A NOTE ON ASYMPTOTIC VALUES OF QUASIREGULAR FUNCTIONS

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Abstract. In this note we give an example of a quasiregular function in $\mathbb{R}^n$ ($n \geq 3$) of order of growth $n-1$ and whose set of asymptotic values is $A \cup \{\infty\}$ for a given Suslin analytic set $A \subset \mathbb{R}^n$. Our example is a modification of Drasin's construction in [4] of a quasiregular function with order of growth $n-1$ and set of asymptotic values $\mathbb{R}^n \cup \{\infty\}$.

1. Introduction

A quasiregular function $f : \mathbb{R}^n \to \mathbb{R}^n$ ($n \geq 2$) is a continuous function such that $f \in W^1_{n,\mathrm{loc}}(\mathbb{R}^n)$ and for some $K \geq 1$,

$$|f'(z)|^n \leq K J_f(z) \text{ a.e.,} \tag{1}$$

where f' is the generalized derivative of f, $|f'(z)|$ is its operator norm and J_f the Jacobian determinant. Often, f is called K-quasiregular. The smallest number K for which the inequality (1) is true is called the (outer) dilatation of f. When f is also a homeomorphism, f is said to be K-quasiconformal (see [1] and [11]). Every L-bilipschitz map in $\mathbb{R}^n$ is K-quasiconformal with $K = L^{2(n-1)}$. For $n \geq 3$ and $K = 1$ the only quasiregular maps are the orientation preserving Möbius transformations. The class of quasiregular maps includes analytic functions in $\mathbb{C}$ and, in this sense, quasiregular maps generalize analytic functions to dimensions $n \geq 3$. Many of the properties of holomorphic functions have a counterpart for quasiregular functions. For example, a quasiregular map defined in $\mathbb{R}^n$ is unbounded (Liouville's theorem in $\mathbb{R}^n$). Standard references on the subject are the books [8] and [12].

A point $a \in \mathbb{R}^n \cup \{\infty\}$ is an asymptotic value for f if there exists a continuous path $\gamma \subset \mathbb{R}^n$ along which $\lim_{z\to\infty,\, z\in\gamma} f(z) = a$. It is well known that for $n = 2$ and f holomorphic (or meromorphic), the set of asymptotic values of f, $\mathrm{As}(f)$, is a Suslin analytic set, see [7], and, conversely, for any analytic set, $A \subset \mathbb{C}$, there exists an entire function, f, for which $\mathrm{As}(f) = A \cup \{\infty\}$, see [5] (or [2] for the meromorphic case with finite order of growth).

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In this note, we show that these latter results can be extended to quasiregular maps. Moreover, in a forthcoming paper, [3], it is shown that in fact, the set of asymptotic values of a quasiregular function is always an analytic set.

Theorem 1. *Let A be any analytic set in $\mathbb{R}^n$, $n \geq 3$. Then there exists a quasiregular function $f : \mathbb{R}^n \to \mathbb{R}^n$ of order of growth $n-1$ such that* $\mathrm{As}(f) = A \cup \{\infty\}$.

The order of growth of f, ρ_f, is defined as

$$\rho_f = \limsup_{r\to\infty} (n-1)\frac{\log\log M(r)}{\log r}$$

where $M(r) = \max_{|z|=r} |f(z)|$. Rickman and Vourinen have shown in [9] that if $\mathrm{As}(f) \neq \emptyset$ there exist a constant $c(n,K) > 0$ so that $\rho_f > c(n,K)$ (in fact, they have proved the bound for the lower order of growth).

The proof of Theorem 1 is a modification of Drasin's construction in [4] of a quasiregular map f in $\mathbb{R}^n$ $(n \geq 3)$ of order of growth $n-1$ with $\mathrm{As}(f) = \mathbb{R}^n \cup \{\infty\}$. His main idea is to define a sine-like quasiregular function, $S : \mathbb{R}^n \to \mathbb{R}^n$, that is "modulated" by a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ that tends to zero along certain paths in $\mathbb{R}^{n-1}$ with the structure of countably many binary trees. By local quasiconformal translations that act in neighborhoods of the branches of the trees, 0 is mapped to any point in $\mathbb{R}^n$. We use this construction, exploiting the representation of a Suslin analytic set in $\mathbb{R}^n$ as the result of the $\mathcal{A}$-operation on closed sets (see [10] or [6] for references on analytic sets and [2] for another instance of this application). The main difference with Drasin's work is the need to show that asymptotic values outside A are not assumed.

To make this note self-contained we will reproduce Drasin's construction and keep to some extent his notation.

If z is a point in $\mathbb{R}^n$ let $z = (x^1, \dots, x^{n-1}, y) = (x, y)$ be its coordinates with respect to an orthonormal basis with $x = (x^1, \dots, x^{n-1}) \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. Denote by $|z|$ its Euclidean norm. As usual, let $B(a,r) = \{z \in \mathbb{R}^n : |z-a| < r\}$, and $d(A,B)$ the Euclidean distance between the closures of the sets A and B. If $z = (x,y) \in \mathbb{R}^n$, its "conjugate" is the point $\overline{z} = (x, -y)$. For $x = (x^1, \dots, x^{n-1}) \in \mathbb{R}^{n-1}$ define $\|x\| = \max_{i=1,\dots,n-1} |x^i|$. If $z = (x,y), \tilde{z} = (\tilde{x}, \tilde{y}) \in \mathbb{R}^n$ denote by $d'(z, \tilde{z}) = |x - \tilde{x}|$ where $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{n-1}$.

In sections 2 and 3 of this paper we reproduce Drasin's construction in [4] of a quasiregular sine function, $S(z)$, and its modulating function $H : \mathbb{R}^n \to \mathbb{R}^+$, although we make some explicit choices of some intermediate functions that do not appear in [4]. In §4, as in [2], the structure of an analytic set is related to infinite sequences of natural numbers. Finally the function f of Theorem 1 is defined in §5 and all its properties are proved therein.

2. Drasin's quasiregular sine function

In his example, Drasin mimics the structure of the analytic function in $\mathbb{C}$, $\sin z$. He extends its action on $\left\{|\mathrm{Re}(z)| < \frac{\pi}{2}\right\}$ to dimensions $n \geq 3$, by radial symmetrization and uses reflections to get its periodic behavior. Concretely, for $n \geq 3$, let $V^+ := \left\{(x, y) \in \mathbb{R}^n : |x| < \frac{\pi}{2}, y > \operatorname{arcsinh} 1\right\}$ and $h : V^+ \to \mathbb{R}^n$ be a quasiconformal function (see [4]) which is a radial extension of $\sin z$ defined as

$$h(z) = h(x, y) = \left(\frac{x}{|x|}\sin|x|\cosh y, \cos|x|\sinh y\right),$$

where $z = (x, y)$ with $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. The function h maps $V^+ \cap \{y = c\}$ ($c > \operatorname{arcsinh} 1$) onto the upper half of the ellipsoid
$E(c) := \left\{(x, y) : \frac{|x|^2}{\cosh^2 c} + \frac{y^2}{\sinh^2 c} = 1\right\}$.

