\mathbb{R}}(\mathbb{S}^{8n+1}) = \mathbb{Z}_2, & \tilde{K}_{\mathbb{R}}(\mathbb{S}^{8n+5}) = 0, \\ \tilde{K}_{\mathbb{R}}(\mathbb{S}^{8n+2}) = \mathbb{Z}_2, & \tilde{K}_{\mathbb{R}}(\mathbb{S}^{8n+6}) = 0, \\ \tilde{K}_{\mathbb{R}}(\mathbb{S}^{8n+3}) = 0, & \tilde{K}_{\mathbb{R}}(\mathbb{S}^{8n+7}) = 0. \end{array}$$

3.3.3 Proof of Theorem 3.4 for $\mathbb{S}^n$

Proposition 3.17. *The map*

$$\{\alpha\}_{\mathbb{F}} : \mathrm{Rep}_{\mathbb{F}}(\mathrm{Spin}(n)) \to S_{\mathbb{F}}(\mathbb{S}^n)$$

is surjective for all $n \in \mathbb{N}$ both in the real and in the complex case. Moreover, the stable classes in the cases in which $\tilde{K}_{\mathbb{F}}(\mathbb{S}^n) \neq 0$ are given by

$$\begin{array}{ll} S_{\mathbb{C}}(\mathbb{S}^{2n}) = \mathbb{Z}\{E_{\Delta_+}\}_{\mathbb{C}}, & \\ S_{\mathbb{R}}(\mathbb{S}^{8n}) = \mathbb{Z}\{E_{\Delta_+}\}_{\mathbb{R}}, & S_{\mathbb{R}}(\mathbb{S}^{8n+2}) = \{\{1\}_{\mathbb{R}}, \{E_{\mathsf{r}(\Delta_+)}\}_{\mathbb{R}}\}, \\ S_{\mathbb{R}}(\mathbb{S}^{8n+1}) = \{\{1\}_{\mathbb{R}}, \{E_{\Delta}\}_{\mathbb{R}}\}, & S_{\mathbb{R}}(\mathbb{S}^{8n+4}) = \mathbb{Z}\{E_{\mathsf{r}(\Delta_+)}\}_{\mathbb{R}}. \end{array}$$

Proof. The surjectivity of $\{\alpha\}_{\mathbb{C}}$ is included in Theorem 3.6. From Corollary 3.16 it follows that $S_{\mathbb{C}}(\mathbb{S}^{2n}) = \mathbb{Z}\{E_{\Delta_+}\}_{\mathbb{C}}$.

The surjectivity of $\{\alpha\}_{\mathbb{R}}$ when $n \equiv 3, 5, 6, 7 \pmod 8$ is trivial since $\tilde{K}_{\mathbb{R}}(\mathbb{S}^n) = 0$.

Now let E be an arbitrary real vector bundle over $\mathbb{S}^n$ for the remaining cases:

- $n \equiv 0 \pmod 8$. By Theorem 5.12 in [50], chapter IV, the map

  $$c : \tilde{K}_{\mathbb{R}}(\mathbb{S}^n) \to \tilde{K}_{\mathbb{C}}(\mathbb{S}^n) \cong S_{\mathbb{C}}(\mathbb{S}^n) = \mathbb{Z}\{E_{\Delta_+}\}_{\mathbb{C}}$$

  is an isomorphism. From Proposition 3.15 we know that Δ_+ is real and therefore

  $$S_{\mathbb{R}}(\mathbb{S}^n) = \mathbb{Z}\{E_{\Delta_+}\}_{\mathbb{R}}.$$

- $n \equiv 2, 4 \pmod 8$. By Theorem 6.1 in [50], chapter IV, the real restriction map for $n \equiv 2 \pmod 8$ (resp. $n \equiv 4 \pmod 8$)

  $$\mathsf{r} : \tilde{K}_{\mathbb{C}}(\mathbb{S}^n) \to \tilde{K}_{\mathbb{R}}(\mathbb{S}^n)$$

