GROMOV HYPERBOLICITY OF PLANAR GRAPHS

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Abstract. This paper shows that under appropriate assumptions adding or removing an infinite amount of edges to a given planar graph preserves its non-hyperbolicity, a result which is shown to be false in general. The authors consider the conjecture which states that every tessellation graph of $\mathbb{R}^2$ with convex tiles is non-hyperbolic; it is shown that in order to prove this conjecture it suffices to consider tessellations graphs of $\mathbb{R}^2$ such that every tile is a triangle and a partial answer to this question is given. A weaker version of this conjecture stating that every tessellation graph of $\mathbb{R}^2$ with rectangular tiles is non-hyperbolic is stated and partially answered. If this conjecture were true, many tessellation graphs of $\mathbb{R}^2$ with tiles which are parallelograms would be non-hyperbolic.

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1. Introduction.

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces. The concept of Gromov hyperbolicity grasps the essence of both negatively curved spaces like the classical hyperbolic space or Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 18, 19]).

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [19] and the references therein), where its practical importance was discussed. This theory was mainly applied to the study of automatic groups (see [33]), which appear in computational science. The concept of hyperbolicity appears also in discrete mathematics, in particular, a few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [14, 15, 17, 30]). Another application of these spaces is secure transmission of information on the internet (see [23, 24, 25]), playing a significant role in the spread of viruses through the network (see [24, 25]). Hyperbolicity is also useful in the study of DNA data (see [7]). It has been shown empirically in [45] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension.

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [3, 4, 5, 7, 8, 10, 16, 23, 24, 25, 26, 27, 29, 31, 32, 35, 36, 37, 38, 42, 43, 44, 46, 47].

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [20]). The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of [2, 6, 21, 22, 38, 39, 40, 43, 44]. In particular, in [38, 43, 44, 46] it is proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a very simple graph. Deciding whether a space is hyperbolic is a difficult problem since the location of geodesics is unknown, and hence, it is useful to know hyperbolicity criteria for graphs. This will be the topic of discussion in what follows.

One of the main questions in the study of any mathematical property is to find transformations which preserve that property. In [8, Theorem 3.15] the authors prove that adding or removing any finite amount of edges of a graph preserves its non-hyperbolicity (or hyperbolicity). It is thus natural to consider what would happen if the amount of edges were infinite. Theorem 3.1 below gives a positive answer to this question under some appropriate hypotheses for planar graphs; Theorem 3.6 shows that the general answer is negative, even for planar graphs.

The papers [9] and [37] study the hyperbolicity of some type of planar graphs. In particular, in [9], the authors conjectured that every tessellation graph of $\mathbb{R}^2$ with convex tiles is non-hyperbolic. Sections 4 and 5 deal with this open problem. Theorem 5.1 shows that in order to prove this conjecture, it suffices to consider tessellation graphs of $\mathbb{R}^2$ such that every tile is a triangle. A weaker conjecture is stated, namely that every tessellation graph of $\mathbb{R}^2$ with rectangular tiles is non-hyperbolic. Theorem 4.6 gives a partial answer to this question. Finally, Theorem 4.2 shows that if this weaker conjecture is true, then many tessellation graphs of $\mathbb{R}^2$ with tiles which are parallelograms are non-hyperbolic.

2. Background on Gromov hyperbolic spaces.

Let (X, d) be a metric space and let $\gamma : [a, b] \longrightarrow X$ be a continuous function. The curve γ is a *geodesic* if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where L denotes the length of a curve; a *geodesic line* is a geodesic with domain $\mathbb{R}$, and a *geodesic ray* is a geodesic with domain $[0, \infty)$. X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; denote by $[xy]$ any of such geodesics (since uniqueness of geodesics is not required, this notation is ambiguous, but convenient). It is clear that every geodesic metric space is path-connected. If the metric space X is a graph, $[u, v]$ denotes the edge joining the vertices u and v.

In order to consider a graph G as a geodesic metric space, one must identify any edge $[u, v] \in E(G)$ with the real interval $[0, l]$ (if $l := L([u, v])$); therefore, any point in the interior of any edge is a point of G and, if the edge $[u, v]$ is considered as a graph with just one edge, then it is isometric to $[0, l]$. A connected graph G is naturally equipped with a distance defined on its points, induced by taking shortest paths in G, inducing in G the structure of a metric graph. Note that edges can have arbitrary lengths.

Throughout the paper only simple, connected and locally finite graphs are considered (i.e., graphs without loops or multiple edges and so that each ball contains a finite number of edges); these properties guarantee graphs are geodesic metric spaces. The study of the hyperbolicity of graphs with loops and multiple edges can be reduced to the study of the hyperbolicity of simple graphs (see [4, Theorems 8 and 10]).

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon with sides $J_j \subseteq X$, then J is said to be *δ-thin* if for every $x \in J_i$ one has that $d(x, \cup_{j \neq i} J_j) \leq \delta$. The sharp thin constant of J, $\delta(J)$, is then $\delta(J) := \inf\{\delta \geq 0 : \ J \text{ is } \delta\text{-thin}\}$. If x_1, x_2, x_3 are points in X, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is *δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. Denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : \ T \text{ is a geodesic triangle in } \ X\}$. The space X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; in this case, $\delta(X) = \inf\{\delta \geq 0 : \ X \text{ is } \delta\text{-hyperbolic}\}$.

Trivially, every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. The real line $\mathbb{R}$ is 0-hyperbolic whereas the Euclidean plane $\mathbb{R}^2$ is not. In general, a normed vector space E is hyperbolic if and only if $\dim \ E = 1$. Every metric tree with arbitrary length edges is 0-hyperbolic; every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -c^2 < 0$ is hyperbolic. More background and further results are given in, *e.g*, [1, 18].

Those spaces X with $\delta(X) = 0$ are precisely the metric trees, and the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is.

There are several definitions of Gromov hyperbolicity, all equivalent in the sense that if X is δ-hyperbolic with respect to definition A, then it is δ'-hyperbolic with respect to definition B for some δ' (see, e.g., [1, 18]).

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometric embedding*, with constants $\alpha \geq 1, \ \beta \geq 0$ if for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function f is *ε-full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.

A map $f : X \longrightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha \geq 1, \ \beta, \varepsilon \geq 0$ such that f is an ε-full (α, β)-quasi-isometric embedding. In that case we say that X and Y are *quasi-isometric*.

Note that a quasi-isometric embedding, in general, is not continuous.

Let X be a metric space, Y a non-empty subset of X and ε a positive number. The *ε-neighborhood* of Y in X, denoted by $\mathcal{V}_\varepsilon(Y)$ is defined as the set $\{x \in X : d_X(x, Y) \leq \varepsilon\}$.

A fundamental property of hyperbolic spaces is the following:

Theorem 2.1 (Invariance of hyperbolicity)**.** *Let $f : X \longrightarrow Y$ be an (α, β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic.*

Besides, if f is ε-full for some $\varepsilon \ge 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic.

