ASYMPTOTIC VALUES OF MEROMORPHIC FUNCTIONS OF FINITE ORDER

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Abstract. The asymptotic values of a meromorphic function (of any order) defined in the complex plane form a Suslin-analytic set. Moreover, given an analytic set A^* we construct a meromorphic function of finite order and minimal growth having A^* as its precise set of asymptotic values.

1. Introduction

A nonconstant meromorphic function $f(z)$ in the plane has the *asymptotic value* a if there is a curve γ tending to ∞ such that $f(z) \to a$ as $z \to \infty$, $z \in \gamma$. Let $As(f)$ be the set of asymptotic values of f; for example, $As(e^z) = \{0, \infty\}$. A classical result of Mazurkiewicz [13] asserts that $As(f)$ is an analytic set in the sense of Suslin [3, 16].

Recall that the order of f is given by

$$\lambda = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic (when f is entire, $T(r, f)$ may be replaced by $\log M(r, f)$, with $M(r, f)$ the maximum modulus function).

Heins [11] showed that given an analytic set A^*, there is a meromorphic function f with $As(f) = A^*$ and, if $\infty \in A^*$, then $A^* = As(f)$ for some entire function f. In general, Heins's function has infinite order. For example, if

$$A := A^* \setminus \{\infty\} = A^* \cap \mathbb{C}, \tag{1}$$

and card $(A) = \infty$ with A bounded, Heins produces a Riemann surface with infinitely many 'logarithmic branch points' over $w = \infty$, so by Ahlfors's theorem $\lambda = \infty$. Note that A, as the intersection of two analytic sets, is analytic.

Eremenko [8] produced meromorphic functions with $\lambda < \infty$ having $As(f) = \hat{\mathbb{C}}$. In fact, if $\psi(r)$ is a given increasing unbounded function, he could arrange that

$$T(r, f) < \psi(r) \log^2 r \qquad \text{as} \quad r \to \infty, \tag{2}$$

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and so f even has order 0. The significance of condition (2) is that when $\psi(r) = O(1)$, Valiron [17] showed that $As(f)$ contains at most one element.

Theorem 1. *Given an analytic set A^* in $\hat{\mathbb{C}}$ and λ, $0 \le \lambda \le \infty$, there is a function f meromorphic in the plane of order λ such that*

$$As(f) = A^*.$$

Indeed, given an increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(r) \to \infty$ as $r \to \infty$, one can arrange that f satisfy (2).

Although questions of this type have been considered in various contexts for many years, the definitive result for meromorphic functions requires additional tools. Our final function f appears only indirectly, although the structure of the asymptotic curves and the asymptotic values assigned to them is presented explicitly. In contrast to [8], a full chapter (§4) is needed to show that no other asymptotic values occur, and it requires new techniques. In [6] there is an informal outline of this work, and full details are given here.

Since there are elementary examples with A^* being empty or having one element, we assume A^* has cardinality at least two, and $0, \infty \in A^*$. A key step is to produce a meromorphic function $g(z)$ whose growth also satisfies (2), with $As(g) = \{0, \infty\}$, with data on the curves on which g tends to its asymptotic values. We then follow ideas going back to Teichmüller and apply quasiconformal compositions to convert g (via the Beltrami equation) to a meromorphic f having $As(f) = A^*$ with growth (2); even here, in §5 we must reformulate the standard definition of analytic set.

This meromorphic function g arises by approximating a specific δ–subharmonic function $U(z)$. The general form of U is very simple, based on the fact that the function $u(z) := A + B\theta$ (with $\theta = \arg z$) is harmonic, and, if $B \neq 0$, of least growth. In §2 we introduce a simple model (called $\hat{U}$ here) whose inadequacies then point to the correct form of U in §2.2. Although our final function is necessarily complicated, the analysis in §4.3 is based on studying the elementary function $w = \sin z$ (despite that $As(\sin z) = \varnothing$).

Throughout, C is a finite positive constant which may change from line to line, unless specified otherwise, although the constants C' and C_0 introduced in (18) and Theorems 2 and 2′ are absolute, associated to the data $\{A^*, \psi\}$ of (1) and (2). In addition to $\mathcal{U} = \{z : \Im z > 0\}$, $S(r) = \{|z| = r\}$, we set $B(a, r) = \{|z - a| < r\}$, $B(r) = B(0, r)$, $\overset{\circ}{E}$ the interior of E, $\overline{E}$ its closure, and $a \wedge b = \min(a, b)$.

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2. The function U and its Laplacian

2.1. The toy function $\hat{U}$.

We introduce $\hat{U}(z)$, a simplified version of U, first for z in the upper half-plane $\mathcal{U}$. Take $0 = \Theta_0 < \Theta_1 < \cdots < \Theta_k < \Theta_{k+1} = \pi$ with data $L > 0$, boundary values $\hat{U}(r) = \hat{U}(re^{i\pi}) = 0$ $(r > 0)$ and constant values $\hat{U}(re^{i\Theta_\ell})$ on the system of rays $\arg z = \Theta_\ell$, $1 \le \ell \le k$, in $\mathcal{U}$. Then for $r > 0$, $0 \le \theta \le \pi$, extend $\hat{U}$ to each sector $\{\Theta_\ell < \arg z < \Theta_{\ell+1}\}$, $0 \le \ell \le k$, by

$$\hat{U}(re^{i\theta}) = \min\{\hat{U}(re^{i\Theta_\ell}) + L(\theta - \Theta_\ell),\ \hat{U}(re^{i\Theta_{\ell+1}}) + L(\Theta_{\ell+1} - \theta)\}. \tag{3}$$

In what follows it will be assumed that data $\hat{U}(re^{i\theta})$ are chosen so that (3) defines $\Psi_{\ell+1} \in [\Theta_\ell, \Theta_{\ell+1}]$, $(0 \le \ell \le k)$ as the θ–value at which each pair of linear functions coincide, and $\hat{U}$ has a local maximum in θ at each $\Psi_{\ell+1}$. Thus $\hat{U}$ is piecewise-linear function in θ, vanishes on the real axis (other that at $z = 0$ where it is not defined), and monotonic on each θ-interval $\{\Theta_\ell < \theta < \Psi_{\ell+1}\}$, $\{\Psi_{\ell+1} < \theta < \Theta_{\ell+1}\}$, $0 \le \ell \le k$. Figure 1 shows one possible graph on $[0, \pi]$ with $k = 3$.

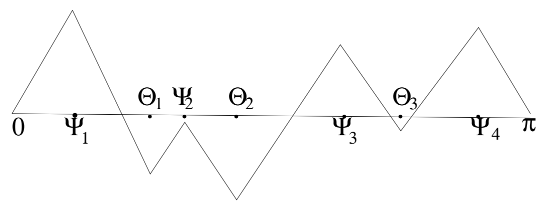

FIGURE 1. Graph of $\hat{U}(re^{i\theta})$ for fixed r.