  is surjective (resp. an isomorphism). Therefore

  $$\begin{array}{l} S_{\mathbb{R}}(\mathbb{S}^n) = \{\{1\}_{\mathbb{R}}, \{E_{\mathsf{r}(\Delta_+)}\}_{\mathbb{R}}\} \ \text{ if } n \equiv 2 \pmod 8. \\ S_{\mathbb{R}}(\mathbb{S}^n) = \mathbb{Z}\{E_{\mathsf{r}(\Delta_+)}\}_{\mathbb{R}} \ \text{ if } n \equiv 4 \pmod 8. \end{array}$$

- $n \equiv 1 \pmod 8$. By Proposition 3.15 the representation Δ is real. We are going to prove that $\{E_\Delta\}_\mathbb{R}$ is not trivial and hence

$$S_\mathbb{R}(\mathbb{S}^{8n+1}) = \mathbb{Z}_2 = \{\{1\}_\mathbb{R}, \{E_\Delta\}_\mathbb{R}\}.$$

Denote by $i^*_\mathbb{F}$ the map $i^*_{8n+1,\mathbb{F}}$. We want to see that there does not exist $\tau \in \mathrm{Rep}_\mathbb{R}(\mathrm{Spin}(8n+2))$ such that $i^*_\mathbb{R}(\tau) = \Delta + k$, for any natural number k. Suppose it does; then $\mathsf{c}(\tau) \in \mathrm{Rep}_\mathbb{C}(\mathrm{Spin}(8n+2))$ is of the form

$$\mathsf{c}(\tau) = \sum_{j_1,\dots,j_{4n+1}} a_{j_1,\dots,j_{4n+1}} (\Lambda^1)^{j_1} \dots (\Lambda^{4n-1})^{j_{4n-1}} (\Delta_+)^{j_{4n}} (\Delta_-)^{j_{4n+1}}.$$

We can rewrite this expression as

$$\mathsf{c}(\tau) = \sum_{l_1,l_2} b_{l_1,l_2}(\Lambda^1, \dots, \Lambda^{4n-1})(\Delta_+)^{l_1}(\Delta_-)^{l_2}$$

for the obvious polynomials $b_{l_1,l_2} \in \mathbb{Z}[\Lambda^1, \dots, \Lambda^{4n-1}]$. Now we have:

$$\begin{aligned} i^*_\mathbb{C}(\mathsf{c}(\tau)) &= \sum_{l_1,l_2} i^*_\mathbb{C}\left(a_{l_1,l_2}(\Lambda^1, \dots, \Lambda^{4n-1})\right) i^*_\mathbb{C}(\Delta_+)^{l_1} i^*_\mathbb{C}(\Delta_-)^{l_2} \\ &= \sum_{l_1,l_2} a_{l_1,l_2}(\Lambda^1 + 1, \dots, \Lambda^{4n-1} + \Lambda^{4n-2})(\Delta)^{l_1+l_2} \end{aligned}$$

On the other hand,

$$\mathsf{c}\left(i^*_\mathbb{R}(\tau)\right) = \mathsf{c}(\Delta + k) = \Delta + k \in \mathfrak{R}_\mathbb{C}(\mathrm{Spin}(8n+1)).$$

From the identity $i^*_\mathbb{C} \circ \mathsf{c} = \mathsf{c} \circ i^*_\mathbb{R}$, it follows that

$$\begin{cases} a_{0,0}(\Lambda^1 + 1, \dots, \Lambda^{4n-1} + \Lambda^{4n-2}) = k \\ a_{1,0}(\Lambda^1 + 1, \dots, \Lambda^{4n-1} + \Lambda^{4n-2}) + a_{0,1}(\Lambda^1 + 1, \dots, \Lambda^{4n-1} + \Lambda^{4n-2}) = 1 \\ a_{i,j}(\Lambda^1 + 1, \dots, \Lambda^{4n-1} + \Lambda^{4n-2}) = 0 \ \text{ if } i + j \geq 2 \end{cases}$$