If D is a closed subset of X, the *inner metric* considered in D is defined as

$$d_D(z, w) := \inf \big\{ L_X(\gamma) : \gamma \subset D \text{ is a continuous curve joining } z \text{ and } w \big\} \ge d_X(z, w) .$$

Consequently, $L_D(\gamma) = L_X(\gamma)$ for every curve $\gamma \subset D$.

In an informal way, a tessellation, T, on a complete Riemannian surface, X, is a partition of X by geometric shapes (called tiles) with no overlaps and no gaps. The tessellation graph associated to T is the union of the boundaries of the tiles. More precisely, for $n \ge 1$, an n-cell is a topological space homeomorphic to the open ball in $\mathbb{R}^n$. A 0-cell is a singleton space. A *tesselation* on a complete Riemannian surface, X, is a CW 2-complex on X such that every point on X is contained in some n-cell of the complex for some $n \in \{0, 1, 2\}$. A *tessellation graph* is the 1-skeleton (the set of 0-cells and 1-cells). The edges (1-cells) of a tessellation graph are just rectifiable paths in X and have the length induced by the metric on X (these paths may or may not be geodesics in X). Throughout the paper $X = \mathbb{R}^2$ with the exceptions of Theorem 3.4 and the proof of Theorem 3.6, where X will stand for the hyperbolic plane.

Along the paper, given a set E contained in a Riemannian surface X, we denote by $A_X(E)$ its area and by $\overline{E}$ its closure.

3. Hyperbolicity of tessellation graphs.

If G_0 is a non-hyperbolic tessellation graph of $\mathbb{R}^2$, a natural question is whether this non-hyperbolicity will be preserved when adding to it any number (possibly infinite) of vertices and edges. The next result gives an affirmative answer to this question under some regularity hypotheses on G_0. Theorem 3.6 below will show this result to be false in general.

The next result shows the connection between the continuous and the discrete frame (see, e.g., [28]).

Theorem 3.1. *Let G_0 be the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}_{n\in I}$. Assume that there exists a partition $I = \Lambda_1 \cup \Lambda_2$ of the set of indices and positive constants c_1, c_2, verifying the following properties:*

(i) $\operatorname{diam}_{G_0} \partial F_n \le c_1$ and $A_{\mathbb{R}^2}(F_n) \ge c_2$ for every $n \in \Lambda_1$,

(ii) $d_{\partial F_n}(x, y) \le c_1 d_{\mathbb{R}^2}(x, y)$ for every $x, y \in \partial F_n$ and for every $n \in \Lambda_2$.

Then G_0 is not hyperbolic. Moreover, any 1-skeleton G of a tessellation of $\mathbb{R}^2$ which contains G_0 as a subgraph is not hyperbolic.

Proof. It will be proven that the inclusion $i : G_0 \longrightarrow \mathbb{R}^2$ is a quasi-isometric embedding; in fact, it is shown that

$$d_{\mathbb{R}^2}(x, y) \le d_{G_0}(x, y) \le \big(2c_1^2 c_2^{-1} + c_1\big) d_{\mathbb{R}^2}(x, y) + \pi c_1^3 c_2^{-1}, \tag{3.1}$$

for every $x, y \in G_0$.

First of all, it is clear that $d_{\mathbb{R}^2}(x, y) = d_{\mathbb{R}^2}(i(x), i(y)) \le d_{G_0}(x, y)$ for every $x, y \in G_0$.

Fix now $x, y \in G_0$ and let σ be the Euclidean segment joining x and y in $\mathbb{R}^2$. If $n \in \Lambda_1$ and $\overline{F_n} \cap \sigma \ne \varnothing$, then $F_n \subseteq \mathcal{V}_{\operatorname{diam} F_n}(\sigma) \subseteq \mathcal{V}_{c_1}(\sigma)$. Since $\mathcal{V}_{c_1}(\sigma)$ is the union of two half-disks and a rectangle, clearly $A_{\mathbb{R}^2}(\mathcal{V}_{c_1}(\sigma)) = 2c_1 L(\sigma) + \pi c_1^2$. Let $\mathcal{N}(\sigma)$ denote the number of $\overline{F_n}$ with $n \in \Lambda_1$ that cross σ, then

$$c_2\, \mathcal{N}(\sigma) \le A_{\mathbb{R}^2}(\mathcal{V}_{c_1}(\sigma)) = 2c_1 L(\sigma) + \pi c_1^2.$$

Therefore,

$$\mathcal{N}(\sigma) \le c_2^{-1}\big(2c_1 d_{\mathbb{R}^2}(x, y) + \pi c_1^2\big).$$

Consider σ as an oriented segment from x to y. A finite set of points will be inductively defined as follows: let y_1 be the first point on σ with $y_1 \in \cup_{n\in\Lambda_1} \overline{F_n}$; then $y_1 \in \overline{F_{r_1}}$ for some $r_1 \in \Lambda_1$; take y_2 to be the last point on $\sigma \cap \overline{F_{r_1}}$. Proceeding this way, assume that $\{y_1, \dots, y_{2j}\}$ have been defined with y_{2s-1} the first point and y_{2s} the last on $\sigma \cap \overline{F_{r_s}}$ for $s = 1, \dots, j$. If $\sigma \setminus [y_1 y_{2j}]$ does not intersect $\cup_{n\in\Lambda_1\setminus\{r_1,\dots,r_j\}} \overline{F_n}$, then this process is stopped. If $\sigma \setminus [y_1 y_{2j}]$ intersects $\cup_{n\in\Lambda_1\setminus\{r_1,\dots,r_j\}}\overline{F_n}$, then define y_{2j+1} as the first point in $(\sigma \setminus [y_1 y_{2j}]) \cup \{y_{2j}\}$ with $y_{2j+1} \in \cup_{n\in\Lambda_1\setminus\{r_1,\dots,r_j\}}\overline{F_n}$; then $y_{2j+1} \in \overline{F_{r_{j+1}}}$ for some $r_{j+1} \in \Lambda_1$ and define y_{2j+2} as the last point in $\sigma \cap \overline{F_{r_{j+1}}}$. Eventually his process will finish and a finite set $\{y_1, \dots, y_{2N}\}$ will be obtained.

Let $[x_1x_2], [x_3x_4], \dots, [x_{2m-1}x_{2m}]$ be the Euclidean segments contained in the closure of $\sigma \setminus \cup_{l=1}^N [y_{2l-1}y_{2l}]$. Notice that $[x_{2j-1}x_{2j}] \subset \sigma \cap (\cup_{n\in\Lambda_2}\overline{F_n}\,)$, and thus

$$\begin{aligned} d_{G_0}(x,y) &\le \sup_{n\in\Lambda_1}\{\operatorname{diam}_{G_0}\partial F_n\}\,\mathcal{N}(\sigma) + \sum_{j=1}^m d_{G_0}(x_{2j-1},x_{2j}) \\ &\le c_1\mathcal{N}(\sigma) + c_1\sum_{j=1}^m d_{\mathbb{R}^2}(x_{2j-1},x_{2j}) \\ &\le \big(2c_1^2c_2^{-1} + c_1\big) d_{\mathbb{R}^2}(x,y) + \pi c_1^3 c_2^{-1}, \end{aligned}$$

which completes the proof of equation (3.1).