The function $\hat{U}$ of (3) is δ–subharmonic in $\mathcal{U}$ (i. e., $\Delta\hat{U}$ is a signed measure (charge)), zero on $\partial\mathcal{U} \setminus \{0\}$, and harmonic off the rays $\{\arg z = \Theta_\ell, \Psi_\ell\}$, and so may be extended to be δ–subharmonic on $\mathbb{C} \setminus \{0\}$ by

$$\hat{U}(-z) = -\hat{U}(z) \qquad (z \in \mathcal{U}), \tag{4}$$

a rigidity we use henceforth, and without which the approximation arguments (§4) would collapse ((4) is the key to (37)). It also produces respectively $k+1$ and k rays in the lower half plane on which $\hat{U}$ has local minima and maxima (in θ) on $S(r)$. For any function $\hat{U}$ considered here (or, later, U), let Γ^0 be the curves which are the locus of local minima in θ of $\hat{U}(re^{i\theta})$ for fixed $r > 0$, Γ^* those which are the locus of local maxima, and

$$\Gamma^\sharp := \Gamma^0 \cup \Gamma^*.$$

Thus for $\hat{U}$, $\Gamma^\sharp$ is a network of $4k+2$ rays, with $\Gamma^\sharp \cap (\mathbb{R} \setminus \{0\}) = \varnothing$.

The Laplacian of $\hat{U}$ has a special nature (at least if $z \ne 0$): $\Delta\hat{U}(z) = 0$ when $z = re^{i\theta} \notin \Gamma^\sharp$, whereas if $z \in \Gamma^\sharp \cup S(r)$, the formula $\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$ shows that if $z = re^{i\theta}$, then

$$\Delta\hat{U}(re^{i\theta}) = \pm 2Lr^{-2}\delta_\varphi(\theta),$$

where $\delta_\varphi(\theta)$ is the Dirac function; the plus sign is used when $z \in \Gamma^0$, and the minus sign when $z \in \Gamma^*$ (much as $|x|'' = 2\delta_0$). In summary,

$$\Delta\hat{U}(re^{i\theta}) = 2r^{-2}L\left[\sum_{\theta^0\in\Gamma^0}\delta_{\theta^0}(\theta) - \sum_{\theta^*\in\Gamma^*}\delta_{\theta^*}(\theta)\right]. \tag{5}$$

To obtain a meromorphic function $\hat{g}$ such that $\log|\hat{g}(z)|$ mimics $\hat{U}(z)$, we approximate $\Delta\hat{U}$ by a measure composed of (positive and negative) unit masses, the principle being that (a) if Δv is a Borel measure consisting exclusively of unit point masses, then $v = \log|\hat{g}|$ for some meromorphic function $\hat{g}$, and (b) we can recover the asymptotic behavior of $\hat{g}$ from graphs as in Figure 1 at points at which $|\hat{U}(z) - \log|\hat{g}(z)||$ is small. We see later (Lemma 9) that g attains its asymptotic values on curves in $\Gamma^\sharp$, but probably not on all curves.

2.2. **What is wrong with $\hat{U}$?** Suppose $\hat{g}$ is meromorphic with $\log|\hat{g}|$ modelled on $\hat{U}$ using (3) in $\mathcal{U}$ and (4) in $\mathbb{C}\setminus\mathcal{U}$. For $r>0$, each $S(r)\cap\Gamma^\sharp$ has $2(2k+1)$ points, so a straight forward computation in §7.1 (based on (5)) will show that

$$T(r,\hat{g}) = (4k+2+o(1))\log^2 r \ (r\to\infty).$$

Thus $T(r,\hat{g})/\log^2 r$ is bounded and in fact since $\hat{U}$ is bounded, we could not expect 0 or ∞ to be asymptotic values of g. To circumvent this, our function U is a 'limit' of functions $\hat{U}$ as $k,L\uparrow\infty$. Then on each $S(r)\cap\mathcal{U}$, the graph of $U(re^{i\theta})$ will be as in Figure 1, but with complexity increasing with r, in a manner that

$$\lim_{r\to\infty}\inf_{S(r)} U(z) = -\infty, \qquad \lim_{r\to\infty}\sup_{S(r)} U(z) = +\infty.$$

The meromorphic function g for which $\log|g|$ approximates U is obtained by 'atomizing' ΔU exactly as described in §2.1 for $\hat{U}$.

We partition $\mathbb{C}$ into the disk $\mathcal{A}_0=\{|z|<r_0\}$ and annuli $\mathcal{A}_k$,

$$\mathcal{A}_k := \{r_{k-1}\le |z| < r_k\} \qquad (k\ge 1), \tag{6}$$

for a rapidly-increasing sequence $\{r_k\}$ with $r_0>1$. The function U is defined on $\mathbb{C}$ so that relative to each $\mathcal{A}_k$ it mimics a toy function $\hat{U}$ of increasing complexity.

Thus, in place of the constant L in (3), let $L(r)\uparrow\infty$ be a smooth function with $L(r)=0$ on $[0,1]$, and for is some fixed constant C', say $C'=20$, suppose that

$$\begin{aligned} &\lim\nolimits_{r\to\infty} r^{-1}L(r)+rL'(r)+r^2|L''(r)| = 0\\ &\sup\nolimits_{r>0} r^{-1}L(r)+rL'(r)+r^2|L''(r)| \le C'. \end{aligned}\tag{7}$$

To satisfy (2), suppose that

$$L^2(r^{21}) = o(\psi(r)) \qquad (r\to\infty) \tag{8}$$

(any large number would work in place of 21), and impose the compatability conditions

$$\log(r_{k+1}/r_k) > (k+L_0)L(r_{k+1}) \quad \text{and} \quad L(r_k) > (k+1)^3 \quad (k\ge 0), \tag{9}$$

for some value L_0 large enough (for example taking $L_0 > ((5/4)^{2/3}-1)^{-1}$ gives constant 10 in (31))

Comment. Conditions such as (7) and, later, (15) play an important role. A helpful way to visualize them is to choose, for each k, suitable numbers L_k and $\delta_k>0$. Then we may arrange that $L(r_k)-L(r_{k-1})=L_k$ with $\sup_{[r_{k-1},r_k]} rL'(r) < \delta_k$ by increasing the ratio r_k/r_{k-1} as needed. In turn, these conditions are compatible with $\sup_{[r_{k-1},r_k]} r^2|L''(r)|$ being small, increasing r_k/r_{k-1} if necessary.

Other restrictions will be given later. They will be of two types. Often the ratios $\{r_k/r_{k-1}\}$ will increase, but not the values $\{L(r_k)\}$, so that (7) – (9) remain valid. In addition, §2.5 introduces additional conditions, many of which might be avoided at the expense of complicating several arguments.