The map $\phi : \mathbb{Z}[\Lambda^1, \dots, \Lambda^{4n-1}] \to \mathbb{Z}[\Lambda^1, \dots, \Lambda^{4n-1}]$ defined by the rule $\phi(\Lambda^k) = \Lambda^k + \Lambda^{k-1}$ for $k \geq 1$, is a ring isomorphism. The inverse is given recursively as $\phi^{-1}(\Lambda^k) = \Lambda^k - \phi^{-1}(\Lambda^{k-1})$, where $\phi^{-1}(\Lambda^1) = \Lambda^1 - 1$. Therefore we have that

$$\begin{cases} a_{0,0}(\Lambda^1, \dots, \Lambda^{4n-1}) = k \\ a_{1,0}(\Lambda^1, \dots, \Lambda^{4n-1}) + a_{0,1}(\Lambda^1, \dots, \Lambda^{4n-1}) = 1 \\ a_{i,j}(\Lambda^1, \dots, \Lambda^{4n-1}) = 0 \ \text{ if } i + j \geq 2 \end{cases}$$

We deduce that $\mathsf{c}(\tau)$ equals either $k + \Delta_+$ or $k + \Delta_-$. It then would follow that either Δ_+ or Δ_- is in the image of the complexification map. But this is a contradiction since as we can see in Proposition 3.15, the representations Δ_+ and Δ_- are not of real type.

Finally, let d be the dimension of the real representation Δ. Observe that

$$2\left(E_\Delta - d_\mathbb{R}\right) = \mathsf{r} \circ \mathsf{c}\left(E_\Delta - d_\mathbb{R}\right) = \mathsf{r}\left(E_\Delta - d_\mathbb{C}\right) = 0,$$

since $\mathsf{r} : \tilde{K}_\mathbb{C}(\mathbb{S}^{8n+1}) \to \tilde{K}_\mathbb{R}(\mathbb{S}^{8n+1})$ is the zero map. It follows that $2\{E_\Delta\}_\mathbb{R} = \{1\}_\mathbb{R}$.

□

3.3.4 Proof of Theorem 3.5

The proof follows from Proposition 3.17 together with Theorem 3.8. Let E be an arbitrary real vector bundle over the sphere $\mathbb{S}^n$. If E is stably trivial, then the Whitney sum $E \oplus k$ is isomorphic to a trivial bundle if $\operatorname{rank}(E \oplus k) \geq n+1$.

- $n \equiv 3, 5, 6, 7 \pmod 8$. Since $\tilde{K}_{\mathbb{R}}(\mathbb{S}^n) = 0$, every bundle is stably trivial so $k_0 \leq n+1$.
- $n \equiv 1 \pmod 8$. Assume that $E \in \{E_\Delta\}_{\mathbb{R}}$. Since $\dim \Delta = 2^n \geq n+1$, it follows that if $\operatorname{rank}(E \oplus k) \geq 2^n$ then $E \oplus k$ is isomorphic to $E_\Delta \oplus k' = E_{\Delta \oplus k'}$, so $k_0 \leq 2^n$.
- $n \equiv 2 \pmod 8$ is analogue to the case $n \equiv 1 \pmod 8$ since $\dim \mathsf{r}(\Delta_+) = 2 \cdot 2^{n-1} = 2^n$.