We shall show next that G_0 is not hyperbolic. To this end it will be first proven that

$$\delta(G_0) \ge \frac{1}{2}\sup_n \operatorname{diam}_{\mathbb{R}^2}\partial F_n. \tag{3.2}$$

For any fixed n, let us consider the set A_n of closed curves in G_0 freely homotopic to ∂F_n in $\mathbb{R}^2 \setminus F_n$. Choose a closed curve $\sigma_n \in A_n$ with $L(\sigma_n) = \min\{L(\sigma) : \sigma \in A_n\}$; it is clear that $L(\sigma_n) \ge 2\operatorname{diam}_{\mathbb{R}^2}\partial F_n$, and that $d_{\sigma_n}(x,y) = d_{G_0}(x,y)$ for every $x,y \in \sigma_n$ since σ_n is a shortest curve in A_n. Let x_n, y_n be points in σ_n with $d_{G_0}(x_n,y_n) = d_{\sigma_n}(x_n,y_n) = L(\sigma_n)/2$. Then there are two different geodesics σ_n^1, σ_n^2 in G_0 joining x_n and y_n with $\sigma_n^1 \cup \sigma_n^2 = \sigma_n$. Therefore the set $B_n = \{\sigma_n^1, \sigma_n^2\}$ is a geodesic bigon (a geodesic triangle having two of its vertices to be the same point). If u_n is the midpoint of σ_n^1, then $\delta(B_n) \ge d_{G_0}(u_n, \sigma_n^2) = d_{G_0}(u_n, \{x_n, y_n\}) = \frac14 L(\sigma_n) \ge \frac12 \operatorname{diam}_{\mathbb{R}^2}\partial F_n$. Taking the supremum on n equation (3.2) above follows.

If $\sup_n \operatorname{diam}_{\mathbb{R}^2}\partial F_n = \infty$, by equation (3.2) G_0 is not hyperbolic. If $\sup_n \operatorname{diam}_{\mathbb{R}^2}\partial F_n = c_1^* < \infty$, then the inclusion $i : G_0 \longrightarrow \mathbb{R}^2$ is a c_1^*-full $(a_0, b_0/a_0)$-quasi-isometry, with $a_0 := 2c_1^2c_2^{-1} + c_1 > c_1 \ge 1$ and $b_0 := \pi c_1^3 c_2^{-1}$, since $\operatorname{diam}_{\mathbb{R}^2} F_n \le \operatorname{diam}_{G_0}\partial F_n$. In this case, Theorem 2.1 implies G_0 is not hyperbolic.

Let us finally show that any 1-skeleton G of a tessellation of $\mathbb{R}^2$ which contains G_0 as a subgraph will not be hyperbolic.

Clearly $d_G(x,y) \le d_{G_0}(x,y)$ for every $x,y \in G_0$, and also $d_{\mathbb{R}^2}(x,y) \le d_G(x,y)$ for every $x,y \in G$,. By equation (3.1),

$$d_G(x,y) \le d_{G_0}(x,y) \le a_0 d_{\mathbb{R}^2}(x,y) + b_0 \le a_0 d_G(x,y) + b_0$$

or, equivalently,

$$\frac{1}{a_0}\, d_{G_0}(x,y) - \frac{b_0}{a_0} \le d_G(x,y) \le d_{G_0}(x,y),$$

for every $x,y \in G_0$,. Thus the inclusion $i_0 : G_0 \longrightarrow G$ is an $(a_0, b_0/a_0)$-quasi-isometric embedding. Therefore, since G_0 is not hyperbolic, by Theorem 2.1 one obtains that G is not hyperbolic. □

The arguments just given in the proof of Theorem 3.1 have the following consequences:

Theorem 3.2. *Let G_0 be the 1-skeleton of a tessellation of $\mathbb{R}^2$ such that there exist non-negative constants a_0, b_0 so that $d_{G_0}(x,y) \le a_0 d_{\mathbb{R}^2}(x,y) + b_0$ for every $x,y \in G_0$. Then any 1-skeleton G of a tessellation of $\mathbb{R}^2$ which contains G_0 as a subgraph is not hyperbolic.*

Theorem 3.3. *Let G be the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$. Then*

$$\delta(G) \ge \frac{1}{2}\sup_n \operatorname{diam}_{\mathbb{R}^2}\partial F_n.$$

A direct consequence of Theorem 3.3 is that, if $\sup_n \operatorname{diam}_{\mathbb{R}^2} \partial F_n = \infty$, then G is not hyperbolic. It will be shown in Theorem 3.6 that if $\sup_n \operatorname{diam}_G \partial F_n = \infty$, this is false.

The following results on hyperbolicity will be needed in the proof of Theorem 3.6, one of the main results of this section:

Theorem 3.4. *([37, Theorem 3.1 and Remark 3.2]) Let G be the 1-skeleton of a tessellation of the hyperbolic plane $\mathbb{H}$ with tiles $\{F_n\}$. If for some positive constants c_1, c_2, one has $\operatorname{diam}_G \partial F_n \le c_1$ and $A_{\mathbb{H}}(F_n) \ge c_2$ for every n, then G is hyperbolic.*

Let us denote by $G \setminus \{v\}$ the metric space obtained by removing the point $\{v\}$ from the metric space G.

A vertex v of a graph G is a *cut vertex* if $G \setminus \{v\}$ is not connected. Note that in a tree, any vertex with degree greater than one is a a cut vertex.

Finally, let us denote by $\{G_r\}_r$ the closures in G of the connected components of the set

$$G \setminus \big\{ v \in V(G) : \; v \; \text{ is a cut vertex of } \; G \big\}.$$

The set $\{G_r\}_r$ is the *canonical T-decomposition* of G.

Example. Let us consider two cycle graphs Γ_1, Γ_2, and $x_1 \in V(\Gamma_1), x_2 \in V(\Gamma_2)$. Define the graph G as the graph with $V(G) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(G) = E(\Gamma_1) \cup E(\Gamma_2) \cup [x_1, x_2]$. Then $\{\Gamma_1, \Gamma_2, [x_1, x_2]\}$ is the canonical T-decomposition of G.

Theorem 3.5. *([4, Theorem 5]) If $\{G_r\}_r$ is the canonical T-decomposition of G, then $\delta(G) = \sup_r \delta(G_r)$.*

The next result will deal with periodic graphs. The tessellation graph G of $\mathbb{R}^2$ is *periodic* if there exist $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that $T(G) = G$, where $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined as $T(x, y) = (x, y) + (u, v)$.

Recall that a geodesic line is a geodesic with domain $\mathbb{R}$. By *Euclidean line* we mean an straight line in $\mathbb{R}^2$, i.e., a geodesic line in the Euclidean plane.

