Conformal images of Borel sets

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Abstract

For any holomorphic map in the unit disk, the set of radial limits at a Borel set on the unit circle is a Suslin-analytic set. Here it is proved that, for a conformal map, this set is, in fact, Borel. As a consequence the sets of accessible boundary points, of cut points and of transition points are Borel. In addition, it is shown that the set of end points is a G_δ-set.

§1. **Introduction**

Although Borel sets are well-behaved under most operations performed in analysis, Suslin observed that the continuous image of a Borel set need not be Borel. He characterized sets obtained as continuous images of Borel sets as *analytic sets* and showed that the class of analytic sets is closed under continuous (or Borel measurable) functions (see for example [15], Theorem 73, p.145). Suslin also noted that every Borel set is an analytic set and, moreover, that Borel sets can be characterized as those analytic sets whose complement is analytic ([15], [1]). He and Lusin furthermore proved that injective Borel functions do preserve Borel sets. More precisely, ([6], Theorem 15.1)

Theorem (Lusin-Suslin) *Let B be a Borel set in $\mathbb{C}$ and let $f : B \to \mathbb{C}$ be Borel measurable and injective. Then $f(B)$ is also a Borel set.*

Let the function f be holomorphic in the unit disk $\mathbb{D}$. We denote by E_f the set of points on the unit circle $\mathbb{T}$ at which f has a radial limit, namely

$$E_f = \{\zeta \in \mathbb{T} : f(\zeta) := \lim_{r\to 1} f(r\zeta) \in \hat{\mathbb{C}} \;\text{ exists}\}.$$

It is known that $f|_{E_f}$ is a Borel function and moreover that E_f is a Borel set (see, e.g. Proposition 6.5 in [12]). Therefore, if $A \subset E_f$ is analytic then $f(A)$ is analytic, and the conclusion cannot be strengthened even if A is assumed to be Borel.

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McMillan [11] showed that, in general, if A is the set of asymptotic values of a function continuous in the disc, then A is analytic. Moreover, Berman and Nishiura [2], proved that, for any nowhere dense perfect subset $B \subset \mathbb{T}$ and any analytic set $A \subset \hat{\mathbb{C}}$, there is a holomorphic function in $\mathbb{D}$ such that $B \subset E_f$ and $f(B) = A$.

The set of *asymptotic values* of a holomorphic function in $\mathbb{D}$ is also an analytic set (see [9]). Here, an asymptotic value of f at the point $\zeta \in \mathbb{T}$ is a limit of f along a simple curve in $\mathbb{D}$ ending at ζ, in analogy with the radius ending at ζ. Ryan in [13], [14], characterized the set of asymptotic values of a holomorphic function in $\mathbb{D}$ as belonging to a certain subclass of the class of analytic sets.

Heins, in [5], showed that every infinite analytic set that contains the point at infinity is the set of asymptotic values of some entire function of infinite order. This is also the case for meromorphic functions defined in $\mathbb{D}$ (see [7]) and for meromorphic functions in $\mathbb{C}$ of any order ρ, $0 \leq \rho < \infty$ (see [3]).

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§2. **The main result**

Consider a conformal map in $\mathbb{D}$ onto a domain $G \subset \hat{\mathbb{C}}$. We show now that, in contrast, the set of the radial limits remains Borel if f is injective.

Theorem 1 *Let f be a conformal map in $\mathbb{D}$ onto $G \subset \hat{\mathbb{C}}$ and let $A \subset \mathbb{T}$ be such that*

$$f(\zeta) = \lim_{r \to 1} f(r\zeta) \in \hat{\mathbb{C}} \quad \text{exists for} \quad \zeta \in A.$$

Then if A is a Borel set, so is $f(A)$.