For the remaining cases we need the so-called Bott Integrability Theorem:

Theorem 3.18 (Corollary 9.8 in [42], Chapter 20). *Let $a \in H^{2n}(\mathbb{S}^{2n}, \mathbb{Z})$ be a generator. Then for each complex vector bundle E over $\mathbb{S}^{2n}$, the n-th Chern class $c_n(E)$ is a multiple of $(n-1)!a$, and for each $m \equiv 0 \pmod{(n-1)!}$ there exists a unique $\{E\}_{\mathbb{C}} \in S_{\mathbb{C}}(\mathbb{S}^{2n})$ such that $c_n(E) = ma$.*

Recall that $H^*(\mathbb{S}^{2n}, \mathbb{Z}) = H^0(\mathbb{S}^{2n}, \mathbb{Z}) \oplus H^{2n}(\mathbb{S}^{2n}, \mathbb{Z})$, thus the total Chern class of a complex vector bundle E over $\mathbb{S}^{2n}$ is of the form $c_T(E) = 1 + c_n(E)$. From the Whitney Product Formula in Section 3.1.2 we get that $c_n(E \oplus F) = c_n(E) + c_n(F)$ for $E, F \in \operatorname{Vect}_{\mathbb{C}}(\mathbb{S}^{2n})$. Since $S_{\mathbb{C}}(\mathbb{S}^{2n}) = \mathbb{Z}\{E_{\Delta_\pm}\}_{\mathbb{C}}$, it follows that $c_n(E_{\Delta_\pm}) = (n-1)!(\pm a)$ and hence

$$c_n(lE_{\Delta_\pm}) = (n-1)!(\pm l)a \tag{3.3.17}$$

for each integer l.

Now we return to the real setting, so let E be again an arbitrary real vector bundle over the sphere $\mathbb{S}^n$.

- $n \equiv 0 \pmod 8$. Assume that $E \in \pm l\{E_{\Delta_+}\}_{\mathbb{R}} = \{E_{l\Delta_\pm}\}_{\mathbb{R}}$, for some positive integer l. Since $\dim l\Delta_\pm = 2^{n-1}l \geq n+1$, it follows that if $\operatorname{rank}(E \oplus k) \geq 2^{n-1}l$, then $E \oplus k$ is isomoprhic to $E_{l\Delta_+} \oplus k' = E_{l\Delta_+ \oplus k'}$, so $k_0 \leq 2^{n-1}l$.

  The $(n/2)$-th Chern class of the complexified vector bundle $\mathsf{c}(E)$ satisfies

  $$c_{n/2}(\mathsf{c}(E)) = c_{n/2}\left(\mathsf{c}\left(E_{l\Delta_\pm}\right)\right) = c_{n/2}\left(E_{l\Delta_\pm}\right) = ((n/2)-1)!(\pm l)a$$

  where the first equality follows from the stability of the Chern classes and the last one from (3.3.17).

- $n \equiv 4 \pmod 8$. Assume that $E \in \pm l\{E_{\mathsf{r}(\Delta_+)}\}_{\mathbb{R}} = \{E_{l\mathsf{r}(\Delta_\pm)}\}_{\mathbb{R}}$, for some positive integer l. Since $\dim l\mathsf{r}(\Delta_\pm) = 2 \cdot 2^{n-1}l \geq n+1$, it follows that if $\operatorname{rank}(E \oplus k) \geq 2^n l$, then $E \oplus k$ is isomorphic to $E_{l\mathsf{r}(\Delta_\pm)} \oplus k' = E_{l\mathsf{r}(\Delta_\pm)\oplus k'}$, so $k_0 \leq 2^n l$.

  The $(n/2)$-th Chern class of the complexified vector bundle $\mathsf{c}(E)$ satisfies

  $$c_{n/2}(\mathsf{c}(E)) = c_{n/2}\left(\mathsf{c}\left(E_{l\mathsf{r}(\Delta_\pm)}\right)\right) = c_{n/2}\left(\mathsf{c} \circ \mathsf{r}\left(E_{l\Delta_\pm}\right)\right)$$

Now recall that $\mathsf{c} \circ \mathsf{r} = 1 + \mathsf{t}$, where t denotes the conjugation of complex vector bundles, so

$$c_{n/2}\left(\mathsf{c} \circ \mathsf{r}\left(E_{l\Delta_\pm}\right)\right) = c_{n/2}\left(E_{l\Delta_\pm} \oplus \mathsf{t}\left(E_{l\Delta_\pm}\right)\right) = c_{n/2}\left(E_{l\Delta_\pm}\right) + c_{n/2}\left(\mathsf{t}\left(E_{l\Delta_\pm}\right)\right)$$