Theorem 3.6. *There exists a periodic hyperbolic 1-skeleton G of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$ verifying $\sup_n \operatorname{diam}_G \partial F_n = \infty$ and containing infinitely many Euclidean lines. Furthermore, there exists a periodic non-hyperbolic subgraph G_0 of G which is also a tessellation graph of $\mathbb{R}^2$.*

Remark 3.7. *The main idea in the construction of such a tessellation is to include in $\mathbb{R}^2$ a tessellation graph quasi-isometric to a periodic model of the hyperbolic plane. The example given in Theorem 3.6 shows that it is not possible to replace $\sup_n \operatorname{diam}_{\mathbb{R}^2} \partial F_n$ by $\sup_n \operatorname{diam}_G \partial F_n$ in Theorem 3.3. Theorem 4.6 shows a large class of non-hyperbolic tessellation graphs containing infinitely many Euclidean lines.*

Proof. The structure of the proof is as follows: first, a hyperbolic graph G_3, which is a tessellation of $\mathbb{H}$, will be defined; based on G_3, define a new hyperbolic graph G_6 which is a tessellation of $\mathbb{R}^2$; finally, the graph G satisfying all conditions in the statement will be defined from G_6.

Let us consider the hyperbolic plane $\mathbb{H}$ with its Fermi coordinates (see, e.g., [11, p. 247]), i.e., the plane $\mathbb{R}^2$ with the Riemannian metric $ds^2 = \cosh^2\! y \, dx^2 + dy^2$ (thus $dA = \cosh y \, dx \, dy$).

Let $[t]$ stand for the integer part of t. Consider the segments $I_{m,n}$ in $\mathbb{H}$ given by $I_{m,n} := \{(x, y) \in \mathbb{H} : n/[\cosh m] \le x \le (n+1)/[\cosh m], y = m\}$ for $m = 0, 1, 2, \dots$ and $0 \le n \le [\cosh m] - 1$, and $J_{m,n} := \{(x, y) \in \mathbb{H} : x = n/[\cosh m], m \le y \le m + 1\}$ for $m \ge 0$ and $0 \le n \le [\cosh m]$.

Let $G_1 := \cup_{m,n} \big\{ I_{m,n} \cup J_{m,n} \big\}$, $S_0(x, y) := (x, -y)$ and $G_2 := G_1 \cup S_0(G_1)$. Also, let $R_k(x, y) := (x + k, y)$ for $k \in \mathbb{Z}$. The graph G_3 is now defined as $G_3 := \cup_k R_k(G_2)$. Clearly, G_3 is a tessellation graph of $\mathbb{H}$. Let us check that it verifies the hypotheses in Theorem 3.4.

To this end, let $\{F_r^*\}$ be the tiles of the tessellation G_3. Since S_0 and R_k are isometries of $\mathbb{H}$,

$$\operatorname{diam}_{G_3} \partial F_r^* \le L_{\mathbb{H}}(\partial F_r^*).$$

A standard computation gives

$$L_{\mathbb{H}}(I_{m,n}) = \int_{n/[\cosh m]}^{(n+1)/[\cosh m]} \cosh m \, dx = \frac{\cosh m}{[\cosh m]} \le 2, \qquad L_{\mathbb{H}}(J_{m,n}) = \int_m^{m+1} dy = 1,$$

$$\int_{n/[\cosh m]}^{(n+1)/[\cosh m]} \cosh(m+1)\, dx = \frac{\cosh(m+1)}{\cosh m} \cdot \frac{\cosh m}{[\cosh m]} \le \frac{e^{m+1}}{e^m/2} \cdot 2 = 4e,$$

$$A_{\mathbb{H}}\big(\{(x,y) \in \mathbb{H} : n/[\cosh m] \le x \le (n+1)/[\cosh m], m \le y \le m+1\}\big) =$$

$$= \int_{n/[\cosh m]}^{(n+1)/[\cosh m]} \int_m^{m+1} \cosh y \, dy\, dx \ge \int_{n/[\cosh m]}^{(n+1)/[\cosh m]} \cosh m \, dx = \frac{\cosh m}{[\cosh m]} \ge 1.$$

Therefore,

$$\operatorname{diam}_{G_3} \partial F_r^* \le L_{\mathbb{H}}(\partial F_r^*) \le 4e+4, \qquad A_{\mathbb{H}}(F_r^*) \ge 1,$$

for every r, and Theorem 3.4 allows to conclude that G_3 is hyperbolic.

Consider now the graph G_3 embedded in the Euclidean plane $\mathbb{R}^2$. Let us define $K_{0,0} := I_{0,0}$; for $m \ge 1$ and $0 \le n \le [\cosh m] - 1$, let $K_{m,n}$ be a polygonal curve joining the endpoints of $I_{m,n}$ which is contained in the rectangle $\{(x,y) \in \mathbb{R}^2 : n/[\cosh m] \le x \le (n+1)/[\cosh m], m - 1/6 \le y \le m + 1/6\}$, where $L_{\mathbb{R}^2}(K_{m,n}) = L_{\mathbb{H}}(I_{m,n})$ and

$$\bigcup_{n=0}^{[\cosh(m-1)]-1} (n/[\cosh(m-1)], m) \subset \bigcup_{n=0}^{[\cosh m]-1} K_{m,n}.$$

Set $G_4 := \cup_{m,n}\{K_{m,n} \cup J_{m,n}\}$, $G_5 := G_4 \cup S_0(G_4)$ and define the next graph G_6 as $G_6 := \cup_k R_k(G_5)$. Clearly G_6 is a tessellation graph of $\mathbb{R}^2$. Since the graphs G_3 (in $\mathbb{H}$) and G_6 (in $\mathbb{R}^2$) are isometric, the graph G_6 is hyperbolic.

Finally, let us define G. For $m \ge 0$, $0 \le n \le [\cosh m] - 1$ and $0 \le s \le m$, let $M_{m,n,s}$ be the cycle graph which is the union of the four Euclidean segments joining the points

$$\Big(\frac{n + s/(4m+4)}{[\cosh m]}, m + \frac{1}{2}\Big), \quad \Big(\frac{n + (2s+1)/(8m+8)}{[\cosh m]}, m + \frac{2}{3}\Big),$$
$$\Big(\frac{n + (s+1)/(4m+4)}{[\cosh m]}, m + \frac{1}{2}\Big), \quad \Big(\frac{n + (2s+1)/(8m+8)}{[\cosh m]}, m + \frac{1}{3}\Big).$$

Set $G_7 := \cup_{m,n,s} M_{m,n,s}$, $G_8 := G_7 \cup S_0(G_7)$, $G_9 := \cup_k R_k(G_8)$ and $G := G_6 \cup G_9$. Clearly G is a tessellation graph of $\mathbb{R}^2$. Note that the sets $M_{m,n,s}$, its images by S_0 and R_k, and G_6, are the canonical T-decomposition of G; hence, Theorem 3.5 gives that $\delta(G) = \max\{\delta(G_6), \sup_{m,n,s} \delta(M_{m,n,s})\}$. One can check that $\delta(M_{m,n,s}) = L_{\mathbb{R}^2}(M_{m,n,s})/4 \le 1$; since $\delta(G) \le \max\{\delta(G_6), 1\} < \infty$, the graph G is hyperbolic.