The periodic behavior characteristic of $\sin z$ is obtained by successive reflections. With this aim the round cylinder, V^+, is viewed as a quasiconformal image of a cubic based prism. Let $C_0 = \{(x, y) \in \mathbb{R}^n : \|x\| \leq 1/2\}$ be an infinite prism and consider $C_0^* = \{(x, y) \in C_0 : y > \frac{1}{2} - \|x\|\}$. Define a bilipschitz map $g : C_0^* \to V^+$,

$$g(z) = g(x, y) = \big(\kappa(x), y + \|x\| - \tfrac{1}{2} + c_0\big),$$

that is, $g = g_2 \circ g_1$ where $g_1(x, y) = \big(x, y + \|x\| - \frac{1}{2}\big)$ maps C_0^* onto $C_0^+ := \{(x, y) \in C_0 : y > 0\}$ and $g_2(x, y) = \big(k(x), y + c_0\big)$ maps C_0^+ onto V^+ with $c_0 = \operatorname{arcsinh} 1$ and κ a bilipschitz homeomorphism from $\left\{\|x\| \leq \frac{1}{2}\right\}$ onto $\left\{|x| \leq \frac{\pi}{2}\right\}$. We choose κ such that if $x \in \left\{\|x\| = \frac{1}{2}\right\}$ then $\kappa(x) = \frac{\pi}{2}\frac{x}{|x|} \in \left\{|x| = \frac{\pi}{2}\right\}$, and hence the image of opposite $(n-2)$-cells of $\left\{\|x\| = \frac{1}{2}\right\}$ have maximal separation on $\left\{|x| = \frac{\pi}{2}\right\}$.

Notice that the composition $S = h \circ g$ defines a quasiconformal function in C_0^* onto $\mathbb{R}^n_+ \setminus E(c_0)$. First extend S to $\overline{C_0^*} := \{z \in \mathbb{R}^n : \overline{z} \in C_0^*\}$ as $S(z) = \overline{S(\overline{z})}$. Since S maps the boundary of the polyhedron $T = C_0 \setminus (C_0^* \cup \overline{C_0^*})$ onto $E(c_0) = \{|x|^2 + 2y^2 = 2\}$ in a bilipschitz way it can be extended into T as a bilipschitz homeomorphism so that $S(\overline{z}) = \overline{S(z)}$ and $S(0) = 0$. Now S is defined in all C_0. Observe that $S(\partial C_0) = \left\{(x, 0) \in \mathbb{R}^n : |x| \geq \sqrt{2}\right\}$.

Consider a partition of $\mathbb{R}^n$ into infinite prisms, all obtained by translating C_0 to $C_{\mathsf{n}} := \{(x + \mathsf{n}, y) \in \mathbb{R}^n : (x, y) \in C_0\}$, $\mathsf{n} \in \mathbb{Z}^{n-1}$. Each $(n-1)$-cell of C_0 defined by $F_0^{i+} := \{z \in \partial C_0 : x^i = 1/2\}$ or $F_0^{i-} := \{z \in \partial C_0 : x^i = -1/2\}$, with $i \in \{1, \ldots, n-1\}$, will be refered to as a $(n-1)$-face of C_0 and analogously for the prisms C_{n}. Extend $S(z)$ to $\mathbb{R}^n$ by repeated reflections across the $(n-1)$-faces of the prisms. More specifically, for $\mathsf{n} = (\mathsf{n}^1, \ldots, \mathsf{n}^{n-1}) \in \mathbb{Z}^{n-1}$ and $x = (x^1, \ldots, x^{n-1}) \in \mathbb{R}^{n-1}$, set $(-1)^{\mathsf{n}} x = \big((-1)^{\mathsf{n}^1} x^1, \ldots, (-1)^{\mathsf{n}^{n-1}} x^{n-1}\big)$. Then the reflections across the $(n-1)$-faces of the prisms yield to $R_{\mathsf{n}} : C_{\mathsf{n}} \to C_0$ given by $R_{\mathsf{n}}(x, y) = \big((-1)^{\mathsf{n}}(x - \mathsf{n}), y\big)$. Thus, define S in C_{n} by $S(z) = S(R_{\mathsf{n}}(z))$. Notice that when $C_{\mathsf{n}} \cap C_{\tilde{\mathsf{n}}} \neq \emptyset$ if

n and $\tilde{\mathsf{n}}$ differ at the i-th coordinate ($i \in \{1, \ldots, n-1\}$) then $\mathsf{n}^i = \tilde{\mathsf{n}}^i \pm 1$. Thus for $z \in C_{\mathsf{n}} \cap C_{\tilde{\mathsf{n}}}$ the i-th coordinate of z, $x^i = \mathsf{n}^i \mp 1/2 = \tilde{\mathsf{n}}^i \pm 1/2$ which implies $R_{\mathsf{n}}(z) = R_{\tilde{\mathsf{n}}}(z)$. Therefore, S is well defined and continuous in $\mathbb{R}^n$.

As shown in [4], $S : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiregular function. Also $\mathrm{As}(S) = \{\infty\}$ since $|S| \to \infty$ uniformly as $y \to \infty$ and it is periodic in the first $n-1$ variables. Moreover, its order of growth is $n-1$ (see [4, Lemma 2]).

3. Modulation of the sine function

As in Drasin's paper, the quasiregular sine function is modified with a regular function $H : \mathbb{R}^{n-1} \to \mathbb{R}^+$ that satisfies conditions below (see [4, §5]).

For some $\varepsilon_0 > 0$ small enough,

$$|\nabla \log H(x)| < \varepsilon_0, \qquad \text{for all } x \in \mathbb{R}^{n-1}, \tag{2}$$

$$0 < H(x) < 1 + |x|, \qquad \text{for all } x \in \mathbb{R}^{n-1}. \tag{3}$$

Extend H to $z = (x, y) \in \mathbb{R}^n$ by $H(z) = H(x)$.

Condition (2) and [4, Lemma 5.4] show that the function $f_0(z) = H(z)S(z)$ is quasiregular. The oscillation of $S(z)$ is inherited by f_0. The next lemma, in some sense, quantifies this oscillation.

Lemma 1. *Consider $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_0 = H \cdot S$ and assume that there exist $z \in F$, $\tilde{z} \in \tilde{F}$, F and $\tilde{F}$ two parallel $(n-1)$-faces of C_{n}, such that $|f_0(z)|, |f_0(\tilde{z})| > \delta$. Then*

$$|f_0(z) - f_0(\tilde{z})| \geq \frac{2\delta}{\sqrt{n-1}}.$$

Proof. Notice that for any $z = (x, y) \in \partial C_0$ then, $\|x\| = 1/2$ and therefore

$$S(z) = S(x, y) = \cosh(|y| + c_0) \left(\frac{x}{|x|}, 0 \right) \in \mathbb{R}^n,$$

with $c_0 = \operatorname{arcsinh} 1$. Consider $z_0, \tilde{z}_0 \in \partial C_0$ defined by $z_0 = R_{\mathsf{n}}(z)$ and $\tilde{z}_0 = R_{\mathsf{n}}(\tilde{z})$, therefore $S(z) = S(z_0)$ and $S(\tilde{z}) = S(\tilde{z}_0)$, and z and z_0 lie on parallel $(n-1)$-faces of ∂C_0. Thus, there exists $i \in \{1, \ldots, n-1\}$ such that $z_0 \in \{z \in \partial C_0 : x^i = 1/2\}$ and $\tilde{z}_0 \in \{z \in \partial C_0 : x^i = -1/2\}$ (or maybe, with opposite signs). This implies,