Proof. Without loss of generality we can assume that $f(0) = \infty$ and therefore, $\partial G \subset \mathbb{C}$. Consider the sets

$$A_1 = \{\zeta \in A : f(\xi) \neq f(\zeta) \quad \text{for} \quad \xi \in A, \xi \neq \zeta\}$$

$$A_2 = \{\zeta \in A : f(\xi) = f(\zeta) \quad \text{for a unique} \quad \xi \in A, \xi \neq \zeta\}$$

$$A_3 = \{\zeta \in A : f(\xi) = f(\zeta) \text{ for two or more } \xi \in A, \xi \neq \zeta\}.$$

It is known that the set $f(A_3)$ is countable and since $f|_A$ is Borel-measurable, A_3 is a Borel set (see Propositions 2.19 and 6.5 [12]).

To show that $f(A_1 \cup A_2)$ is Borel, we shall decompose $A_1 \cup A_2$ into a finite collection of disjoint sets in such a way that the restrictions of f to these sets is 1-1, and then get the desired conclusion by means of the Lusin-Suslin theorem.

We write $A_1 \cup A_2 = L$ and define a function $g : L \times L \to \mathbb{C}$ by

$$g(\zeta, \xi) = f(\zeta) - f(\xi)$$

and note that L, $L \times L$ and $g^{-1}(\{0\}) = \{(\zeta, \xi) \in L \times L : g(\zeta, \xi) = 0\}$ are Borel. Indeed, $L = A \setminus A_3$ is Borel, consequently so is $L \times L$ (see Proposition 3.1.23 in [16]) and

$$g^{-1}(\{0\}) = \bigcap_n \bigcup_{m>n} V_{nm},$$

where $V_{nm} := \{(\zeta, \xi) \in L \times L : |f((1 - \frac{1}{m})\zeta) - f((1 - \frac{1}{m})\xi)| < 1/n\}$ is an open set relative to $L \times L$ and thus Borel.

Therefore,

$$Z^+ := \{(\zeta, \xi) \in L \times L : \arg \zeta > \arg \xi,\ g(\zeta, \xi) = 0\}$$
$$Z^- := \{(\zeta, \xi) \in L \times L : \arg \zeta < \arg \xi,\ g(\zeta, \xi) = 0\}$$

are also Borel, where we consider the branch of the argument that takes values in $[0, 2\pi)$.

Define $A_2^\pm$ as the projection of $Z^\pm$, that is,

$$A_2^\pm := \{\zeta \in L : \text{ there exists } \xi \in L, \xi \neq \zeta \text{ such that } (\zeta, \xi) \in Z^\pm\}.$$

The projection of Z^+ onto A_2^+ is continuous and injective. For if not, there would exist $\xi, \xi' \in L$ such that, $\arg \xi, \arg \xi' < \arg \zeta$ and $(\zeta, \xi), (\zeta, \xi') \in Z^+$, where $\xi \neq \xi'$. Then $f(\zeta) = f(\xi) = f(\xi')$ and therefore, $\zeta \notin L$.

A_2^+ is a 1-1 continuous image of a Borel set, and by the Lusin-Suslin theorem, a Borel set. Similarly A_2^- is Borel, and hence so are $A_2 = A_2^+ \cup A_2^-$ and $A_1 = L \setminus A_2$.

Since $f|_{A_2^+}$ is a Borel function, we are left to show that $f|_{A_2^+}$ is injective. Assume otherwise, then there exist $\eta, \zeta \in A_2^+$ such that, $\arg \eta < \arg \zeta$ and $f(\eta) = f(\zeta)$ and therefore $g(\eta, \zeta) = 0$. Since $\eta \in A_2^+$ there exists another point $\xi \in L$ such that $\arg \xi < \arg \eta$ and $f(\xi) = f(\eta)$. So there are points, $\xi \neq \eta \neq \zeta \neq \xi$ in L with $f(\xi) = f(\eta) = f(\zeta)$, a contradiction.