2.3. **Graph of U.** Our fundamental function U is modelled on (3) in each $\mathcal{A}_k$. For $z\in\mathcal{U}\cap\mathcal{A}_0$, first set

$$U(re^{i\theta}) = L(r)\min\{\theta,\pi-\theta\} \qquad (0\le r\le r_0,\ 0\le\theta\le\pi), \tag{10}$$

and then use (4) on $\mathcal{A}_0\setminus\mathcal{U}$. Since $L(r)\equiv 0$ on $[0,1]$, $U(z)\equiv 0$ for $z\in B(1)$. Define,

$$U(r_0e^{i\theta}) = L(r_0)(\min\{\theta,\pi-\theta\}) \qquad (0\le\theta\le\pi)$$

on $B(r_0)\cap\mathcal{U}=\partial\mathcal{A}_0\cap\mathcal{U}$. When $\theta=\pi/2$, $\theta=\pi-\theta$ so that $\Gamma^\sharp\cap(\mathcal{U}\cap B(r_0))=\Gamma^*\cap(\mathcal{U}\cap B(r_0))=\{re^{i\pi/2},1\le r\le r_0\}$, and $z_0=ir_0$ will be the initial point of $\Gamma^0\cap\mathcal{U}$.

Since $U=0$ on $\mathbb{R}$, and U is odd (see (4)) we need only define U on $\mathcal{U}\cap\{|z|\ge r_0\}$. In fact, relative to $\mathcal{A}_k,\ k\ge 1$, U will depend on how it is specified on the arcs of $\Gamma^0\cap(\mathcal{A}_k\cap\mathcal{U})$. Hence (see Figure 1 or Figure 2), for each $k\ge 1$, mark k arguments Θ_ℓ on each of the two arcs of $\partial\mathcal{A}_k\cap\mathcal{U}$ augmented by $\Theta_0=0,\Theta_{k+1}=\pi$, with

$$\begin{aligned}0=\quad&\Theta_0^-(k)<\Theta_1^-(k)\le\cdots\le\Theta_k^-(k)<\Theta_{k+1}^-(k)=\pi\in S(r_{k-1}),\\ 0=\quad&\Theta_0^+(k)<\Theta_1^+(k)<\cdots<\Theta_k^+(k)<\Theta_{k+1}^+(k)=\pi\in S(r_k).\end{aligned}\tag{11}$$

Relative to $\mathcal{A}_k\cap\mathcal{U}$, the k arcs of Γ^0 joining its boundary components connect $r_{k-1}e^{i\Theta_\ell^-(k)}$ to $r_ke^{i\Theta_\ell^+(k)}$, $1\le\ell\le k$. Since $S(r_k)=\partial\mathcal{A}_k\cap\partial\mathcal{A}_{k+1}$, we require for $k\ge 2$ that the sets

$$\{\Theta_\ell^-(k)\}=\{\Theta_\ell^+(k-1)\},\tag{12}$$

which with the second line of (11) forces $\Theta_\ell^-(k)=\Theta_{\ell+1}^-(k)$ for (at least) one $1\le\ell=\ell(k)\le k$; see (26) and Figure 2. (Notice the strict inequalities of the first line of (11)).

Now suppose some given values are assigned to each of the points

$$U(r_pe^{i\Theta_\ell^+(k)}),\quad U(r_pe^{i\Theta_\ell^-(k)})\qquad(p\in\{k-1,k\},\ 0\le\ell\le k+1)\tag{13}$$

in $\partial\mathcal{A}_k\cap\mathcal{U}$ so that whenever $z\in S(r_k)$ has representations $z=r_ke^{i\Theta_\ell^+(k)}$ and $z=r_ke^{i\Theta_{\ell'}^-(k+1)}$ from (11), then

$$U(r_ke^{i\Theta_\ell^+(k)})=U(r_ke^{i\Theta_{\ell'}^-(k+1)}),$$

thus defining U unambiguously on $\Gamma^0\cap(S(r_k)\cap\mathcal{U})$.

These boundary values (13) will be made explicit in §5, (59)–(61), and depend only on the data A (the analytic set) and ψ (see (1), (2) and Theorem 1). This means that we may choose $\{\Lambda_k\}\uparrow\infty$ depending only on data A and ψ, and arrange *ab initio* that

$$\begin{aligned}&\max\nolimits_\ell|U(r_ke^{i\Theta_\ell})|<\frac{7}{k}L(r_k)\qquad(k\ge 1),\\ &\max\nolimits_\ell\left|U(r_ke^{i\Theta_\ell(r_k)})-U(r_{k-1}e^{i\Theta_\ell(r_{k-1})})\right|<\Lambda_k,\\ &\frac{\Lambda_k}{\log(r_k/r_{k-1})}\searrow 0,\quad(k\to\infty)\end{aligned}\tag{14}$$

if the ratios r_k/r_{k-1} $(k\ge 1)$ are chosen large enough.

To extend U to $\mathcal{A}_k$ given its boundary values on $\Gamma^0\cap(\mathcal{A}_k\cap\mathcal{U})$, for each $r\in(r_{k-1},r_k)$ set $\Theta_0(r)=0$, $\Theta_{k+1}(r)=\pi$, and if $1\le\ell\le k$, select arguments $\Theta_\ell(r)$ with $\Theta_\ell(r)<\Theta_{\ell+1}(r)$, so that as $r\downarrow r_{k-1}$, $\Theta_\ell(r)\to\Theta_\ell^-(k)$ and as $r\uparrow r_k$, $\Theta_\ell(r)\to\Theta_\ell^+(k)$, while uniformly in ℓ

$$r|\Theta_\ell'(r)|+r^2|\Theta_\ell''(r)|<2L(r_k)^{-7/6}\qquad(r_{k-1}\le r\le r_k),\tag{15}$$

so that $\Theta_\ell(r)$ is continuous at $r\in[r_{k-1},r_k]$. The estimate (15) can be guaranteed if the ratios r_k/r_{k-1} $(k\ge 1)$ are sufficiently large. Then, (recall (13)), we define U

on each $\{re^{i\Theta_\ell(r)},\ r_{k-1}<r<r_k\}$ as

$$U(re^{i\Theta_\ell(r)}) = U(r_{k-1}e^{i\Theta_\ell(r_{k-1})}) + \frac{\log(r/r_{k-1})}{\log(r_k/r_{k-1})}\left(U(r_ke^{i\Theta_\ell(r_k)}) - U(r_{k-1}e^{i\Theta_\ell(r_{k-1})})\right), \tag{16}$$

and use (3), (4) to extend U to all of $\mathcal{A}_k$ (note from (12) that U is continuous). We have already set $z_0 = ir_0 = r_0e^{i\Theta_1^-(1)}$, now viewing z_0 as a point of $\Gamma^0\cap\partial\mathcal{A}_1$.

As noted in §2.1, (3) also yields functions $\Psi_\ell(r),\ r_{k-1}\le r\le r_k,\ 1\le\ell\le k+2$ with $U(re^{i\Psi_\ell(r)})$ a local maximum in each $S(r)\cap\mathcal{U}$.

2.4. **On ΔU.** Further progress depends on analyzing the charge ΔU.

Lemma 1. *Let U be continuous in $\mathbb{C}$ and $|\partial U/\partial\theta| = L(r)$ when $z\in(S(r)\cap\mathcal{A}_k)\setminus\Gamma^\sharp$ for $k\ge 0$.*

Let the arcs of each $\mathcal{A}_k\cap\Gamma^\sharp := \{re^{i\Theta_\ell(r)}, re^{i\Psi_\ell(r)}\},\ r_{k-1}\le r\le r_k$ satisfy (15) and U be assigned on $\mathcal{A}_k$ using its values on $\Gamma^0\cap(\partial\mathcal{A}_k\cap\mathcal{U})$ as in (3), (4), (13) and (16). Finally, let $\delta_a(\theta)$ be the Dirac function (point mass) supported at $\theta=a$.