$$\begin{aligned}
|f_0(z) - f_0(\tilde{z})| &= |H(z)S(z_0) - H(\tilde{z})S(\tilde{z}_0)| \\
&= \left| H(z) \cosh(|y| + c_0) \frac{x_0}{|x_0|} - H(\tilde{z}) \cosh(|\tilde{y}| + c_0) \frac{\tilde{x}_0}{|\tilde{x}_0|} \right| \\
&\geq \left| H(z) \cosh(|y| + c_0) \frac{x^i}{|x_0|} - H(\tilde{z}) \cosh(|\tilde{y}| + c_0) \frac{\tilde{x}^i}{|\tilde{x}_0|} \right| \\
&= \frac{1}{2} \left(\frac{H(z) \cosh(|y| + c_0)}{|x_0|} + \frac{H(\tilde{z}) \cosh(|\tilde{y}| + c_0)}{|\tilde{x}_0|} \right),
\end{aligned}$$

where $\tilde{x}^i$ is the i-th coordinate of $\tilde{x}_0$. Using that $|f_0(z)| = H(z)|S(z_0)| = H(z)\cosh(|y| + c_0) > \delta$ and $|f_0(\tilde{z})| = H(\tilde{z})\cosh(|\tilde{y}| + c_0) > \delta$, then

$$|f(z) - f(\tilde{z})| \geq \frac{\delta}{2}\left(\frac{1}{|x_0|} + \frac{1}{|\tilde{x}_0|}\right) \geq \frac{2\delta}{\sqrt{n-1}},$$

since $|x_0|, |\tilde{x}_0| \leq \sqrt{n-1}/2$ because $\|x_0\| = \|\tilde{x}_0\| = 1/2$. □

Now we will force f_0 be close to 0 in certain sets by making $H : \mathbb{R}^n \to \mathbb{R}$ very small. These sets will eventually contain the asymptotic paths of f in Theorem 1. These sets are labeled after the sets S_m in the definition of the analytic set below (see Theorem A). We introduce some notation. Let $\mathcal{N}_0$ denote the countable collection of finite sequences of positive natural numbers. If $m = \langle n_1, \dots, n_p\rangle \in \mathcal{N}_0$ and $k \in \mathbb{N}$, the extension of m with k is represented by $m^\frown k = \langle n_1, \dots, n_p, k\rangle \in \mathcal{N}_0$ and its truncation at the l-th entry ($l \leq p$) by $m|_l = \langle n_1, \dots, n_l\rangle$.

Let $D' = \{(x, 0) : (x^2)^2 + \cdots + (x^{n-1})^2 < x^1\}$ be a region in $\mathbb{R}^{n-1} \times \{0\}$ and consider the subset $D'_0 \subset D'$ the translation of D' by $(R, 0, \dots, 0) \in \mathbb{R}^n$ for some large $R > 0$. Let $\{D'(m) : m \in \mathcal{N}_0\}$ be a collection of proper simply connected unbounded subsets of D'_0, $D'(m) \neq \emptyset$ such that

a) For all $m \in \mathcal{N}_0$, $D'(m^\frown k) \subsetneq D'(m)$ for all $k \in \mathbb{N}$.
b) There are only a finite number of sets $D'(m)$, $m \in \mathcal{N}_0$ that intersect any ball $B(0, r)$. Moreover, for every $l \in \mathbb{N}$ there exists $r > 0$ such that $B(0, r) \cap D'(m) \neq \emptyset$ if $m = \langle n_1, \dots, n_p\rangle$ with $n_1 + \cdots + n_p \leq l$ and $B(0, r) \cap D'(m) = \emptyset$ if $n_1 + \cdots + n_p > l$.
c) Let $r_0 = 2^{n+1}\sqrt{n-1}$ and consider a sequence $\{r_l\}_{l\geq 0}$ with $r_l > r_{l-1}$, $l \geq 1$ such that $r_l \uparrow \infty$. If $m = \langle n_1, \dots, n_p\rangle$ with $l = n_1 + \cdots + n_p$ then,

$$d\big(\partial D'(m), D'(m^\frown k)\big) > r_l, \text{ for all } k \in \mathbb{N} \tag{4}$$

and

$$d\big(D'(m^\frown k_1), D'(m^\frown k_2)\big) > r_l, \text{ for all } k_1 \neq k_2,\ k_1, k_2 \in \mathbb{N}. \tag{5}$$

Finally take the sets D'_0 and $D'(m)$, $m \in \mathcal{N}_0$ such that there exists a constant $c_1 > 1$ for which

$$|S(z)| \geq c_1 > 1, \text{ for all } z \in \partial D'(m) \text{ and } m \in \mathcal{N}_0 \text{ or for all } z \in \partial D'_0. \tag{6}$$

To control the asymptotic paths of f_0 we will give more specific values of H in the domains defined above. Concretely, given a decreasing sequence $\{\delta_p\}_{p\geq 0}$ so that $\delta_p \downarrow 0$, then conditions (2)-(3) of $H : \mathbb{R}^n \to \mathbb{R}^+$ are complemented with

$$H(x, 0) = 1, \text{ if } \mathcal{B}\big((x, 0), 1\big) \cap D'_0 = \emptyset, \tag{7}$$

$$\delta_0 < H(x, 0) \leq 2\delta_0, \qquad (x, 0) \in D'_0 \setminus \bigcup_{k\in\mathbb{N}} D'(k), \tag{8}$$

and if p is the number of entries of $m \in \mathcal{N}_0$ then

$$(9) \qquad \begin{aligned} \delta_p < H(x,0) \le \delta_{p-1}, &\quad (x,0) \in D'(m) \setminus \textstyle\bigcup_{k\in\mathbb{N}} D'(m^\frown k), \\ H(x,0) = \delta_{p-1}, &\quad (x,0) \in D'(m) \text{ and } d'((x,0), \partial D'(m)) < r_0, \end{aligned}$$

where $d'(z, A)$ is the Euclidean distance from the projection of z to the projection of A on $\mathbb{R}^{n-1} \times \{0\}$.

The sequence $\{r_l\}_{l\ge1}$ in (4) and (5) above is chosen so that conditions (2)-(3) and (7)-(9) of H hold for the given sequence $\{\delta_p\}_{p\ge0}$.

Remark 1. By condition (3) the order of growth of f_0 is that of S, that is, $n-1$.

4. Analytic sets and bilipschitz maps

In this section some bilipschitz transformations are defined in such a way that 0 is mapped to any point in the analytic set $A \subset \mathbb{R}^n$ as done in [2]. Without loss of generality it can be assumed that $0 \in A$ and for simplicity suppose $\operatorname{diam} A \le 1$ (at the end of the next section it will be explained how to remove this latter condition).

Let $\mathbb{N}^{\mathbb{N}}$ be the set of all infinite sequences of natural numbers. If $\nu \in \mathbb{N}^{\mathbb{N}}$, the truncation of ν at its first p ($p \in \mathbb{N}$) entries is denoted by $\nu|_p \in \mathcal{N}_0$. We will use the characterization of analytic sets in $\mathbb{R}^n$ via the $\mathcal{A}$-operation in the following terms.