The Chern class of the conjugate bundle satisfies (see Proposition 11.1 in [42], Chapter 17):

$$c_{n/2}\left(\mathsf{t}\left(E_{l\Delta_\pm}\right)\right) = (-1)^{n/2} c_{n/2}\left(E_{l\Delta_\pm}\right) = c_{n/2}\left(E_{l\Delta_\pm}\right)$$

since $n/2$ is even. So from (3.3.17) we get that

$$c_{n/2}\left(\mathsf{c}(E)\right) = 2c_{n/2}\left(E_{l\Delta_\pm}\right) = ((n/2)-1)!(\pm 2l)a,$$

which together with the inequality $k_0 \leq 2^n l$ above proves the Theorem.

Finally, recall that the k-th Pontryagin class $p_k(E) \in H^{4k}(M, \mathbb{Z})$ of a real vector bundle E over a compact manifold M is defined as:

$$p_k(E) = (-1)^k c_{2k}(\mathsf{c}(E)).$$

Therefore when M is the sphere $\mathbb{S}^n$ of dimension $n \equiv 0 \pmod 8$ (resp. $n \equiv 4 \pmod 8$), we get that $p_{n/4}(E) = c_{n/2}(\mathsf{c}(E))$ (resp. $p_{n/4}(E) = -1 c_{n/2}(\mathsf{c}(E))$). Anyway, in both cases

$$p_{n/4}(E) = ((n/2)-1)!(\pm \mathsf{p}_E)a$$

for some natural number p_E, where a is a generator of $H^n(\mathbb{S}^n, \mathbb{Z})$.

3.4 Grassmannian manifolds

In this section $\mathbb{F}$ will stand for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $\mathrm{U}_\mathbb{F}(n)$ denote the orthogonal group $\mathrm{O}(n)$, the unitary group $\mathrm{U}(n)$ or the symplectic group $\mathrm{Sp}(n) \subset \mathrm{U}(2n)$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively. Throughout this section we will consider each of the groups $\mathrm{U}_\mathbb{F}(n)$ endowed with its canonical biinvariant metric.

The Grassmannian manifold $\mathrm{G}_\mathbb{F}(k, n)$ is defined as the set of k-dimensional subspaces W of $\mathbb{F}^n$ (right subspaces in the case of $\mathbb{H}^n$). It can be viewed as the homogeneous space $\mathrm{U}_\mathbb{F}(n)/(\mathrm{U}_\mathbb{F}(k) \times \mathrm{U}_\mathbb{F}(n-k))$ under the isomorphism

$$\begin{array}{ccc} \mathrm{U}_\mathbb{F}(n)/(\mathrm{U}_\mathbb{F}(k) \times \mathrm{U}_\mathbb{F}(n-k)) & \longrightarrow & \mathrm{G}_\mathbb{F}(k,n) \\ [M] & \longmapsto & M \begin{pmatrix} \mathbb{F}^k \\ 0 \end{pmatrix} \end{array}$$

This way, $\mathrm{G}_\mathbb{F}(k, n)$ inherits a quotient metric with nonnegative sectional curvature.