Then if $k\ge 0$ and $z\in\mathcal{A}_k$, $\Delta U(z)$ may be represented

$$\Delta U(re^{i\theta}) = 2r^{-2}L(r)\left[\sum_{\theta_0\in\Gamma^0}\delta_{\theta_0}(\theta) - \sum_{\theta^*\in\Gamma^*}\delta_{\theta^*}(\theta)\right] + H(r,\theta) + H_{\mathcal{A}}(r,\theta). \tag{17}$$

In (17), H is differentiable,

$$r^2|H(r,\theta)|\le C', \qquad \lim_{r\to\infty}\sup_\theta\{r^2|H(r,\theta)|\} = 0. \tag{18}$$

In addition, $H_{\mathcal{A}}(r,\theta)$ has support on $\cup\partial\mathcal{A}_k$, with $H_{\mathcal{A}}(r_k,\theta) = \varepsilon_k(\theta)$ its density with respect to the Lebesgue measure on $S(r_k)$, where

$$\sup_{r_ke^{i\theta}\in S(r_k)}|\varepsilon_k(\theta)| = o(1) \quad (k\to\infty). \tag{19}$$

Proof. Since $U=0$ for $\{|z|<1\}$, (3) and (4) show that $\Delta U = 0$ on the real axis (including at $z=0$). When $z\in\Gamma^\sharp\cap\mathcal{A}_k$, (5) produces the bracketed term on the right side of (17) which is the main contribution to ΔU.

We first study the primary error term $H(r,\theta)$ in (17). Thus suppose that $z\in(\mathcal{U}\cap\mathring{\mathcal{A}}_k)\setminus\Gamma^\sharp$. As in §2.1, we compute using polar coordinates. Assume for concreteness that

$$U(re^{i\theta}) = U(re^{i\Theta_\ell(r)}) + L(r)(\theta-\Theta_\ell(r)), \tag{20}$$

(see (3)). Then $\Delta U(re^{i\Theta_\ell(r)}) = 0$ (recall (16)). Hence

$$\Delta U(re^{i\Theta_\ell(r)}) = L''(r)(\theta-\Theta_\ell(r)) - 2L'(r)\Theta_\ell'(r) - L(r)\Theta_\ell''(r) + r^{-1}[L'(r)(\theta-\Theta_\ell(r)) - L(r)\Theta_\ell'(r)] := H(r,\theta).$$

(with a change of sign when $U(re^{i\theta}) = U(re^{i\Theta_\ell(r)}) - L(r)(\theta-\Theta_\ell(r))$). Moreover, since $|\theta-\Theta_\ell(r)|<\pi$, we obtain (18) for $z\in\mathcal{U}$ by estimating $r^2H(r,\theta)$ using (7), (9) and (15). For example, if $r_{k-1}<r<r_k$ then $r^2(r^{-1}L(r)|\Theta_\ell'(r)|) = L(r)r|\Theta_\ell'(r)|\le L(r_k)^{-1/6} = o(1)$ and $r^2L'(r)\Theta_\ell'(r) = O(rL'(r)L(r_k^{-7/6})) = o(1)$ as $k\to\infty$. The second item of (18) follows again from (4), (7), (9) and (15).

It remains to consider (19), so let $\mathcal{A}(k,\eta)=\{z:||z|-r_k|<\eta\}$. Let $\rho<\eta/2$ and for $z\in\mathcal{A}(k,\eta/2)$ we compute the Laplacian using the formula

$$\Delta U(z)=\varliminf_{\rho\to 0}\frac{1}{4\pi\rho^2}\left[\frac{1}{2\pi}\int_0^{2\pi}(U(z+\rho e^{i\phi})-U(z))\,d\phi\right]:=\varliminf_{\rho\to 0}\frac{1}{4\pi\rho^2}U_\rho(z),$$

with (for the moment) $z\notin\Gamma^\sharp$. Since U_ρ is a Lipschitz function for each $\rho>0$, (19) is a consequence of the estimate (uniform in ρ for $z\in\mathcal{A}(k,\eta/2)$)

$$\varliminf_{\rho\to 0}\left|\rho^{-2}\int_{\mathcal{A}(k,\eta)}U_\rho(z)\,rdrd\theta\right|=o(1),\quad (k\to\infty). \tag{21}$$

To show (21), we follow Baernstein [2] and write, for $z=re^{i\theta}$, $|z+\rho e^{i\phi}|=r(\phi)$, $\arg(z+\rho e^{i\phi})=\alpha(\phi)$, so that $re^{i\theta}+\rho e^{i(\theta+\phi)}=r(\phi)e^{i(\theta+\alpha(\phi))}$. Since $z_0\notin\Gamma^\sharp$, we may assume that U is given by (20) near z. Note that $r(\phi)=r(-\phi)$, $(\alpha(\phi)+\alpha(-\phi))=0$, so on collecting $\phi,-\phi$, the integrand in this computation of ΔU becomes

$$\begin{aligned}&U(r(\phi)e^{i(\theta+\alpha(\phi))})+U(r(\phi)e^{i(\theta-\alpha(\phi))})-2U(re^{i\theta})\\&=2\left[U(r(\phi)e^{i\Theta_\ell(r(\phi))})+L(r(\phi))(\theta+((\alpha(\phi)+\alpha(-\phi))/2)-\Theta_\ell(r(\phi)))\right.\\&\quad\left.-U(re^{i\theta})\right]\\&=2\left[(U(r(\phi)e^{i\Theta_\ell(r(\phi))})-U(re^{i\Theta_\ell(r)}))+(U(re^{i\Theta_\ell(r)})-U(re^{i\theta}))\right.\\&\quad\left.+(L(r(\phi))(\theta-\Theta_\ell(r(\phi)))\right]:=I_1+I_2+I_3.\end{aligned}\tag{22}$$

For concreteness take $r=|z|>r_k,\ z\in\mathcal{A}(k,\eta/2)$. Then if $|\phi|<\pi/2$, both z and $r(\phi)e^{i(\theta+\alpha(\phi))}$ are in $\mathcal{A}_{k+1}$, and so (16) applies. The main contribution to (22) will be from I_1. Our assumptions on z and $\alpha(\phi)$ with (16) imply that

$$\begin{aligned}\tfrac12|I_1|&=|U(r(\phi)e^{i\Theta_\ell(r(\phi))})-U(re^{i\Theta_\ell(r)})|\\&=\left|\frac{\log(r(\phi)/r)}{\log(r_{k+1}/r_k)}\right|\left(U(r_{k+1}e^{i\Theta_\ell(r_{k+1})})-U(r_ke^{i\Theta_\ell(r_k)})\right).\end{aligned}$$

Hence by (14)

$$|I_1|\le 2\frac{\Lambda_{k+1}}{\log(r_{k+1}/r_k)}\log(r(\phi)/r)\le C\frac{\Lambda_{k+1}}{\log(r_{k+1}/r_k)}\cdot\frac{\rho}{r_k},$$

where we have used that $z\in\mathcal{A}(k,\eta/2)$, $r(\phi)>r_k$ and $|r-r(\phi)|<\rho$ to obtain the last inequality. If $|\phi-\pi|<\pi/2$ and $r_k+\rho<r$, the same estimate holds for I_1.