Theorem A. *Let $A \subset \mathbb{R}^n$ be an analytic set and $\{\delta_p\}_{p\ge1}$, $\delta_p \downarrow 0$, a decreasing sequence. Then there exists a collection of sets $\{\mathsf{S}_m\}_{m\in\mathcal{N}_0}$ with the following properties:*

1)

$$A = \bigcup_{\nu\in\mathbb{N}^{\mathbb{N}}} \bigcap_{p\in\mathbb{N}} \mathsf{S}_{\nu|_p}.$$

2) $\mathsf{S}_m \ne \emptyset$ *is closed for all* $m \in \mathcal{N}_0$.
3) $\mathsf{S}_{m^\frown k} \subset \mathsf{S}_m$ *where* $m^\frown k = \langle n_1, \dots, n_p, k\rangle \in \mathcal{N}_0$ *for* $m = \langle n_1, \dots, n_p\rangle \in \mathcal{N}_0$ *and* $k \in \mathbb{N}$.
4) $\operatorname{diam} \mathsf{S}_m \le \delta_{p+1}$ *if* $m \in \mathcal{N}_0$ *has* p *entries.*

For the proof of the theorem in this form we refer to the reader to [10] and [2].

The sequence $\{\delta_p\}_{p\ge0}$ that appears when defining conditions (8) and (9) of H and also in Theorem A arises from applying repeatedly the following lemma. Similar lemmas have been previously used in [4].

Lemma 2. *Given $L > 1$ there exists a positive number $\delta < 1/2$ such that for any $a \in \mathbb{R}^n$ with $|a| \leq \delta$ the map*

$$\varphi(z) = \begin{cases} z, & |z| \geq 1, \\ z + a, & |z| \leq \delta, \\ z + \dfrac{a}{1-\delta}(1 - |z|), & \delta < |z| < 1, \end{cases}$$

is L-bilipschitz.

Proof. Set $\delta := (L-1)/(2L-1) < 1/2$. In $\{\delta \leq |z| \leq 1\}$, φ is L-bilipschitz as a consequence of the triangle inequality and the fact that $|a| \leq \delta$. Since φ is obviously 1-bilipschitz (an isometry) in $\{|z| \leq \delta\}$, to check that it is bilipschitz in the ball $\{|z| \leq 1\}$ it is enough to consider $z \in B(0, \delta)$ and $w \in \{\delta < |z| \leq 1\}$. Let ξ be the point of intersection of $\partial B(0, \delta)$ with the line segment that joins z and w. Since both $\xi, w \in \{\delta \leq |z| \leq 1\}$ and $\varphi(z) = z + a$ if $|z| \leq \delta$,

$$\begin{aligned} |\varphi(z) - \varphi(w)| &\leq |\varphi(z) - \varphi(\xi)| + |\varphi(\xi) - \varphi(w)| \leq |z - \xi| + L|\xi - w| \\ &\leq L(|z - \xi| + |\xi - w|) = L|z - w|. \end{aligned}$$

For the lower bound, choose $\zeta' = \varphi(\zeta)$ the point on the straight segment that joins $\varphi(z)$ to $\varphi(w)$ that lies on $\partial B(a, \delta)$ (notice that this is possible since $\varphi(z) \in B(a, \delta)$ and $\varphi(w) \notin \overline{B}(a, \delta)$). Hence

$$\begin{aligned} |\varphi(z) - \varphi(w)| &= |\varphi(z) - \zeta'| + |\zeta' - \varphi(w)| = |\varphi(z) - \varphi(\zeta)| + |\varphi(\zeta) - \varphi(w)| \\ &\geq |z - \zeta| + \tfrac{1}{L}|\zeta - w| \geq \tfrac{1}{L}(|z - \zeta| + |\zeta - w|) \geq \tfrac{1}{L}|z - w|. \end{aligned}$$

We proceed in a similar way to show that φ is L-bilipschitz no only in $\overline{B}(0, 1)$ but in $\mathbb{R}^n$. □

By adequate re-scaling one obtains the following

Corollary 1. *Given $L > 1$ and $\rho > 0$ (or respectively, given $L > 1$ and $\delta > 0$) there exists $0 < \delta < \rho/2$ (or respectively $\rho > 2\delta$) such that if $|a| \leq \delta$ there is a L-bilipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ for which $\varphi(z) = z + a$ if $|z| \leq \delta$ and $\varphi(z) = z$ if $|z| \geq \rho$.*

We use this lemma (in fact, its corollary) in an iterative way. For a given $L > 1$, take a sequence $L_1 > L_2 > \cdots$ with

$$\prod_{p \geq 1} L_p < L \tag{10}$$

(thus $L_p \downarrow 1$). Apply Corollary 1 with $\delta = 2$ and $L = L_1$ obtaining ρ and define $\delta_1 = \delta/2 = 1 = \operatorname{diam} A$ and $\delta_0 = \rho$. For $p \geq 2$ take $L = L_p$, $\rho = \delta_{p-1}$, apply Corollary 1 to obtain δ and define $\delta_p = \delta/2$. Clearly $\delta_p \downarrow 0$ and $2\delta_p < \delta_{p-1}$ for $p \geq 1$.

As in [4] successive applications of Corollary 1 will produce a large collection of bilipschitz maps. At each point $z = (x, y) \in \mathbb{R}^n$ the final L-bilipschitz map Ψ on $w = f_0(z)$ will be of the form $\Psi(w) = \Psi_p(w) =$

$\varphi_p \circ \varphi_{p-1} \circ \cdots \circ \varphi_0(w)$, the various φ_j, $0 \le j \le p$, depend on the region $D'(m)$ $(m \in \mathcal{N}_0)$ where x may lay.

Now, we relate these translations to the analytic set. For every $m \in \mathcal{N}_0$ pick $a_m \in \mathsf{S}_m$ where S_m is the set in Theorem A. This choice will be fixed for the rest of the note. To each $m = \langle n_1, \ldots, n_p \rangle \in \mathcal{N}_0$ there will be associated a "chain" of bilipschitz transformations, $\varphi_0, \ldots, \varphi_p$. Concretely, let $a_{m|_0} := 0$ and consider the points $a_{m|_j} \in \mathsf{S}_{m|_j}$ (recall that $m|_j = \langle n_1, \ldots, n_j \rangle$ is the truncation of m at its first j entries, $1 \le j \le p$). Notice that $|a_{m|_1}| \le \delta_1$ since $0, a_{m|_1} \in A$ and $\operatorname{diam} A \le 1 = \delta_1$. Also, for $j = 1, \ldots, p-1$, $|a_{m|_{j+1}} - a_{m|_j}| \le \delta_{j+1}$ since $a_{m|_{j+1}}, a_{m|_j} \in \mathsf{S}_{m|_j}$ and $\operatorname{diam} \mathsf{S}_{m|_j} \le \delta_{j+1}$. Thus, for $j = 0, \ldots, p-1$, take $\varphi_j : \mathbb{R}^n \to \mathbb{R}^n$ the L_{j+1}-bilipschitz transformation obtained in Corollary 1 with

$$\varphi_j(z) = \begin{cases} z + a_{m|_{j+1}} - a_{m|_j}, & |z - a_{m|_j}| \le 2\delta_{j+1}, \\ z, & |z - a_{m|_j}| \ge \delta_j. \end{cases}$$

Notice the abuse of notation since the chain $\varphi_0, \ldots, \varphi_p$ depends on $m \in \mathcal{N}_0$ and it maybe different for different points in $\mathcal{N}_0$ with the same number of entries. Nevertheless, the bilipschitz constants, L_{j+1} $(j = 0, \ldots, p-1)$, do not depend on the particular choice $m \in \mathcal{N}_0$.