3.4.1 Tautological bundle

The *tautological vector bundle* $\mathscr{T}_{\mathbb{F}}(k,n)$ over $\mathrm{G}_{\mathbb{F}}(k,n)$ is defined as

$$\mathscr{T}_{\mathbb{F}}(k,n) = \{(W,w) \in \mathrm{G}_{\mathbb{F}}(k,n) \times \mathbb{F}^n : w \in W\},$$

where the bundle projection map is given by $(W,w) \mapsto W$. Define the representation:

$$\begin{array}{rcl} \rho_{\mathbb{F}} : \mathrm{U}_{\mathbb{F}}(k) \times \mathrm{U}_{\mathbb{F}}(n-k) & \longrightarrow & \mathrm{U}_{\mathbb{F}}(k) \\ (A,B) & \longmapsto & A \end{array}$$

It turns out that $\mathscr{T}_{\mathbb{F}}(k,n)$ is isomorphic to the homogeneous vector bundle $E_{\rho_{\mathbb{F}}}$. The isomorphism is given by:

$$\begin{array}{rcl} E_{\rho_{\mathbb{F}}} & \longrightarrow & \mathscr{T}_{\mathbb{F}}(k,n) \\ [M,v] & \longmapsto & \left(M \begin{pmatrix} \mathbb{F}^k \\ 0 \end{pmatrix}, M \begin{pmatrix} v \\ 0 \end{pmatrix} \right) \end{array}$$

Notice that, although $\mathscr{T}_{\mathbb{H}}(k,n)$ is defined as a quaternionic vector bundle, here we are only considering its underlying complex structure. As such, it is isomorphic to the complex vector bundle

$$E_{\rho_{\mathbb{H}}} = (\mathrm{U}_{\mathbb{F}}(n) \times \mathbb{C}^{2k})/(\mathrm{U}_{\mathbb{F}}(k) \times \mathrm{U}_{\mathbb{F}}(n-k)).$$

Observe that $\mathrm{G}_{\mathbb{F}}(k,n)$ is diffeomorphic to $\mathrm{G}_{\mathbb{F}}(n-k,n)$ under the map $W \mapsto W^{\perp}$, where $\mathbb{F}^n$ is endowed with the Euclidean metric. Clearly, the Whitney sum of $\mathscr{T}_{\mathbb{F}}(n-k,n)$ with $\mathscr{T}_{\mathbb{F}}(k,n)$ is the trivial bundle of rank n. From now on we will write just $\mathscr{T}_{\mathbb{F}}$ to denote the bundle $\mathscr{T}_{\mathbb{F}}(k,n)$.

3.4.2 Proof of Theorem 3.4 for projective spaces

Recall that $\mathrm{G}_{\mathbb{F}}(1,n+1)$ is the projective space $\mathbb{FP}^n$. In these cases, the quotient metric inherited from $\mathrm{U}_{\mathbb{F}}(n+1)$ is the one giving $\mathbb{FP}^n$ the structure of compact rank one symmetric space.

Proposition 3.19. *For $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, the following maps are surjective:*

$$\{\alpha\}_{\mathbb{R}} : \mathrm{Rep}_{\mathbb{R}}(\mathrm{U}_{\mathbb{F}}(1) \times \mathrm{U}_{\mathbb{F}}(n)) \to S_{\mathbb{R}}(\mathbb{FP}^n)$$

$$\{\alpha\}_{\mathbb{C}} : \mathrm{Rep}_{\mathbb{C}}(\mathrm{U}_{\mathbb{F}}(1) \times \mathrm{U}_{\mathbb{F}}(n)) \to S_{\mathbb{C}}(\mathbb{FP}^n)$$

Proof. The real and complex K-theory of projective spaces are well known, see for example [1] and [64]. The rings $\tilde{K}_{\mathbb{R}}(\mathbb{FP}^n)$ and $\tilde{K}_{\mathbb{C}}(\mathbb{FP}^n)$ are respectively generated by the following elements:

$$\begin{array}{rr} \mathscr{T}_{\mathbb{R}} - 1_{\mathbb{R}} \in \tilde{K}_{\mathbb{R}}(\mathbb{RP}^n), & \mathsf{c}(\mathscr{T}_{\mathbb{R}}) - 1_{\mathbb{C}} \in \tilde{K}_{\mathbb{C}}(\mathbb{RP}^n), \\ \mathsf{r}(\mathscr{T}_{\mathbb{C}}) - 2_{\mathbb{R}} \in \tilde{K}_{\mathbb{R}}(\mathbb{CP}^n), & \mathscr{T}_{\mathbb{C}} - 1_{\mathbb{C}} \in \tilde{K}_{\mathbb{C}}(\mathbb{CP}^n), \\ \mathsf{r}(\mathscr{T}_{\mathbb{H}}) - 4_{\mathbb{R}} \in \tilde{K}_{\mathbb{R}}(\mathbb{HP}^n), & \mathscr{T}_{\mathbb{H}} - 2_{\mathbb{C}} \in \tilde{K}_{\mathbb{C}}(\mathbb{HP}^n). \end{array}$$

Since the tautological bundle $\mathscr{T}_{\mathbb{F}}$ is homogeneous, the map

$$\alpha_{\mathbb{R}} : \tilde{\mathfrak{R}}_{\mathbb{R}}(\mathrm{U}_{\mathbb{F}}(1) \times \mathrm{U}_{\mathbb{F}}(n)) \to \tilde{K}_{\mathbb{R}}(\mathbb{FP}^n)$$

is surjective, and by Lemma 3.11,

$$\{\alpha\}_{\mathbb{R}} : \mathrm{Rep}_{\mathbb{R}}(\mathrm{U}_{\mathbb{F}}(1) \times \mathrm{U}_{\mathbb{F}}(n)) \to S_{\mathbb{R}}(\mathbb{FP}^n)$$

is also surjective. The same arguments work for the map $\alpha_{\mathbb{C}}$.

□

Proposition 3.19 proves that there is a homogeneous vector bundle in every stable class of real and complex vector bundles over each projective space. Now apply Lemma 3.13 to get Theorem 3.4 for projective spaces.

3.5 The Cayley plane

In this section we consider the Cayley plane $\mathrm{Ca}\mathbb{P}^2$. Recall that the Cayley plane is a 16-dimensional CW-complex consisting of three cells of dimensions 0, 8 and 16. As a homogeneous space, it can be viewed as the quotient of the 52-dimensional exceptional Lie group F_4 under the action of the spin group Spin(9). Let us endow F_4 with its canonical biinvariant metric, so that $\mathrm{Ca}\mathbb{P}^2$ with the quotient metric is a compact rank one symmetric space.

3.5.1 Representation rings $\mathfrak{R}_{\mathbb{F}}(F_4)$ and $\mathfrak{R}_{\mathbb{F}}(\mathrm{Spin}(9))$

The representation rings of F_4 are known (see [2], [52] and [73]). Denote by λ^k the k-th exterior product of the irreducible 26-dimensional representation λ given in Corollary 8.1 in [2], and by κ the adjoint action of F_4 on its Lie algebra $\mathfrak{f}_4$. It turns out that the representations λ^k and κ are real. We denote their complexifications in the same way. The real and complex representation rings of F_4 are the polynomial ring

$$\mathfrak{R}_{\mathbb{F}}(F_4) = \mathbb{Z}[\lambda^1, \lambda^2, \lambda^3, \kappa],$$

where $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$, and the complexification map

$$\mathsf{c} : \mathfrak{R}_{\mathbb{R}}(F_4) \to \mathfrak{R}_{\mathbb{C}}(F_4)$$

is an isomorphism.

The representation rings of Spin(9) have been described in Section 3.3.1. Observe that the complexification map

$$\mathsf{c} : \mathfrak{R}_{\mathbb{R}}(\mathrm{Spin}(9)) \to \mathfrak{R}_{\mathbb{C}}(\mathrm{Spin}(9))$$

is surjective.