When $r_k<r<r_k+\rho$, the point $r(\phi)e^{i(\theta+\alpha(\phi))}$ will be either in $\mathcal{A}_k$ or $\mathcal{A}_{k-1}$. In the former case, we repeat what was just done. Otherwise, the index ℓ may change in the sense that $U(r(\phi)e^{i(\theta+\alpha(\phi))})$ may be given by (20) using $\Theta_{\ell'}(r(\phi))$ with (perhaps) $\ell'\ne\ell$ if $r(\phi)e^{i(\theta+\alpha(\phi))}\in\mathcal{A}_{k-1}$. However, since

$$\begin{aligned}&U(r(\phi)e^{i\Theta_{\ell'}(r(\phi))})-U(re^{i\Theta_\ell(r)})=(U(r(\phi)e^{i\Theta_{\ell'}(r(\phi))})-U(r_ke^{i\Theta_\ell(r_k)}))\\&+(U(r_ke^{i\Theta_\ell(r_k)})-U(re^{i\Theta_\ell(r)})),\end{aligned}$$

we still may arrange that

$$\frac12|I_1|\le\frac{\Lambda_k}{\log(r_k/r_{k-1})}\log(r_k/r(\phi))+\frac{\Lambda_{k+1}}{\log(r_{k+1}/r_k)}\log(r/r_k)\le C\frac{\Lambda_k}{\log(r_k/r_{k-1})}\cdot\frac{\rho}{r_k},$$

since $r(\phi)<r_k<r$, $r-r(\phi)<\rho$, and (14).

Analogous estimates apply when $r_k - \eta < r < r_k$. We integrate this over $\mathcal{A}(k,\eta)$, whose area is $O(r_k\eta)$, and recall that $\rho < \eta/2$. Hence

$$\int_{\mathcal{A}(k,\eta)} \rho^{-2}|I_1| r dr d\theta \le C\frac{\Lambda_k}{\log(r_k/r_{k-1})}\cdot\frac{\eta}{\rho}.$$

As for I_2 and I_3 from (22), (16) and (20) show that

$$\begin{aligned}&\tfrac12\,(I_2+I_3) = U(r_k e^{i\Theta_\ell(r_k)}) - U(re^{i\theta}) + L(r(\phi))(\theta-\Theta_\ell(r(\phi)))\\ &= -L(r)(\theta-\Theta_\ell(r)) + L(r(\phi))(\theta-\Theta_\ell(r(\phi))\\ &= (L(r(\phi))-L(r))(\theta-\Theta_\ell(r(\phi)) + L(r)(\Theta_\ell(r)-\Theta_\ell(r(\phi)).\end{aligned}$$

The estimates of the first derivatives of $L(r)$, $\Theta_\ell(r)$ from (7) and (15) are exploited in a manner similar to that used in estimating I_1, and so

$$\int_{\mathcal{A}(k,\eta)} \rho^{-2}(|I_2|+|I_3|)\, r dr d\theta = \frac{\eta}{\rho}\, o(1), \quad (k\to\infty)$$

yielding the first estimate of (19).

(This argument also shows that the contribution to (19) from the $O(k)$ points of $S(r)\cap\Gamma^\sharp$ can also be absorbed in this type of estimate.) □

2.5. **Refined properties of U.** That $w=a$ be an asymptotic value of f on a curve γ requires information for all large $r=|z|\in\gamma$, and one needs equally precise information on a significant portion of the plane to ensure that if $a\notin A^*$, then a cannot be an asymptotic value. To surmount problems arising from the inevitable exceptional sets which arise in approximation theory, we impose conditions on the functions $\{\Theta_\ell\}$ of (15). Some of these might be weakened or perhaps avoided at the price of complicating the proofs of the key Theorem 3 (§3.6) and Lemma 8.

For each $k\ge 1$, define ρ_{k-1} by

$$\log(\rho_{k-1}/r_{k-1}) = L^{1/4}(r_{k-1}), \tag{23}$$

so that $S(\rho_{k-1})\subset\mathcal{A}_k$, while (9) shows that $\rho_{k-1}/r_{k-1} = o(r_k/r_{k-1})$. Note from the first term in (7) and (23) that $L(r_{k-1})\asymp L(\rho_{k-1})$:

$$L(r_{k-1}) \le L(\rho_{k-1}) = L(r_{k-1}) + o(1)\log(\rho_{k-1}/r_{k-1}) = (1+o(1))L(r_{k-1}).$$

In addition, we define r_k', r_k'', where $\rho_{k-1} < r_k' < r_k'' < r_k$ so that

$$\rho_{k-1} = o(r_k'); \quad r_k' = o(r_k'') \quad r_k'' = o(r_k).$$

In particular, let

$$\log(r_k'/\rho_{k-1}) = L^{1/3}(r_{k-1}),$$

and set

$$\mathcal{K} = \bigcup_k \{r_k' \le |z| \le r_k''\}, \tag{24}$$

the *core* of $\cup\mathcal{A}_k$.

Note from (10) that U is known on $\mathcal{A}_0$, and by (11) and (4)

$$\operatorname{card}\big(S(r_k)\cap\mathcal{A}_{k+1}\cap\Gamma^0\big) = \operatorname{card}\big(S(r_k)\cap\overline{\mathcal{A}_k}\cap\Gamma^0\big) + 2 \quad (k\ge 1).$$

Thus the condition (12) will be satisfied by requiring that two arcs $\{\Theta_\ell(r)\}$ in $\mathcal{A}_k$ emerge from a common point of $S(r_k)\cap\mathcal{U}$ $(k\ge 2)$. Hence $\{\Gamma^0\}$ undergoes a bifurcation on $S(r_k)\cap\mathcal{U}$ (in turn creating another bifurcation of Γ^* on $S(r_k)\setminus\mathcal{U}$). The bifurcation points $\pm z_k\in S(r_k)$ are called *nodes* of Γ^0, so that $\Gamma^0\cap\mathcal{U}$ is a dyadic tree. In §5 we identify the branches of $\Gamma^0\cap\mathcal{U}$ in terms of the nodes $\{z_k\}$

through which they pass. As an arc $\gamma \subset \Gamma^0$ recedes, its index Θ_ℓ relative to $\mathcal{A}_k$ will also depend on k (see Figure 2 which represents $\Gamma^\sharp$ in $\mathcal{U} \cap \{r_4 < |z| < r_8\}$) . On the outer boundary $S(r_k)$ of each $\partial\mathcal{A}_k \cap \mathcal{U}$, the arguments Θ_ℓ^+ in (11) are chosen to have the form

$$\Theta_\ell^+(k) = \frac{\ell}{k+1}\pi \qquad (0 \le \ell \le k+1). \tag{25}$$

We then locate the bifurcation node $z_k \in S(r_k)$, now viewed as the inner boundary of $\mathcal{A}_{k+1} \cap \mathcal{U}$ so that if $k = 2^n + p,\ 0 \le p \le 2^n - 1$, then

$$\begin{aligned} \Theta_\ell^-(k) &= \frac{\ell}{k+1}\pi \qquad && \text{for } 1 \le \ell \le 2p+1, \\ \Theta_\ell^-(k) &= \frac{\ell-1}{k+1}\pi \qquad && \text{for } 2p+2 \le \ell \le k+2; \end{aligned} \tag{26}$$

thus $\Theta_{2p+1}^-(k) = \Theta_{2p+2}^-(k)$, guaranteeing (11) and (12). We then use (13)–(16) with (3) and (4) to extend U to $\mathbb{C} \cap \{|z| > r_0\} = \cup_{k \ge 1}\mathcal{A}_k$.