Observe that

$$0 \xrightarrow{\varphi_0} a_{m|_1} \xrightarrow{\varphi_1} a_{m|_2} \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{p-1}} a_{m|_p} = a_m,$$

$$B(0, 2\delta_p) \xrightarrow{\varphi_0} B(a_{m|_1}, 2\delta_p) \xrightarrow{\varphi_1} B(a_{m|_2}, 2\delta_p) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{p-1}} B(a_m, 2\delta_p),$$

and moreover, if $\Psi_{p-1} = \varphi_{p-1} \circ \cdots \circ \varphi_0$, then by (10),

$$|\Psi_{p-1}(z) - a_m| \le L|z|, \quad \text{for all } z \in \mathbb{R}^n. \tag{11}$$

If $m' = m^\frown k$ then the chain associated to m' is obtained by adding an extra bilipschitz transformation to the chain of m. Thus, in general, if $m = \langle n_1, \ldots, n_p \rangle$ and $m' = \langle n_1, \ldots, n_p, \ldots, n_k \rangle$ with $k > p$, then the chain of bilipschitz transformations associated to m' is an ampliation (by $k - p$ functions) of the chain associated to m.

The action of the chain $\varphi_0, \ldots, \varphi_p$ associated to $m \in \mathcal{N}_0$ on z depends on the size of $|z|$.

Lemma 3. *Let $\nu \in \mathbb{N}^{\mathbb{N}}$ and $p \ge 0$. Consider the chain associated to $\nu|_{p+1}$ given by $\varphi_0, \ldots, \varphi_p$ and denote by $\Psi_p = \varphi_p \circ \varphi_{p-1} \circ \cdots \circ \varphi_0$. If $\delta_{k+1} < |z| \le \delta_k$, then for any $p \ge k$*

$$\Psi_p(z) = \varphi_k(z + a_{\nu|_k}).$$

Proof. To simplify notation denote by $\boldsymbol{a}_k := a_{\nu|_k}$. Since $|z| \le \delta_k$, then by the definition of the φ_i's, $\Psi_{k-1}(z) = \varphi_{k-1}\big(\varphi_{k-2}\left(\cdots(\varphi_0(z))\cdots\right)\big) = z + \boldsymbol{a}_k$ since they all are translations in $\{|z| \le \delta_k\}$.

Let $\tilde{z} = z + \boldsymbol{a}_k$ and notice that $\tilde{z} \notin B(\boldsymbol{a}_k, \delta_{k+1})$, therefore, $\varphi_k(\tilde{z}) \notin \varphi_k\left(B(\boldsymbol{a}_k, \delta_{k+1})\right) = B(\boldsymbol{a}_{k+1}, \delta_{k+1})$. Then, by the definition of φ_{k+1},

$$\Psi_{k+1}(z) = \varphi_{k+1}(\varphi_k(\tilde{z})) = \varphi_k(\tilde{z}).$$

Since $B(\boldsymbol{a}_{p+1}, \delta_{p+1}) \subset B(\boldsymbol{a}_p, \delta_p)$ for all $p \geq 0$ (recall $2\delta_{p+1} < \delta_p$ and $|\boldsymbol{a}_{p+1} - \boldsymbol{a}_p| \leq \delta_{p+1}$) and $\varphi_k(\tilde{z}) \notin B(\boldsymbol{a}_{k+1}, \delta_{k+1})$ then $\varphi_k(\tilde{z}) \notin B(\boldsymbol{a}_i, \delta_i)$ for $i = k+1, \ldots, p$. Arguing inductively, $\Psi_p(z) = \varphi_k(\tilde{z})$. □

Corollary 2. *Using the notation of Lemma 3. If* $|z| > \delta_{k+1}$ *then*

$$\Psi_{p+1}(z) = \Psi_p(z), \qquad \text{for all } p \geq k.$$

5. The definition of the quasiregular function

After all the preparation in the previous sections, we construct the function f of Theorem 1 as it was done in [4]. Concretely, consider the vertical extension of the sets defined in section 3: $D = \{(x,y) \in \mathbb{R}^n : (x,0) \in D'\}$, $D_0 = \{(x,y) \in \mathbb{R}^n : (x,0) \in D_0'\}$ and, $D(m) = \{(x,y) \in \mathbb{R}^n : (x,0) \in D'(m)\}$, $m \in \mathcal{N}_0$. Define $f : \mathbb{R}^n \to \mathbb{R}^n$ recursively as follows:

$$f(z) = f_0(z), \qquad z \in \mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} D(k),$$

and

$$f(z) = \Psi_{p-1}(f_0(z)), \quad z \in \overline{D(m)} \setminus \bigcup_{k=1}^{\infty} D(m^\frown k),$$

where $\Psi_{p-1} = \varphi_{p-1} \circ \cdots \circ \varphi_0$ is the chain associated to $m \in \mathcal{N}_0$, m with p entries.

To show the continuity of f, observe that the functions, $f_0, \varphi_0, \ldots, \varphi_{p-1}$ are continuous therefore it suffices to check it on the boundary of $\overline{D(m)} \setminus \cup_k D(m^\frown k)$.

Assume first that $z = (x,y) \in \partial D(k)$, $k \in \mathbb{N}$ (an element of $\mathcal{N}_0$ with just one entry). By (6) and (9)

$$|f_0(z)| = H(z)|S(z)| \geq c_1 H(x,0) = c_1\delta_0 > \delta_0.$$

If φ_0 is the bilipschitz transformation that maps $0 \mapsto a_k$, $a_k \in \mathsf{S}_k$, then $f(z) = \varphi_0(f_0(z)) = f_0(z)$ and thus f is continuous on $\partial D(k)$.

In general, if $z = (x,y) \in \partial D(m^\frown k)$ with $m \in \mathcal{N}_0$ and m with p entries, then (6) and (9) imply

$$|f_0(z)| = H(z)|S(z)| \geq c_1\delta_p > \delta_p$$

and by Corollary 2, $\Psi_p(f_0(z)) = \Psi_{p-1}(f_0(z))$, where $\Psi_p = \varphi_p \circ \cdots \circ \varphi_0$ and $\Psi_{p-1} = \varphi_{p-1} \circ \cdots \circ \varphi_0$ are respectively the chain associated to $m^\frown k$ and m. Therefore f continuous on $\partial D(m^\frown k)$ for all $m \in \mathcal{N}_0$ and $k \in \mathbb{N}$.

Remark 2. In fact, as in [4], $f = \Psi \circ f_0$ is K_1-quasiregular with $K_1 = K_1(n, K, L)$ where f_0 is K-quasiregular and $L > \Pi_{j\geq 1} L_j$ as in (10). Also, the order of growth of f, ρ_f, is $n-1$, since $\Psi(z) = z$ if $|z| > \delta_0$ and $\rho_{f_0} = n-1$.