3.5.2 The $K_{\mathbb{F}}$-theory of $\mathrm{Ca}\mathbb{P}^2$

The cohomology of $\mathrm{Ca}\mathbb{P}^2$ is well known, in particular we have:

$$H^k(\mathrm{Ca}\mathbb{P}^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 8, 16 \\ 0 & \text{otherwise} \end{cases}$$

Hence $H^*(\mathrm{Ca}\mathbb{P}^2, \mathbb{Z})$ is torsion-free and Theorem 3.14 gives us the following:

$$K_{\mathbb{C}}(\mathrm{Ca}\mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

The real K-theory of $\mathrm{Ca}\mathbb{P}^2$ follows from Lemma 2.5 in [40], which states that if M is a finite CW-complex with cells only in dimensions 0 (mod 4) then

$$K_{\mathbb{R}}(M) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}},$$

where n is the number of cells in M. In particular, we have

Proposition 3.20. *$K_{\mathbb{R}}(\mathrm{Ca}\mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.*

Now consider the induced map $\mathsf{r} \circ \mathsf{c} : K_{\mathbb{R}}(\mathrm{Ca}\mathbb{P}^2) \to K_{\mathbb{R}}(\mathrm{Ca}\mathbb{P}^2)$. By Proposition 3.20 we know that $K_{\mathbb{R}}(\mathrm{Ca}\mathbb{P}^2)$ is torsion-free, and since the map $\mathsf{r} \circ \mathsf{c}$ is nothing but multiplication by 2, it must be injective.

Lemma 3.21. *The induced map $\mathsf{r} \circ \mathsf{c} : K_{\mathbb{R}}(\mathrm{Ca}\mathbb{P}^2) \to K_{\mathbb{R}}(\mathrm{Ca}\mathbb{P}^2)$ is injective. In particular, $\mathsf{c} : K_{\mathbb{R}}(\mathrm{Ca}\mathbb{P}^2) \to K_{\mathbb{C}}(\mathrm{Ca}\mathbb{P}^2)$ is also injective.*

3.5.3 Proof of Theorem 3.4 for $\mathrm{Ca}\mathbb{P}^2$

First we construct homogeneous bundles in every stable class.

Proposition 3.22. *The map*

$$\{\alpha\}_{\mathbb{F}} : \mathrm{Rep}_{\mathbb{F}}(\mathrm{Spin}(9)) \to S_{\mathbb{F}}(\mathrm{Ca}\mathbb{P}^2)$$

is surjective for $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$.

Proof. The surjectivity of $\{\alpha\}_{\mathbb{C}}$ is included in Theorem 3.6 since F_4 is simply connected and contains $\mathrm{Spin}(9)$ as a subgroup of maximal rank.

For the real case let E be an arbitrary real vector bundle over $\mathrm{Ca}\mathbb{P}^2$. By the discussion above we have that

$$\mathsf{c}(E - \mathrm{rank}_{\mathbb{R}} E) = E_\rho - \dim \rho \tag{3.5.1}$$

for some $\rho \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{Spin}(9))$. On the other hand

$$\mathsf{c} : \mathfrak{R}_{\mathbb{R}}(\mathrm{Spin}(9)) \to \mathfrak{R}_{\mathbb{C}}(\mathrm{Spin}(9))$$

is surjective, so there exists $\rho' \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Spin}(9))$ such that $\mathsf{c}(\rho') = \rho$, and hence

$$\mathsf{c}(E_{\rho'} - \dim \rho') = E_\rho - \dim \rho. \tag{3.5.2}$$

By Lemma 3.21, the complexification map $\mathsf{c} : K_{\mathbb{R}}(\mathrm{Ca}\mathbb{P}^2) \to K_{\mathbb{C}}(\mathrm{Ca}\mathbb{P}^2)$ is injective, so from (3.5.1) and (3.5.2) it follows that

$$E_{\rho'} - \dim \rho' = E - \mathrm{rank}_{\mathbb{R}} E$$

in $K_{\mathbb{R}}(\mathrm{Ca}\mathbb{P}^2)$ and hence $\{E\}_{\mathbb{R}} = \{E_{\rho'}\}_{\mathbb{R}}$. □

Finally, the proof of Theorem 3.4 for the Cayley plane is a direct consequence of Proposition 3.22 together with Lemma 3.13.

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