Let us check the condition on the tiles of this graph. Denote by $\{F_r\}$ the tiles of G; if ∂F_r contains $M_{m,n,0}, M_{m,n,1}, \dots, M_{m,n,m}$, then

$$\operatorname{diam}_G \partial F_r \ge \frac{1}{2} \sum_{s=0}^{m} L_{\mathbb{R}^2}(M_{m,n,s}) \ge \frac{1}{2}(m+1)\frac{2}{3} = \frac{m+1}{3},$$

and one concludes that $\sup_r \operatorname{diam}_G \partial F_r = \infty$.

Furthermore, the graph G is periodic and contains infinitely many Euclidean lines by construction.

Finally, let us construct a periodic non-hyperbolic subgraph G_0 of G which is also a tessellation graph of $\mathbb{R}^2$. Let us define $K_m := \cup_{n=0}^{[\cosh m]-1} K_{m,n}$ $G_{10} := \cup_m \{K_m \cup J_{m,0} \cup J_{m,[\cosh m]}\}$, $G_{11} := G_{10} \cup S_0(G_{10})$ and $G_0 := \cup_k R_k(G_{11})$. It is clear that G_0 is a tessellation graph of $\mathbb{R}^2$ and a subgraph of G. For each $m \ge 0$, consider the midpoint p_m of K_m, i.e., the point with $d_{G_0}(p_m, (0,m)) = d_{G_0}(p_m, (1,m)) = L_{\mathbb{R}^2}(K_m)/2$, and the geodesic bigon B_m in G_0 with two different geodesics γ_m^1, γ_m^2, joining p_m and p_{m+1}; then $\gamma_m^1 \cup \gamma_m^2 =$

$K_m \cup K_{m+1} \cup J_{m,0} \cup J_{m,[\cosh m]}$. If q_m is the midpoint of γ_m^1, then

$$\begin{aligned}\delta(B_m) &\geq d_{G_0}(q_m, \gamma_m^2) = d_{G_0}(q_m, \{p_m, p_{m+1}\}) = \frac{1}{2}\, L_{\mathbb{R}^2}(\gamma_m^1)\\ &= \frac{1}{2}\Big(\frac{1}{2}\, L_{\mathbb{R}^2}(K_m) + L_{\mathbb{R}^2}(J_{m,0}) + \frac{1}{2}\, L_{\mathbb{R}^2}(K_{m+1})\Big)\\ &= \frac{1}{2}\Big(\frac{1}{2}\, \frac{\cosh m}{[\cosh m]}\, [\cosh m] + 1 + \frac{1}{2}\, \frac{\cosh(m+1)}{[\cosh(m+1)]}\, [\cosh(m+1)]\Big)\\ &= \frac{1}{4}\big(\cosh m + \cosh(m+1) + 2\big),\end{aligned}$$

and one concludes $\delta(G_0) \geq \sup_m \delta(B_m) = \infty$. □

A corollary for 2-quasiperiodic graphs follows. Recall that the tessellation graph G of $\mathbb{R}^2$ is 2-*periodic* if there exist two linearly independent vectors $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$ such that

$$T_j(G) = G\,, \qquad \text{for } j = 1, 2\,,$$

where $T_j : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are defined as

$$T_j(x, y) = (x, y) + (u_j, v_j)\,, \qquad j = 1, 2\,.$$

The graph G is 2-*quasiperiodic* if there exists a 2-periodic subgraph G_0 of G.

Corollary 3.8. *If G is 2-quasiperiodic then G is not hyperbolic*

Proof. If G_0 is a 2-periodic subgraph with tiles $\{F_n\}_{n\in I}$, then one can take the partition $\Lambda_1 = I$, $\Lambda_2 = \varnothing$ of the set of indices in the statement of Theorem 3.1. Therefore, G is not hyperbolic. □

4. Tessellations with parallelograms and rectangles

In this section it is shown that the hyperbolicity of certain tessellations with parallelograms is equivalent to the hyperbolicity of tessellations with rectangles. It is also shown that under some hypotheses rectangular tessellations are not hyperbolic.

4.1. **Tessellations with parallelograms.** Next it will be shown that considering tessellations of parallelograms with bounded inclinations is equivalent to considering rectangular tessellations with sides parallel to the axis in order to study hyperbolicity.

Consider the standard basis in $\mathbb{R}^2$ defined by $\{\overrightarrow{e_1}, \overrightarrow{e_2}\}$ and, given α, β, let $\overrightarrow{U_\alpha}$ and $\overrightarrow{V_\beta}$ be the vectors defined by

$$\overrightarrow{U_\alpha} := (\cos\alpha, \sin\alpha), \qquad \overrightarrow{V_\beta} := (\sin\beta, \cos\beta).$$

Fix real numbers a and b with $a < b < \pi + a$. A tessellation $\mathcal{T}$ is a p-tessellation of $\mathbb{R}^2$ if its tiles satisfy the following conditions:

(1) $\mathcal{F}$ is a parallelogram, for all $\mathcal{F} \in \mathcal{T}$.
(2) For each $\mathcal{F} \in \mathcal{T}$ there exists a pair of angles α, β satisfying that $\alpha \in (a, b)$, $\beta \in (-\frac{\pi}{2} - a, \frac{\pi}{2} - b)$, such that the sides of $\mathcal{F}$ are parallel to $\overrightarrow{U_\alpha}$ and $\overrightarrow{V_\beta}$ respectively.

Notice that the second condition above implies that if two adjacent tiles in $\mathcal{T}$ partially share a side, then (for both of them) there is a side that is either parallel to $\overrightarrow{U_\alpha}$ (for some α) or to $\overrightarrow{V_\beta}$ (for some β).

Theorem 4.1. *Given a p-tessellation $\mathcal{T}$ of $\mathbb{R}^2$, there exists a tessellation T of $\mathbb{R}^2$ with rectangular tiles and a bijective continuous function $f : \mathcal{T} \longrightarrow T$ so that $f|_{\mathcal{G}}$ is an isometry from the 1-skeleton $\mathcal{G}$ of $\mathcal{T}$ to the 1-skeleton G of T.*

Proof. Applying a rotation, without loss of generality, $\alpha \in (-c, c)$ where $c = (b-a)/2$. Then the vectors $\overrightarrow{U_\alpha}$ and $\overrightarrow{V_\beta}$ give, respectively, the "almost-horizontal" and "almost-vertical" directions of a tile. In fact, if P is a vertex of a tile, $\mathcal{F} \in \mathcal{T}$, the lines with directions $\overrightarrow{U_c}$ and $\overrightarrow{U_{-c}}$ through P divide the plane in four sectors. Since the sides of every $\mathcal{F} \in \mathcal{T}$ are parallel to $\overrightarrow{U_\alpha}$ and $\overrightarrow{V_\beta}$ with $\alpha \in (-c, c)$ and $\beta \in (-\frac{\pi}{2} + c, \frac{\pi}{2} - c)$, no more than four tiles can share a vertex and therefore $\mathcal{T}$ has the structure of a rectangular tessellation.