If two points $z, \tilde{z} \in \mathbb{R}^n$ are horizontally close then $f(z)$ and $f(\tilde{z})$ are defined using the same chain of functions $\varphi_0, \ldots, \varphi_p$. Recall the sequence $\{r_l\}_{l\geq 1}$ of conditions (4) and (5) and the notation $d'(z, \tilde{z}) = |x - \tilde{x}|$ if $z = (x, y)$ and $\tilde{z} = (\tilde{x}, \tilde{y})$.

Lemma 4. *Let $z, \tilde{z} \in \mathbb{R}^n$ such that $d'(z, \tilde{z}) < r_0$ and $z, \tilde{z} \in \bigcup_{\mathsf{n}\in\mathbb{Z}^{n-1}} \partial C_{\mathsf{n}}$. Then*

$$|f(z) - f(\tilde{z})| \geq \frac{1}{L}|f_0(z) - f_0(\tilde{z})|.$$

Proof. Define $D(0) := \mathbb{R}^n$ and $0^\frown k := k$, $k \in \mathbb{N}$. Then the sets $D(m) \setminus \cup_{k\in\mathbb{N}} D(m^\frown k)$, with $m \in \mathcal{N}_0 \cup \{0\}$ form a partition of $\mathbb{R}^n$ and thus for $z, \tilde{z} \in \mathbb{R}^n$ there exist $m, \tilde{m} \in \mathcal{N}_0 \cup \{0\}$ with p and $\tilde{p}$ entries respectively, so that

$$z \in D(m) \setminus \cup_{k\in\mathbb{N}} D(m^\frown k), \quad \tilde{z} \in D(\tilde{m}) \setminus \cup_{k\in\mathbb{N}} D(\tilde{m}^\frown k).$$

Since $d'(z, \tilde{z}) < r_0$ conditions (4) and (5) imply that $\tilde{m} = m$ or $\tilde{m} = m^\frown l$ or $m = \tilde{m}^\frown l'$ for some l or l' in $\mathbb{N}$. Assume first that $\tilde{m} = m$ then, by the definition of f,

$$f(z) = \Psi_{p-1}(f_0(z)), \quad \text{and} \quad f(\tilde{z}) = \Psi_{p-1}(f_0(\tilde{z})),$$

where $\Psi_{p-1} = \varphi_{p-1} \circ \cdots \circ \varphi_0$ and $\varphi_{p-1}, \ldots, \varphi_0$ is the chain of bilipschitz transformations associated to m. Thus,

$$\begin{aligned}|f(z) - f(\tilde{z})| = |\Psi_{p-1}(f_0(z)) - \Psi_{p-1}(f_0(\tilde{z}))| &\geq \left(\prod_{j=1}^{p} \frac{1}{L_j}\right) |f_0(z) - f_0(\tilde{z})| \\ &\geq \frac{1}{L}|f_0(z) - f_0(\tilde{z})|,\end{aligned}$$

since each φ_j function is L_{j+1}-bilipschitz, $j = 0, \ldots, p-1$, and $L > \prod_{j\geq 1} L_j$ by (10).

If, otherwise, $\tilde{m} = m^\frown l$ (the case $m = \tilde{m}^\frown l'$ is symmetrical) then by the definition of f,

$$f(z) = \Psi_{p-1}(f_0(z)), \quad \text{and} \quad f(\tilde{z}) = \varphi_p\big(\Psi_{p-1}(f_0(\tilde{z}))\big),$$

where again $\Psi_{p-1} = \varphi_{p-1} \circ \cdots \circ \varphi_0$ and $\varphi_{p-1}, \ldots, \varphi_0$ is the chain of bilipschitz transformations associated to m and φ_p is the extra function in the chain associated to $\tilde{m}$. Since $d'(z, \tilde{z}) < r_0$, $z \in D(m) \setminus \cup_{k\in\mathbb{N}} D(m^\frown k)$ and $\tilde{z} \in D(m^\frown l)$ then $d'(\tilde{z}, \partial D(m^\frown l)) < r_0$. By condition (9), $H(\tilde{z}) = \delta_p$ and thus $|f_0(\tilde{z})| \geq \sqrt{2}\delta_p > \delta_p$. Hence, by Corollary 2,

$$\varphi_p(\Psi_{p-1}(f_0(\tilde{z}))) = \Psi_{p-1}(f_0(\tilde{z})),$$

and as above,

$$|f(z) - f(\tilde{z})| = |\Psi_{p-1}(f_0(z)) - \Psi_{p-1}(f_0(\tilde{z})) \geq \frac{1}{L}|f_0(z) - f_0(\tilde{z})|.$$

□

Now it will be shown that f_0 cannot be large on the asymptotic paths of f with finite asymptotic value. For this purpose we are going to replace the curve γ for an unbounded sequence of points on γ. Recall the notation F_0^{i+} and F_0^{i-} for the $(n-1)$-faces of the prism C_0 given in section 2 and the maps $R_{\mathsf{n}} : C_{\mathsf{n}} \to C_0$ of compositions of reflections across the $(n-1)$-faces of the prisms. The set of all $(n-1)$-faces of the prisms is divided into $2(n-1)$ sets according to their equivalence class with respect to the reflections. Concretely, for $i \in \{1, \dots, n-1\}$ let

$$\mathcal{F}^{i+} := \bigcup_{\mathsf{n}\in\mathbb{Z}^{n-1}} \{F_{\mathsf{n}} : F_{\mathsf{n}} = R_{\mathsf{n}}^{-1}(F_0^{i+})\}, \quad \mathcal{F}^{i-} := \bigcup_{\mathsf{n}\in\mathbb{Z}^{n-1}} \{F_{\mathsf{n}} : F_{\mathsf{n}} = R_{\mathsf{n}}^{-1}(F_0^{i-})\}.$$

Lemma 5. *Let γ be an asymptotic path of f with finite asymptotic value. Then there exists an unbounded sequence of points on γ, $\{z_j\}_{j\geq 1}$, such that, for all $j \geq 1$,*

(1) there exists $i = \{1, \dots, n-1\}$ such that $z_{2j-1} \in \mathcal{F}^{i\xi}$ and $z_{2j} \in \mathcal{F}^{i(-\xi)}$ with $\xi \in \{+, -\}$,

(2) $d'(z_j, z_{j+1}) \leq 2^n(n-1)^{1/2} = r_0/2$ (where $d'(\cdot,\cdot)$ is the Euclidean distance of the vertical projection of the points. See the introduction).

Proof. Assume first that γ intersects a finite number of prisms C_{n}. Then for $z = (x, y) \in \gamma$, x lies in a compact subset of $\mathbb{R}^{n-1}$, and since $H(z) = H(x, 0)$ is continuous and positive for $x \in \mathbb{R}^{n-1}$ (see conditions (7)–(9)) then there exists $c > 0$ such that $H(z) > c$ for $z \in \gamma$. Moreover, since $|z| \to \infty$ on γ then $|y| \to \infty$ on γ which implies $|S(z)| \to \infty$ on γ. Thus, $|f(z)| = |f_0(z)| = |S(z)|H(z) \to \infty$ on γ. In this situation, γ could not be an asymptotic curve with finite asymptotic value. Thus γ intersects infinitely many prisms $\{C_{\mathsf{n}} : \mathsf{n} \in \mathbb{Z}^{n-1}\}$. Since $\gamma \to \infty$, without loss of generality we can assume that γ visits each prism C_{n} ($\mathsf{n} \in \mathbb{Z}^{n-1}$) at most once.