In Figure 2 (not to scale) Γ^0 is indicated with solid lines and Γ^* with dashed lines. The symbols $\Theta_\ell(k)$ are labeling the nodes with argument $\Theta_\ell(k)$.

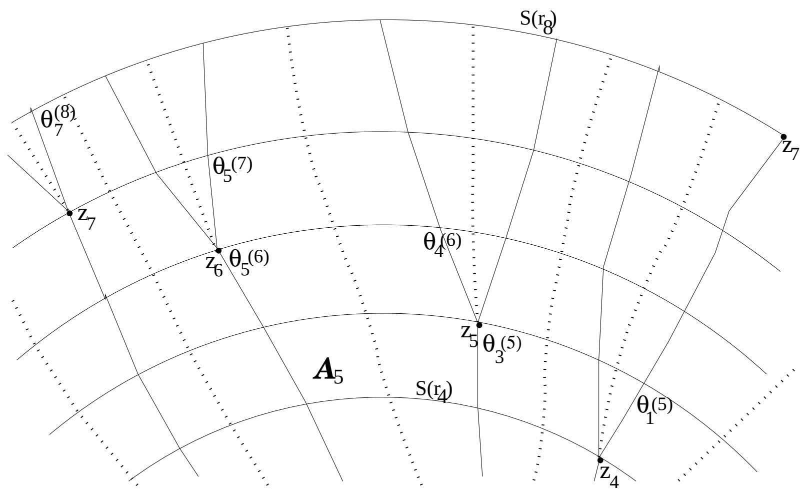

FIGURE 2. The trace of $\Gamma^\sharp$.

With ρ_{k-1} from (23), construct $\Gamma^0 \cap \mathcal{A}_k$ with initial conditions (26) and (consistent with (7))

$$\begin{aligned} &r\Theta'_{2p+1}(r) = -r\Theta'_{2p+2}(r) = L^{-3/4}(r_{k-1}) \quad && (r_{k-1} \le r \le \rho_{k-1}) \\ &\text{and if } \ell \ne 2p+1,\, 2p+2, && \\ &\Theta'_\ell = 0 && (r_{k-1} < r < \rho_{k-1}), \end{aligned} \tag{27}$$

(where $'$ is differentiation with respect to r) as illustrated in Figure 2.

It follows using (27), (23) and the first condition of (9) that

$$|\Theta_{2p+1}(\rho_{k-1}) - \Theta_{2p+1}(r_{k-1})| = L^{-3/4}(r_{k-1}) \log\left(\frac{\rho_{k-1}}{r_{k-1}}\right) = (1+o(1))L^{-1/2}(\rho_{k-1}).$$

Moreover, since (25) and the second property of (7) guarantee that on $S(r_k)$ distinct points of Γ^0 have angular separation

$$\pi/(k+1) > \pi L(r_k)^{-1/3} > \pi(1+o(1))L(\rho_k)^{-1/3},$$

the second line of (27) will show that if $re^{i\tau(r)}, re^{i\tau'(r)} \in \Gamma^0 \cap (S(r) \cap \mathcal{U})$ then

$$|\tau(r) - \tau'(r)| > L^{-2/9}(r) \qquad (\rho_k < r < r_{k+1}). \tag{28}$$

On recalling (3) and (15), it is not difficult to see that (28) then holds as well on $\Gamma^\sharp \cap S(r)$ when $r_k \le r \le r_{k+1}$ except for Θ_{2p} and Θ_{2p+1}.

Finally, in the core $\{r_k' < |z| < r_k''\}$ of each $\mathcal{A}_k$ (recall (24)) we require that

$$\Theta_\ell'(r) = 0 \qquad (1 \le \ell \le k+1,\ r_k' < r < r_k). \tag{29}$$

3. Approximation by a meromorphic function

The idea that the behaviour of a general δ-subharmonic function U can be captured by another of the special form $\log|g|$ with g meromorphic goes back several decades (a survey is in [6], additional interesting references are [18], [12], [9], among others).

In our situation the error $|\log|g(z)| - U(z)|$ must be carefully controlled which is formalized in the next theorem.

Theorem 2. *Let $L(r)$ be a function which satisfies (7), let the system $\{\mathcal{A}_k\}_{k\ge 0}$ satisfy (6) and (9), where (increasing each of the ratios r_{k+1}/r_k if necessary)*

$$\int_{r_k}^{r_{k+1}} L(t)t^{-1}dt \text{ is an integer,} \tag{30}$$

and let U be constructed relative to the system $\{\mathcal{A}_k\}$ so that $U(z) = 0$ for z real and $z \in B(0,1)$, U is assigned to the network $\Gamma^0 \cap \mathcal{U}$ as in (16) so that U is continuous relative to $\Gamma^0 \cap \mathcal{U}$, and then extended to each $\mathcal{A}_k$ using (3) and (4).

Then there is a meromorphic function $g(z)$ and an absolute constant $C_0 > 0$ such that if

$$E = \bigcup B(\zeta_p, |\zeta_p|/10L(|\zeta_p|)) \tag{31}$$

with $\{\zeta_p\}$ the zeros and poles of g, then

(a) $\mathrm{meas}(E \cap S(r)) = o(r) \qquad (r \to \infty)$*;*
(b) *if $z \notin E$ then $\big|\log|g(z)| - U(z)\big| < C_0$;*
(c) *if E' is a component which contains one point-mass ζ_p, then for sufficiently large r*

$$\begin{aligned} \log|g(z)| &\le U(z) + C_0, && z \in E',\ \zeta_p \text{ zero of } g,\\ \log|g(z)| &\ge U(z) - C_0, && z \in E',\ \zeta_p \text{ pole of } g. \end{aligned}$$

The behavior of g on components E' of E which are not disks is more delicate, and requires the additional structure introduced in §2.5: see §3.6.