For a given $\overrightarrow{U_\alpha}$ and $\overrightarrow{V_\beta}$, let

$$S(\alpha, \beta) = \big\{\overline{\mathcal{F}} : \mathcal{F} \in \mathcal{T} \text{ is a parallelogram of angles } \alpha, \beta\big\}.$$

The tessellation $\mathcal{T}$ induces a "partition" of $\mathbb{R}^2$, $\mathcal{S}$, given by the connected components of $S(\alpha, \beta)$, with $\alpha \in (-c, c)$, $\beta \in (-\frac{\pi}{2} + c, \frac{\pi}{2} - c)$. If $B \neq \emptyset$ is a connected component of $S(\alpha, \beta)$ then B is a union of closures of tiles of $\mathcal{T}$. If $B = \mathbb{R}^2$, then there exists α and β so that all the tiles in $\mathcal{T}$ have sides parallel to $\overrightarrow{U_\alpha}$ and $\overrightarrow{V_\beta}$. In this situation define $f = f_{\alpha\beta}$ where $f_{\alpha\beta} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the linear map such that $f_{\alpha\beta}(\overrightarrow{U_\alpha}) = \overrightarrow{e_1}$ and $f_{\alpha\beta}(\overrightarrow{V_\beta}) = \overrightarrow{e_2}$. Clearly f is an isometry from the 1-skeleton $\mathcal{G}$ of $\mathcal{T}$ to the 1-skeleton G of T.

In what follows it is assumed that $B \neq \mathbb{R}^2$. Since each $B \neq \emptyset$ is the union of parallelograms whose sides have a fixed inclination, then its boundary components are polygonal lines with two possible angles. Moreover, B is a convex set and therefore it is either a parallelogram (if B bounded) or otherwise, it is a generalized parallelogram with a side at infinity, that is, a half-strip (if there is one side at infinity), a strip or a sector (if there are two) or half-plane (if there are three). Indeed, if it is not convex, there is a tile $\mathcal{F} \in \mathcal{T}$, $\overline{\mathcal{F}} \notin B$, that shares two sides with B and therefore $\mathcal{F}$ is a parallelogram with sides parallel to those of B, thus $\overline{\mathcal{F}} \in B$. The same argument implies that if $B' \neq B$ are two connected components of $S(\alpha, \beta)$ and $S(\alpha', \beta')$ such that $B \cap B' \neq \emptyset$, then B and B' share a whole side or a vertex.

The function $f : \mathcal{T} \longrightarrow T$ will be a piecewise linear function defined on the sets B inductively and so that if $B \in S(\alpha, \beta)$, then

$$f(x) - f(y) = f_{\alpha\beta}(x) - f_{\alpha\beta}(y) = f_{\alpha\beta}(x - y), \qquad \text{for all } x, y \in B, \tag{4.3}$$

where $f_{\alpha\beta} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the linear map such that $f_{\alpha\beta}(\overrightarrow{U_\alpha}) = \overrightarrow{e_1}$ and $f_{\alpha\beta}(\overrightarrow{V_\beta}) = \overrightarrow{e_2}$.

To start the induction, let $O = (0, 0)$ and define $f(O) = O$. Denote by B_O one of the sets which contains O, and let α_0 and β_0 be so that $B_O \in S(\alpha_0, \beta_0)$. If $x \in B_O$ then

$$f(x) := f_{\alpha_0\beta_0}(x) = f(O) + f_{\alpha_0\beta_0}(x - O).$$

Notice that, for all $x, y \in B_O$, relation (4.3) trivially holds by the linearity of $f_{\alpha_0\beta_0}$. Let $C_0 := B_O$. Assume now that f is defined and continuous on a connected set C_n which is a finite union of blocks $B \in \mathcal{S}$ defined as $C_n := \{B \in \mathcal{S} : B \cap C_{n-1} \neq \emptyset\}$ and that (4.3) holds for every set $B \in C_n$. Extend f from C_n onto $C_{n+1} = \{B \in \mathcal{S} : B \cap C_n \neq \emptyset\}$, in the following way: for $B \in C_{n+1} \setminus C_n$, $B \in S(\alpha, \beta)$, take any point $P \in \partial B \cap C_n$, and define

$$f(x) := f(P) + f_{\alpha\beta}(x - P), \quad x \in B.$$

Notice that (4.3) holds for points $x, y \in B$ by the linearity of $f_{\alpha\beta}$. We are left to show that the extension is well defined. Indeed, since no more than four tiles of $\mathcal{T}$ can meet at a vertex, and since different B's share a whole side, at each vertex exactly four different sets $B \in \mathcal{S}$ meet. The function f straightens the sides of each B and places it adjacent to the images of its neighbors. Concretely, if $B \in C_{n+1} \setminus C_n$ and $x \in \partial B \cap C_n$ then considering x as a point on $B \in C_{n+1} \setminus C_n$,

$$f_B(x) = f(P) + f_{\alpha\beta}(x - P),$$

for a point $P \in \partial B \cap C_n$ where $f(P)$ was already defined. If $x = P$ there is nothing to prove. If $x \neq P$, then there exists $B' \in C_n$ such that $x \in \partial B'$, thus B and B' share a side the one with P and x. By (4.3)

$$f_{B'}(x) = f(P) + f_{\alpha'\beta'}(x - P).$$

Since both $x, P \in \partial B' \cap \partial B$ then, $f_{\alpha\beta}(x - P) = f_{\alpha'\beta'}(x - P)$ and therefore f is well defined on $B \cap C_n$. To see that is well defined on C_{n+1}, consider now $B, B' \in C_{n+1} \setminus C_n$ so that $B \cap B' \neq \emptyset$. Then, there exists a point $Q \in B \cap B' \cap C_n$ and by (4.3)

$$f_B(x) = f(Q) + f_{\alpha\beta}(x - Q); \qquad f_{B'}(x) = f(Q) + f_{\alpha'\beta'}(x - Q).$$

Since f is well defined on C_n, $f(Q)$ is the same in both definitions, and since $x, Q \in B \cap B'$ then $f_{\alpha\beta}(x-Q) = f_{\alpha'\beta'}(x-Q)$. Thus $f_B(x) = f_{B'}(x) = f(x)$ and f is well defined on C_{n+1}. An induction argument gives that f is continuous in $\mathbb{R}^2$.

Notice that, by construction, f maps each B to a rectangle with sides parallel to the axes, and each $\mathcal{F} \in B$ to a rectangle inside $f(B)$ also with sides parallel to the axes. Also if B_1 and B_2 are adjacent to B on opposite sides (that is, $B \cap B_i \neq \emptyset$, $i = 1, 2$ and $B_1 \cap B_2 = \emptyset$), then $f(B_1)$ and $f(B_2)$ are also adjacent to $f(B)$ on opposite sides. Therefore, the function f is both injective and surjective. Finally, since f is linear on B each tile $\mathcal{F} \in \mathcal{T}$ is mapped to a rectangle and its side lengths are preserved. That is, when restricted to the 1-skeleton $\mathcal{G}$ of $\mathcal{T}$ the function f is an isometry. $\square$

The next result is a consequence of the previous theorem.