Define a sequence $\{w_k\}_{k\geq 1} \subset \gamma \cap \left(\cup_{\mathsf{n}} \partial C_{\mathsf{n}}\right)$, ordered according to the parametrization of γ, with the following property: if γ intersects an $(n-1)$-face, say F_{n}, then there is a unique point in the sequence $\{w_k\}_{k\geq 1}$ that belongs to that $(n-1)$-face, F_{n}. Clearly by construction,

a) if $w_k \in \mathcal{F}^{i+}$ then $w_{k+1} \notin \mathcal{F}^{i+}$, (and the same replacing $+$ by $-$),

b) $d'(w_k, w_{k+1}) \leq \sqrt{n-1}$ for all $k \geq 1$.

Let $N = 2^{n-1} + 1$. Then among $w_1, \dots, w_N$ there exist two points w_k and $w_{k'}$, and an index $i \in 1, \dots, n-1$ such that $w_k \in \mathcal{F}^{i+}$ and $w_{k'} \in \mathcal{F}^{i-}$. Otherwise there will be two points among $w_1, \dots, w_N$ in the same $(n-1)$-face of a prism. Take z_1 and z_2 to be w_k and $w_{k'}$ keeping the ordering of the labels. By the same reasoning for any $j > 1$, there exist an $i \in 1, \dots, n-1$ and two points $w_k, w_{k'} \in \{w_{jN+1}, \dots, w_{(j+1)N}\}$ such that $w_k \in \mathcal{F}^{i+}$ and $w_{k'} \in \mathcal{F}^{i-}$. Define z_{2j+1} and z_{2j+2} to be w_k and $w_{k'}$ keeping the ordering of the labels.

Since $d'(w_j, w_{j+1}) \leq \sqrt{n-1}$ then $d'(z_j, z_{j+1}) \leq 2(N-1)\sqrt{n-1} = 2^n(n-1)^{1/2}$. □

Lemma 6. *Let γ be an asymptotic path of f with finite asymptotic value and $\{z_j\}_{j\geq 1}$ the unbounded sequence of points on γ given by Lemma 5. Then*

$$\lim_{j\to\infty} |f_0(z_j)| = 0.$$

Proof. The proof is by contradiction. Suppose that $\limsup_{j\to\infty} |f_0(z_j)| > 0$. Without loss of generality it can be assumed that there exists $\delta > 0$ such that $\limsup_{j\to\infty} |f_0(z_{2j-1})| > \delta > 0$. Consider a subsequence $\{w_{2\ell-1}\}_{\ell\geq 1} \subset \{z_{2j-1}\}_{j\geq 1}$ with $w_{2\ell-1} = z_{2j_\ell-1}$ such that $\lim_{\ell\to\infty} |f_0(w_{2\ell-1})| > \delta$. Define $w_{2\ell} := z_{2j_\ell}$. Then either $\lim_{\ell\to\infty} |f_0(w_{2\ell})| = 0$ or $\limsup_{\ell\to\infty} |f_0(w_{2\ell})| > 0$.

In the first case, for ℓ large enough $|f_0(w_{2\ell-1})| > \delta$ and $|f_0(w_{2\ell})| < \delta/2$. By Lemma 5, $d'(w_{2\ell-1}, w_{2\ell}) < r_0$ and $w_{2\ell-1}, w_{2\ell} \in \cup\partial C_{\mathsf{n}}$ (since $w_{2\ell-1} = z_{2j_\ell-1}$ and $w_{2\ell} = z_{2j_\ell}$). Then Lemma 4 implies

$$|f(w_{2\ell-1}) - f(w_{2\ell})| \geq \frac{1}{L}|f_0(w_{2\ell-1}) - f_0(w_{2\ell})| > \frac{\delta}{2L} > 0.$$

Since this inequality holds for ℓ large enough and, w_ℓ's are points on γ, the curve γ cannot be an asymptotic path with finite asymptotic value.

If otherwise, $\limsup_{\ell\to\infty} |f_0(w_{2\ell})| > 0$, by taking again another subsequence if necessary, there can be found $\delta' > 0$ such that $\lim_{\ell\to\infty} |f_0(w_{2\ell-1})| > \delta'$ and $\lim_{\ell\to\infty} |f_0(w_{2\ell})| > \delta'$ with $w_{2\ell-1} = z_{2j_\ell-1}$ and $w_{2\ell} = z_{2j_\ell}$. Again Lemma 5 implies that $d'(w_{2\ell-1}, w_{2\ell}) < r_0$ and by Lemma 4

$$|f(w_{2\ell-1}) - f(w_{2\ell})| \geq \frac{1}{L}|f_0(w_{2\ell-1}) - f_0(w_{2\ell})|,$$

which by Lemma 1

$$|f(w_{2\ell-1}) - f(w_{2\ell})| \geq \frac{1}{L}|f_0(w_{2\ell-1}) - f_0(w_{2\ell})| > \frac{2\delta'}{L\sqrt{n-1}} > 0,$$

for ℓ large enough. □

We are ready to complete the final step of the proof of Theorem 1:

Proposition 1. *The set of asymptotic values of f is $A \cup \{\infty\}$.*

Proof. First we are going to show that $\mathrm{As}(f) \subset A \cup \{\infty\}$. Assume that γ is an asymptotic curve of f with finite asymptotic value $b \in \mathbb{R}^n$. By Lemmas 5 and 6, there exists an unbounded sequence of points on γ, $\{z_j\}_{j\geq 1}$, for which $\lim_{j\to\infty} |f_0(z_j)| = 0$ and $d'(z_j, z_{j+1}) \leq 2^n(n-1)^{1/2}$. By the construction of f (see (11)) for each z_j there is a point $\boldsymbol{a}_j \in A$ ($\boldsymbol{a}_j = a_m$ if $z_j \in \overline{D(m)} \setminus \cup_k D(m^\frown k)$ or $\boldsymbol{a}_j = 0$ if $z_j \in \mathbb{R}^n \setminus \cup_k D(k)$) so that $f(z_j) \in B(\boldsymbol{a}_j, L\varepsilon_j)$ with $\varepsilon_j := |f_0(z_j)|$. Since $|f_0(z_j)| \to 0$ then $\boldsymbol{a}_j \to b$. We are going to show that $b \in A$.

We claim that there exists an unbounded subsequence $\{z_{j_k}\}_{k\geq 1} \subset \{z_j\}_{j\geq 1}$ for which one of the following statements holds:

a) $\{z_{j_k}\}_{k\geq 1} \subset \mathbb{R}^n \setminus \cup_{l=1}^{\infty} D(l)$,

b) there exists $m \in \mathcal{N}_0$ such that $\{z_{j_k}\}_{k\geq 1} \subset \overline{D(m)} \setminus \cup_{l=1}^{\infty} D(m^\frown l)$,

c) for each $k \geq 2$, there exists $m_k \in \mathcal{N}_0$ such that $z_{j_k} \in \overline{D(m_k)}$ and $m_k = m_{k-1}{}^\frown l_{k-1}$ for some $l_{k-1} \in \mathbb{N}$.