Results such as Theorem 2 depend on analysis of the (signed) measure ΔU, so we prove Theorem 2 as formulated in Theorem 2′. Write ΔU from Lemma 1 as

$$\Delta U = \mu - \mu^* + \mu_e, \tag{32}$$

with support on $\{|z| \geq 1\}$, where where $\mu \geq 0$ is supported on $\Gamma^0, \mu^* \geq 0$ on Γ^* and $d\mu_e(z) = H(r,\theta)\, rdrd\theta + H_{\mathcal{A}}(r,\theta)\, d\theta$, with $H_{\mathcal{A}}$ supported on $\cup_k \mathcal{A}_k$. Since g is meromorphic, $\Delta \log |g|$ is a network of unit masses, so that $\Delta \log |g| = \sigma - \sigma^* + \sigma_e$, each summand corresponding to a term of ΔU.

By construction, each component of $\Gamma^\sharp \cap \mathcal{A}_k$ is an arc joining the boundary components of $\mathcal{A}_k$, relative to which ΔU becomes one of the terms in the first two summands of (17). Using (30), each component γ is the union of mutually disjoint arcs $\{J\}$ of 'measure' ± 1. Since L vanishes on $[0,1]$, $\mu + \mu^* + \mu_e$ vanishes on $B(0,1)$, and (4) shows that

$$\mu(S) = \mu^*(-S) \quad \text{for all measureable sets}\, S. \tag{33}$$

Let $J \subset \Gamma^0$ such that $\mu(J) = 1$ and recall that the density $d\mu$ is given by (17), that is $d\mu \sim (2L(r)/r)dr$. Then conditions (7), (9) on the growth of $L(r)$ and (15), (27) and (29) (that show that J is almost a radial segment) imply that

$$J \subset \left\{ r \leq |z| \leq r\left(1 + \frac{1}{L(r)}\right)\right\} \qquad \text{and} \qquad \frac{r}{3L(r)} \leq |J| \leq \frac{3r}{2L(r)}, \tag{34}$$

for some $r = r(J) > r_0$. The same estimates hold when $J \in \Gamma^*$ with $\mu^*(J) = 1$.

3.1. **A reformulation.** The logarithmic potential of a signed measure Σ of compact support is defined as

$$P(z, \Sigma) = \int_{\mathbb{C}} \log |1 - z/\zeta| d\Sigma(\zeta),$$

which is δ-subharmonic (subharmonic when $\Sigma \geq 0$). Our measures do not have compact support which means the formula has to be carefully interpreted, which we achieve by appropriate pairing of measures. We recall measures μ, μ^* and (the signed measure) μ_e in (32) and follow a standard procedure (c.f. [6]) to "atomize" the first two measures obtaining σ and σ^*. This leads to the expressions:

$$G(z) := U(z) + V(z),$$

where

$$\begin{aligned} &V(z) := V_{\Gamma^\sharp}(z) + V_e(z),\\ &V_{\Gamma^\sharp}(z) := P(z, \sigma - \mu) - P(z, \sigma^* - \mu^*),\\ &V_e(z) := -P(z, \mu_e). \end{aligned}$$

We will show directly that V is well-defined: each of the two summands defining $V_{\Gamma^\sharp}$ converges, while not only does V_e converge, but $V_e(z) = o(1)$. Thus there is a meromorphic function in the plane g with $G(z) = \log |g(z)|$. Our estimates will show that for most z, $|G(z)|$ is small, where we apply techniques such as in [12], [14] or [6].

Recall that $\Gamma^\sharp \cap \mathcal{A}_k$ is a union of intervals J and J^* so that $|\mu(J)| = 1$ and $\mu^*(J^*) = 1$. To construct σ we consider an interval $J \subset \gamma \subset \Gamma^0 \cap \mathcal{A}_k$, with $\mu(J) = 1$. Following [18] we place the associated point mass at its *centroid* ζ_J,

$$\int_J (\zeta - \zeta_J) d\mu(\zeta) = 0, \tag{35}$$

so that δ_{ζ_J} is a term of σ. The same principle yields $\{\zeta_{J^*}\} \subset \Gamma^* \cap \mathcal{A}_k$ using μ^*. Notice from (33) and (35) that the $\{\zeta_J, \zeta_{J^*}\}$ may be put into correspondence with

$$\zeta_{J^*} = -\zeta_J, \quad \text{when } J^* = -J. \tag{36}$$

The measure μ_e does not need atomization since it is very small. The analysis of V_e is presented in §3.3.

We thus restate the assertions of Theorem 2 in terms of these approximating measures. To simplify notation, we often let I be a generic choice of J or J^*. In Theorem 2′, the centers $\{\zeta_p\}$ of (31) are the $\{\zeta_J,\, \zeta_{J^*}\}$. Assertion (a) in these theorems is equivalent, but assertions (b) and (d) of Theorem 2′ correspond to (b) in Theorem 2, and (c) and (d) in Theorem 2′ to (c) in Theorem 2.

Theorem 2′. *Under the assumptions of Theorem 2, let $\{\zeta_I\}$ be the centroids of the intervals I, where $I \in \Gamma^0$ or Γ^*. Let E be as in (31) and $\zeta_p = \zeta_I$. Then*

(a) $\operatorname{meas}(E \cap S(r)) = o(r)$, *as* $r \to \infty$,
(b) $|V_{\Gamma^\sharp}(z)| < C_0 \quad (z \notin E)$,
(c) *if* $z \in B(\zeta_I, |\zeta_I|/5L(|\zeta_I|))$ *and* $B(\zeta_I, |\zeta_I|/5L(|\zeta_I|))$ *is a component of* E, *then*

$$\begin{aligned} V_{\Gamma^\sharp}(z) &\le C_0, \quad \text{if } \zeta_I \text{ a zero of } \ g,\\ V_{\Gamma^\sharp}(z) &\ge -C_0 \quad \text{if } \zeta_I \text{ a pole of } \ g, \end{aligned}$$

(d) $|V_e(z)| = o(1) \quad (z \to \infty)$.

Note, since $L(r) \uparrow \infty$, that (28) and (34) imply that all balls $B(z_p, |z_p|/5L(|z_p|) \subset \mathcal{K}$ (from (24)) are disjoint, and so (23) implies that (d) holds in most of $\mathbb{C}$. The situation in $\mathbb{C} \setminus \mathcal{K}$ is settled in Theorem 3 in §3.6.

3.2. **Proof of Theorem 2′(a).** The description of $\Gamma^\sharp$ in §2 implies that the number of points in $S(r) \cap \Gamma^\sharp$ for $r \in \mathcal{A}_k$ is at most $4k + 2$, and the angular measure of each ball in E is $O(1/L(r))$. Thus the total angular measure of $E \cap S(r)$ for $r_k \le r \le r_{k+1}$ is $O(k/L(r))$, so (9) gives

$$\operatorname{meas}(E \cap S(r)) = O\left(rL^{-1+1/3}(r)\right) = o(r) \qquad r \to \infty.$$

3.3. **Proof of Theorem 2′(d).** It is simple to estimate V_e from (3.1). That μ_e is uniformly small follows from (18) and the first of (19). Hence assertion (d) follows from the next lemma.