Theorem 4.2. *All p-tessellation graphs of $\mathbb{R}^2$ are non-hyperbolic if and only if all tessellation graphs of $\mathbb{R}^2$ whose tiles are rectangles are non-hyperbolic.*

4.2. **Tessellations with infinitely many parallel rays.**

Lemma 4.3. *Let T be a rectangular tessellation in $\mathbb{R}^2 \approx \mathbb{C}$ with tiles parallel to the coordinate axes and with infinitely many vertical rays in the upper-half plane. If F is any tile on the tessellation, L and l are the lengths of its longest and shortest sides respectively, consider the following two conditions:*

(1) There exists an increasing function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that for every tile F

$$\frac{L}{l} \leq g(d_{\mathbb{R}^2}(F, i\mathbb{R})),$$

where $i\mathbb{R}$ denotes the imaginary axis.

(2) There exists a constant C so that for every tile F

$$l\, d_G(F, 0) \geq C.$$

If (1) or (2) hold, then for any point x lying on a vertical ray and any other vertical ray γ with $\operatorname{Re} x < \operatorname{Re} \gamma$, there is a geodesic advancing always rightwards and upwards which joins x to γ.

Remark. Note that any curve advancing always rightwards and upwards is a geodesic.

Proof. Let γ_0 and γ_1 be any two of these vertical rays, and without loss of generality suppose γ_0 lies on the left of γ_1. Let D be defined as $D := d(\gamma_0, i\mathbb{R}) \geq 0$. Let σ be the geodesic ray starting in x defined as follows: $\sigma(t) = x + it$ for $t \in [0, t_0]$, where $t_0 := \max\{0, \inf_{z\in\gamma_1} \operatorname{Im} z - \operatorname{Im} x\}$; after that σ advances rightwards when it is possible and otherwise upwards. Denote by $\{F_k\}$ a choice of (ordered) tiles with $\sigma \subset \cup_k \partial F_k$.

For any tile F_k, let h_k denote the length of its horizontal side, and v_k the length of its vertical side. The goal is to show that, in any case, there exists N so that

$$\sum_{k=1}^{N} h_k \geq d(\gamma_0, \gamma_1).$$

Suppose not. Then, there exists C_1 so that $\sum_{k=1}^{\infty} h_k \leq C_1$. It will be shown that this implies that there exists C_2 so that $\sum_{k=1}^{\infty} v_k \leq C_2$, contradicting the fact that T is a tessellation.

Assume (1) holds. Without loss of generality, we can assume that $g(t) \geq 1$ for every $t > 0$; then $v_k \leq h_k g(d_{\mathbb{R}^2}(F_k, i\mathbb{R}))$ for all k. Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} v_k &\leq \sum_{k=1}^{\infty} h_k\, g(d(F_k, i\mathbb{R})) \leq \sum_{k=1}^{\infty} h_k\, g\left(\sum_{n=1}^{k-1} h_n + D\right) \leq \\ &\leq \sum_{k=1}^{\infty} h_k\, g(C_1 + D) \leq C_1\, g(C_1 + D) := C_2\,. \end{aligned}$$

Assume (2) holds. Without loss of generality, by Theorem 3.3 we can assume that $\sup_k L(\partial F_k) = C_0 < \infty$. Then,

$$d_G(F_k, 0) \le d_G(F_1, 0) + \sum_{j=1}^{k-1}(v_j + h_j) \le d_G(F_1, 0) + \sum_{j=1}^{k-1} C_0 \le d_G(F_1, 0) + C_0(k-1).$$

Thus, by hypothesis,

$$h_k \ge \frac{C}{d_G(F_k, 0)} \ge \frac{C}{d_G(F_1, 0) + C_0(k-1)},$$

and therefore one concludes

$$\sum_{j=1}^{\infty} h_k = \infty.$$

□

As it was mentioned above, there are several equivalent definitions of hyperbolicity. For the proof of the next result, the one involving uniformity in the divergence of the geodesics is used. Namely:

Definition 4.4. *The function $e : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a divergence function for the geodesic metric space X if for all $x \in X$, all $R \in \mathbb{R}^+$ and all geodesics $\gamma = [xy]$, $\gamma' = [xz]$, e satisfies the following condition: if $r > 0$, $R + r \le \min\{d(x, y), d(x, z)\}, d(\gamma(R), \gamma'(R)) \ge e(0) > 0$ and α is a path in $\overline{X \setminus B(x, R + r)}$ from $\gamma(R + r)$ to $\gamma'(R + r)$, then $L(\alpha) \ge e(r)$.*

Definition 4.1. *Let X be a geodesic metric space. The geodesics diverge in X if there is a divergence function $e(r)$ such that $\lim_{r\to\infty} e(r) = \infty$.*

In [1] and [34] it was shown the following result.

Theorem 4.5. *A geodesic metric space X is hyperbolic if and only if geodesics diverge in X.*

Theorem 4.6. *Let T be a rectangular tessellation of $\mathbb{R}^2 \approx \mathbb{C}$ with tiles parallel to the coordinate axes and with infinitely many vertical rays. If F is any tile on the tessellation, L and l are the lengths of its longest and shortest sides respectively, consider the following two conditions:*

(1) There exists an increasing function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that for every tile F

$$\frac{L}{l} \le g(d_{\mathbb{R}^2}(F, i\mathbb{R})),$$

where $i\mathbb{R}$ denotes the imaginary axis.

(2) There exists a constant C so that for every tile F

$$l\, d_G(F, 0) \ge C.$$

If (1) or (2) hold, then the 1-skeleton of T is not hyperbolic.

Remark. Theorem 3.6 shows that the existence of infinitely many vertical rays (or even infinitely many vertical lines) does not guarantee the non-hyperbolicity of a tessellation graph.

Proof. Seeking for a contradiction assume that the 1-skeleton G of the tessellation T is hyperbolic. Then, there exists a divergence function, $e : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$.

Denote by $\{\gamma_k\}_k$ the vertical rays in G. Without loss of generality we can assume that $\operatorname{Re}\gamma_k$ increases with k and that $\lim_{t\to\infty} \operatorname{Im}\gamma_k(t) = \infty$ for every k. Let $x \in \gamma_0$.

Let η be a geodesic starting at x which is the union of horizontal and vertical displacements and such that $\eta \cap \gamma_k \ne \varnothing$ for every $k \ge 0$ (recall Lemma 4.3). Denote by η_k the segment of η which starts at the point x and finishes at the first point z_k of γ_k.

Fix n be so that $d_G(\gamma_0, \gamma_n) > e(0)$. Let R be so that $\eta_n(R) = z_n$; then $d_G(\gamma_0(R), \eta_n(R)) \ge d_G(\gamma_0, \eta_n) > e(0)$.

Consider a new geodesic μ which starts at $\gamma_0(R)$ and which is the union of horizontal and vertical displacements, and such that $\mu \cap \gamma_n \neq \varnothing$; let us fix $w_n \in \mu \cap \gamma_n$ and let μ_n be the segment of μ which finishes at $w_n \in \gamma_n$. Denote by $\eta_{n,1}$ and $\mu_{n,1}$ the vertical rays starting at z_n and w_n, respectively.