In the first case, $f(z_{j_k}) = f_0(z_{j_k}) \to 0$ therefore $b = 0 \in A$. Analogously, in the second case for all $k \geq 1$ and $m \in \mathcal{N}_0$ given in b), $f(z_{j_k}) \in B(a_m, L\varepsilon_{j_k})$, that is $\boldsymbol{a}_{j_k} = a_m$ for all $k \geq 1$ and therefore, $\boldsymbol{a}_j \to a_m$ which implies $b = a_m \in A$.

Finally in the last case define $a := \cap_{k\geq 1} \mathsf{S}_{m_k} \in A$ (the sets S_{m_k} as defined in Theorem A). Notice that a exists and it is a point in A since $m_{k+1} = m_k^\frown l_k$ for all $k \geq 1$ which implies $\mathsf{S}_{m_{k+1}} \subset \mathsf{S}_{m_k}$. By the construction of f, $\boldsymbol{a}_{j_k} = a_{m_k} \in \mathsf{S}_{m_k}$. Thus $\boldsymbol{a}_{j_k} \to a \in A$, which implies $\boldsymbol{a}_j \to a \in A$, that is, $b = a \in A$.

So we are left to show what is claimed above. Recall that $r_0 = 2^{n+1}(n-1)^{1/2}$ in (4) and (5). For the sequence $\{z_j\}_{j\geq 1}$ obtained in Lemmas 5 and 6 there are two mutually exclusive possibilities:

1) For all $m \in \mathcal{N}_0$, $\{z_j\}$ visits $D(m)$ at most a finite number of times.
2) There exists $\tilde{m} \in \mathcal{N}_0$ such that $\{z_j\}$ visits $D(\tilde{m})$ infinitely many times.

In case 1), $\{z_j\}$ visits at most a finite number of times each $D(l)$, $l \in \mathbb{N}$. If it passes through finitely many of them, then there exists a subsequence that $\{z_{j_k}\}_{k\geq 1} \subset \{z_j\}_{j\geq 1}$ such that $\{z_{j_k}\} \subset \mathbb{R}^n \setminus \cup_l D(l)$. Otherwise, by condition (4) and the fact that $d'(z_j, z_{j+1}) < r_0$ the sequence $\{z_j\}$ passes through $\mathbb{R}^n \setminus \cup_l D(l)$ going from $D(l)$ to $D(l')$ ($l \neq l'$) and since this happens infinitely many times, there exists a subsequence $\{z_{j_k}\}_{k\geq 1} \subset \mathbb{R}^n \setminus \cup_l D(l)$. In both situations we obtain the subsequence in a).

In case 2), write $\tilde{m} = m^\frown k_0$ for some $m \in \mathcal{N}_0$ and $k_0 \in \mathbb{N}$. If $\{z_j\}$ visits $D(\tilde{m})$ infinitely many times, and also leaves $D(\tilde{m})$ infinitely many times, then conditions (4) and (5), and the fact that $d'(z_j, z_{j+1}) < r_0$ imply that $\{z_j\}$ visits $\overline{D(m)} \setminus \cup_l D(m^\frown l)$ infinitely many times. Therefore, in this situation, there exists a subsequence $\{z_{j_k}\}_{k\geq 1} \subset \overline{D(m)} \setminus \cup_l D(m^\frown l)$, that is, we have found a subsequence in b). Otherwise, without loss of generality it can be assumed that $\{z_j\}_{j\geq 1} \subset D(\tilde{m})$. Pick $m \in \mathcal{N}_0$ be the one with the largest number of entries with such property. Then there exists $j_0 > 1$, $j_0 \in \mathbb{N}$, such that $z_{j_0} \in D(m) \setminus \cup_l D(m^\frown l)$. Using that there can only be a finite number of points of $\{z_j\}_{j\geq j_0}$ in $D(m) \setminus \cup_l D(m^\frown l)$ together with conditions (4) and (5) there can be found $j_1 > j_0$, $j_1 \in \mathbb{N}$, and $m_1 = m^\frown l_0$ (with $l_0 \in \mathbb{N}$) so that $\{z_j\}_{j\geq j_1} \subset D(m_1)$ and $z_{j_1} \in D(m_1) \setminus \cup_l D(m_1{}^\frown l)$. Again, since there can only be a finite number of points of $\{z_j\}_{j\geq j_1}$ in $D(m_1) \setminus \cup_l D(m_1{}^\frown l)$ there exist $j_2 > j_1$, and $m_2 = m_1{}^\frown l_1$ (with $l_1 \in \mathbb{N}$) so that $\{z_j\}_{j\geq j_2} \subset D(m_2)$ and $z_{j_2} \in D(m_2) \setminus \cup_l D(m_2{}^\frown l)$. So, by induction, we get a subsequence $\{z_{j_k}\}_{k\geq 1}$ with $z_{j_k} \in D(m_k) \setminus \cup_l D(m_k{}^\frown l)$ where $m_k = m_{k-1}{}^\frown l_{k-1}$, that is, we have found a sequence in c).

To prove the converse implication of the proposition, take $a \in A$ so that $a = \cap_\nu \mathsf{S}_{\nu|_p}$ for some $\nu \in \mathbb{N}^{\mathbb{N}}$. Therefore $a = \lim_{p\to\infty} a_{\nu|p}$. Write $\boldsymbol{a}_p := a_{\nu|_p}$.

Let γ be a continuous curve in $\mathbb{R}^{n-1} \times \{0\}$ such that for every $p \in \mathbb{N}$ there exists $r > 0$ with $\gamma \cap \{|z| > r\} \subset D(\nu|_p)$. For any $z \in \gamma \cap \{|z| > r\}$, such that $z \in D(\nu|_{p+1}) \setminus D(\nu|_{p+2})$ then $|S(z)| \leq \sqrt{2}$ and by condition (9), $H(z) \leq \delta_p$. Therefore $|f_0(z)| \leq \sqrt{2}\delta_p < \delta_{p-1}$ and by the definition of f, $f(z) = \Psi_p(f_0(z)) \in B\left(\boldsymbol{a}_{p-1}, \delta_{p-1}\right)$. Thus as $z \to \infty$, $f(z) \to a$. □

In the case $\operatorname{diam} A > 1$, write $A = \cup_{j\geq 1} A_j$, as (at most) a countable union of analytic sets, each of them satisfying $\operatorname{diam} A_j \leq 1$, for all $j \geq 1$. For each $j \geq 1$, construct a domain D_j congruent by a rigid motion to D_0, where the asymptotic values in $A_j \cup \{0\}$ will be attained. Construct a sequence $\{\delta_p^j\}_{p\geq 1}$ as in section 4 but replacing in the initial step $\operatorname{diam} A$ by $\operatorname{diam}(A_j \cup \{0\})$ and taking $\delta_1^j = \min\{1/2, \operatorname{diam}(A_j \cup \{0\})\}$. Each set D_j will be placed in $\{|z| > 2\delta_0^j\}$, far apart from each other so that if $r \geq 0$ only a finite number of them and a finite number of subsets $D_j(m)$, $m \in \mathcal{N}_0$, $j \geq 1$ intersect $B(0, r)$ but in a way that they are all eventually exhausted. Since the composition in section 5 act in disjoint regions the proof of Lemmas 5 and 6 and of Proposition 1 can be applied.

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