We have shown that $f|_{A_1}, f|_{A_2^+}$ and $f|_{A_2^-}$ are Borel and injective, and again, by the Lusin-Suslin theorem $f(A_1), f(A_2^+)$ and $f(A_2^-)$ are Borel. Thus $f(A)$ is Borel. $\square$

§3. **Some applications**

It is known that for a general continuum in $\mathbb{C}$ the sets of accessible and cut points are Borel (see [10], also [8], p.176 and [17], p.52 Thm. 5.2, respectively). In the particular case of the continuum being the boundary of a simply connected domain, these results are consequences of Theorem 1. Also, we will show that the set of end points is a G_δ-set.

Let $f : \mathbb{D} \to G$ be a conformal map and recall that E_f denotes the set of points on $\mathbb{T}$ for which the radial limit of f exists.

A point $w \in \partial G$ is *accessible* if there exists a Jordan arc Γ such that $\Gamma \subset G \cup \{w\}$.

Corollary 1 *The set of accessible points is a Borel set.*

Proof. The point $w \in \partial G$ is accessible if and only if $w = f(\zeta)$ for some $\zeta \in E_f$ (see [12], Cor. 2.17). Since E_f is a Borel set, the set of accessible points is also Borel. $\square$

A point $w \in \partial G$ is called a *cut point* if $\partial G \setminus \{w\}$ is not connected.

Corollary 2 *The set of cut points is Borel.*

Proof. As a consequence of the Plane Separation Theorem ([17], p.108), w is a cut point if and only if there exist $\zeta, \xi \in E_f$, $\zeta \neq \xi$ with $f(\zeta) = f(\xi) = w$. Let E_1, E_2 and E_3 be subsets of E_f defined as in the proof of the Theorem 1. Then each E_j $(j = 1, 2, 3)$ is Borel, and therefore so is the set of cut points, $f(E_2 \cup E_3)$. $\square$

A point $w \in \partial G$ is a *transition point* with respect to the component $H \subset \hat{\mathbb{C}} \setminus \partial G$, where $H \neq G$, if there exist Jordan arcs $\Gamma \subset G \cup \{w\}$ and $\Gamma' \subset H \cup \{w\}$ (see [4]).

Corollary 3 *The set of transition points is Borel.*

Proof. Let $H_k,\ k = 1, 2, \ldots$ be the components of $\hat{\mathbb{C}} \setminus \partial G$ such that $H_k \neq G$. Since G is a domain, the domains H_k are simply connected domains. Therefore, by Corollary 1, the intersection of the set of points accessible from H_k with the set of points accesible from G is Borel. Consequently, the set of transition points, being the union of these intersections, is also Borel. □

A point $w \in \partial G$ is called an *end point* if w lies within Jordan curves J_n contained in G that intersect ∂G exactly once and satisfy that $\mathrm{diam} J_n \to 0$ as $n \to \infty$.

Propostion *The set of end points is a G_δ-set.*

Proof. Let T be the set of end points on ∂G. Denote by $\mathrm{Int}\,(J_n(w))$ the open domain that contains w whose boundary is J_n.

Define

$$C := \bigcap_{n \geq 1} \bigcup_{w \in T} \mathrm{Int}\,(J_n(w))\,,$$

where $\mathrm{diam}\,(\mathrm{Int}\,(J_n(w))) < 1/n$. Clearly C is G_δ-set and $T \subset C$.

Next it will be shown that $C \subset T$. Indeed, if $w \in C$, then for every $n \in \mathbb{N}$ there exists $w_n \in T$ so that $w \in \mathrm{Int}\,(J_n(w_n))$ and since $\mathrm{diam}\,(\mathrm{Int}\,(J_n(w))) < 1/n$, $w_n \to w$ as $n \to \infty$. Moreover, $w \in \partial G$ because ∂G is closed.

Thus, in brief, w is a point in ∂G so that $w \in \mathrm{Int}\,(J_n(w_n))$, $J_n(w_n)$ intersects ∂G once and $\mathrm{diam}\,(\mathrm{Int}\,(J_n(w_n))) < 1/n$. That is, $w \in T$. □

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