The curve Γ_1 given by the geodesic segments $[x\gamma_0(R)] \cup \mu_n \cup \mu_{n,1}$ is a geodesic; similarly, the curve Γ_2 defined as $\Gamma_2 := \eta_n \cup \eta_{n,1}$ is also a geodesic. Note that if $w_n = \Gamma_1(t_0)$, then $\Gamma_1(t) = \Gamma_2(t)$ for every $t \geq t_0$, and $d_G(\Gamma_1(R), \Gamma_2(R)) > e(0)$. This contradicts the hyperbolicity assumption. $\square$

5. Tessellations with convex tiles

In [9], the authors conjectured that every tessellation graph of $\mathbb{R}^2$ with convex tiles is non-hyperbolic. Our next result shows that in order to prove this conjecture, it suffices to consider tessellation graphs of $\mathbb{R}^2$ with triangular tiles.

Theorem 5.1. *All tessellation graphs of $\mathbb{R}^2$ whose tiles are convex polygons are non-hyperbolic if and only if all tessellation graphs of $\mathbb{R}^2$ whose tiles are triangles are non-hyperbolic.*

Proof. Let G be a tessellation graph of $\mathbb{R}^2$ whose tiles are convex polygons, and consider its tiles F_n. If $\sup_n \operatorname{diam}_{\mathbb{R}^2} \partial F_n = \infty$, then G is non-hyperbolic and the conclusion holds. Therefore, assume that $c := \sup_n \operatorname{diam}_{\mathbb{R}^2} \partial F_n < \infty$. For each n, let $P_{n,1}$ and $P_{n,2}$ be two vertices of F_n accomplishing the maximum Euclidean distance between the vertices of F_n. Let us consider a new tessellation graph of $\mathbb{R}^2$, G', obtained from G by adding in each tile F_n new edges which join each vertex of F_n with $P_{n,1}$ by the Euclidean segment between them. That is, all the tiles of G' are triangles and therefore, by hypothesis, G' is non-hyperbolic. We shall show that the inclusion $\iota : G \longrightarrow G'$ is a c-full $(1 + \pi/2, 0)$-quasi-isometry and, therefore, by Theorem 2.1, G will also be non-hyperbolic.

Let us consider a tile F_n and its corresponding vertices $P_{n,1}$ and $P_{n,2}$. Then F_n is contained in the closure of the Euclidean circle with center $P_{n,2}$ and radius equal to the Euclidean distance between $P_{n,1}$ and $P_{n,2}$. Without loss of generality one can assume that $P_{n,2}$ is the origin of coordinates and $P_{n,1}$ is the point with coordinates $(1, 0)$. Let P_n be a point of ∂F_n; since F_n is a convex polygon, P_n is contained in the right half-plane, i.e., if (r, θ) are the polar coordinates of P_n, then $0 \leq r \leq 1$ and $-\pi/2 \leq \theta \leq \pi/2$. Let P_n' be the projection of P_n over the circumference $\{x^2 + y^2 = 1\}$. The goal is to compare the Euclidean distance between P_n and $P_{n,1}$ and the sum of the Euclidean distance between P_n and P_n' plus the length of the arc of the circumference $\{x^2 + y^2 = 1\}$ between P_n' and $P_{n,1}$. To this end, one needs to bound the function

$$f(r, \theta) := \frac{\theta + 1 - r}{|1 - re^{i\theta}|} = \frac{\theta + 1 - r}{\sqrt{1 + r^2 - 2r\cos\theta}}\,, \qquad 0 \leq r \leq 1\,,\ -\pi/2 \leq \theta \leq \pi/2\,.$$

Let us consider the functions:

$$g_1(r, \theta) := \frac{\theta^2}{1 + r^2 - 2r\cos\theta}\,, \qquad g_2(r, \theta) := \frac{(1 - r)^2}{1 + r^2 - 2r\cos\theta}\,.$$

For fixed θ, the function $g(r) = 1 + r^2 - 2r\cos\theta$ attains its minimum value when $r = \cos\theta$, therefore $g_1(r, \theta) \leq \theta^2/\sin^2\theta$. Since the ratio $\theta/\sin\theta$ increases for $\theta \in [0, \pi/2]$,

$$g_1(r, \theta) \leq \pi^2/4\,, \qquad \text{for } 0 \leq r \leq 1\,,\ -\pi/2 \leq \theta \leq \pi/2\,.$$

Also, since $-2r \leq -2r\cos\theta$,

$$g_2(r, \theta) \leq 1\,, \qquad \text{for } 0 \leq r \leq 1\,,\ -\pi/2 \leq \theta \leq \pi/2\,.$$

Therefore it follows

$$\sup_{0 \leq r \leq 1,\ -\pi/2 \leq \theta \leq \pi/2} f(r, \theta) \leq 1 + \pi/2\,.$$

The tile F_n is convex, thus

$$\begin{aligned} d_{\partial F_n}(P_n, P_{n,1}) &\leq \theta + 1 - r \leq (1 + \pi/2)\sqrt{1 + r^2 - 2r\cos\theta} \\ &= (1 + \pi/2)\, d_{\mathbb{R}^2}(P_n, P_{n,1}) = (1 + \pi/2)\, d_{G'}(P_n, P_{n,1})\,. \end{aligned} \tag{5.4}$$

For any points P, Q on the graph G, let us consider a geodesic γ in G' joining P and Q. Let $\gamma_n = \gamma \cap F_n'$, where F_n' is the subgraph of G' obtained by adding to ∂F_n the new edges joining the corresponding point $P_{n,1}$ with the other vertices of ∂F_n. If γ_n is contained in ∂F_n, then the length of γ_n in G' coincides with its length in G. If γ_n is not contained in ∂F_n, then $\gamma_n = \gamma_n' \cup \gamma_n''$ where $\gamma_n' = \gamma_n \cap \partial F_n$, $\gamma_n'' = \gamma_n \setminus \gamma_n'$. Note that the closure of γ_n'' is connected and its endpoints are vertices in $\partial F_n \cap V(G)$. Let σ_n be a geodesic in G joining the endpoints of γ_n''; since F_n is convex, σ_n is contained in ∂F_n. ¿From (5.4) one gets

$$d_{G'}(P,Q) = L(\gamma) = \sum_n L(\gamma_n) = \sum_n [L(\gamma_n') + L(\gamma_n'')] \geq \sum_n [L(\gamma_n') + (1+\pi/2)^{-1} L(\sigma_n)] \geq (1+\pi/2)^{-1} d_G(P,Q)\,.$$

In any case one concludes that

$$(1+\pi/2)^{-1} d_G(P,Q) \leq d_{G'}(P,Q) \leq \; d_G(P,Q)\,,$$

which means that the inclusion $\iota : G \longrightarrow G'$ is a $(1+\pi/2, 0)$-quasi-isometric embedding. It is clear that ι is c-full, with $c := \sup_n \operatorname{diam}_{\mathbb{R}^2} \partial F_n < \infty$. By hypothesis the graph G' is non-hyperbolic, and by Theorem 2.1 it follows that G is also non-hyperbolic. □

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