Lemma 2. *The function $V_e(z)$ satisfies*

$$|V_e(z)| = \left|\int_{\mathbb{C}} \log|1 - z/\zeta| d\mu_e(\zeta)\right| = o(1) \qquad (|z| \to \infty).$$

Proof. First consider the contribution to $d\mu_e$ from $d\mu_e^1 := H(r,\theta)\, r dr d\theta$. Since (4) implies that $H(r,\theta) = -H(r, \theta + \pi)$ $(0 \le \theta < \pi)$.

$$\int_{\mathbb{C}} \log|1 - z/\zeta| d\mu_e^1(\zeta) = \int_{\mathbb{C}} (\log|1 - z/\zeta| - \log|1 + z/\zeta|) d(\mu_e^1)^+(\zeta),$$

where $(\mu_e^1)^+(\zeta)$ is the positive part of $\mu_e^1(\zeta)$. Standard estimates then yield that

$$\left|\log\left|\frac{1-z/\zeta}{1+z/\zeta}\right|\right| \le C\left|\frac{z}{\zeta}\right| = C\frac{r}{|\zeta|} \quad (2r<|\zeta|),$$

(37)

$$\left|\log\left|\frac{1-z/\zeta}{1+z/\zeta}\right|\right| \le C\left|\frac{\zeta}{z}\right| = C\frac{|\zeta|}{r} \quad (2|\zeta|<r).$$

By (18), given $\epsilon>0$ there exists r_ϵ with $r^2H<\epsilon$ for $r>r_\epsilon$. Then when $r>r_\epsilon/\epsilon$,

$$\int_{\{|\zeta|>2r\}}\left|\log\left|\frac{1-z/\zeta}{1+z/\zeta}\right|\right| d(\mu_e^1)^+(\zeta) + \int_{\{|\zeta|<r/2\}}\left|\log\left|\frac{1-z/\zeta}{1+z/\zeta}\right|\right| d(\mu_e^1)^+(\zeta)$$
$$\le C\epsilon r\int_{2r}^\infty \frac{1}{t^2}\,dt + \frac{C}{r}\int_0^{r(\epsilon)} dt + C\frac{\epsilon}{r}\int_{r(\epsilon)}^{r/2} dt \le C\epsilon.$$

Now $d\mu_e$ is smooth and satisfies (18), and so

$$\int_{\{|\log|\zeta/z||<\log 2\}} \log|1-z/\zeta|\, d(\mu_e^1)^+(\zeta) = o(1) \qquad (r\to\infty).$$

Estimate (19) and the fact that the sequence $\{r_k\}_{k\ge0}$ is rapidly increasing give the same bound for the contribution to $d\mu_e$ from $H_{\mathcal{A}}(r,\theta)$, with $H_{\mathcal{A}}$ from Lemma 1. □

3.4. **Proof of Theorem 2′(*b*).** Controlling $V_{\Gamma^\sharp}$ is more complicated and needs several lemmas. The first estimates a single term, with z not too near the centroid, based on work from [7].

Lemma 3. *Let $J\in\Gamma^0$ be an interval of μ-measure one. Let $J^*=-J\in\Gamma^*$ and ζ_J and ζ_{J^*} the associated centroids as in (35). Denote by $\mathbf{J}$ the ordered pair $\mathbf{J}=(J,J^*)$ and define*

$$h_{\mathbf{J}}(z) := \int_J \log\left|\frac{1-z/\zeta_J}{1-z/\zeta}\right| d\mu(\zeta) - \int_{J^*}\log\left|\frac{1-z/\zeta_{J^*}}{1-z/\zeta}\right| d\mu^*(\zeta). \tag{38}$$

Then if

$$d(z,\zeta_J\cup\zeta_{J^*}) \ge 3\frac{|\zeta_J|}{L(|\zeta_J|)}, \tag{39}$$

there exists an absolute constant $C>0$ with

$$|h_{\mathbf{J}}(z)| \le C\left(\frac{|J|}{|z-\zeta_J|\wedge|z-\zeta_{J^*}|}\right)^2.$$

Proof. Let $\mathcal{B}=B(\zeta_J,\delta/2)$ be the smallest disk centered at ζ_J which contains J, so that by (34), $\delta\le 2|\zeta_J|/L(|\zeta_J|)$. Then, by (39), $z\notin B(\zeta_J,\delta)$, so we expand the function $\log((\zeta-z)/(\zeta+z))$ about ζ_J, with remainder of second order. The first-order term drops out due to (35), and thus

$$|h_{\mathbf{J}}(z)| = \left|\int_J \log\left|\frac{\zeta-z}{\zeta+z}\right| - \log\left|\frac{\zeta_J-z}{\zeta_J+z}\right| d\mu(\zeta)\right|$$
$$\le C\max_{\mathcal{B}}\left|\frac{1}{(\zeta+z)^2}-\frac{1}{(\zeta-z)^2}\right|\int_J|\zeta-\zeta_J|^2 d\mu(\zeta).$$

However $|\zeta-\zeta_J|\le|J|$, $\mu(J)=1$ and the factor with the max is comparable to $(|z-\zeta_J|\wedge|z-\zeta_{J*}|)^{-2}$. This proves the lemma. □

Lemma 3 leads to the main estimate.

Lemma 4. *Let $z \in \mathbb{C}$ satisfy (39) for all intervals J, J^* in $\Gamma^\sharp$ (so by (31) $z \notin E$), let $\mathbf{J} = (J, J^*)$ and, using the notation in (38), write*

$$V_{\Gamma^\sharp}(z) = \sum_{\mathbf{J}} h_{\mathbf{J}}(z).$$

Then there exists an absolute constant C so that

$$|V_{\Gamma^\sharp}(z)| \leq \sum_{\mathbf{J}} |h_{\mathbf{J}}(z)| \leq C.$$

Proof. Since we are assuming (39) holds for all J, J^*, let $\mathbf{J} = (J, J^*)$ and apply Lemma 3 to each term in the sum. Given $r = |z|$, divide the sum into three groups: $\mathcal{I}_1$ contains the pairs of intervals that are in $B(rL^{-3}(r))$, $\mathcal{I}_2$ those pairs of intervals with null intersection with $B(rL^3(r))$, and $\mathcal{I}_3$ the others.

The estimate for $\mathcal{I}_1$ follows routinely from grouping the pairs of intervals as in the proof of Lemma 3 and using (37) combined with (17), (36) and the fact (cf. (9)) that $O(L^{1/3}(r))$ points of $\Gamma^\sharp$ meet each $S(r)$:

$$\sum_{\mathbf{J} \subset \mathcal{I}_1} |h_{\mathbf{J}}(z)| = \sum_{J \subset B(rL^{-3}(r))} \left| \int_J \log \left| \frac{1 - z/\zeta_J}{1 + z/\zeta_J} \right| - \log \left| \frac{1 - z/\zeta}{1 + z/\zeta} \right| d\mu(\zeta) \right|$$
$$\leq C \int_0^{2rL^{-3}(r)} \frac{t}{r} L^{1/3}(t) \frac{L(t)}{t} dt < CL^{-8/3}(r) = o(1) \qquad (r \to \infty).$$

Next, consider the pairs of intervals in $\mathcal{I}_2$, and choose $m \in \mathbb{N}$ with $2^m \leq L^3(r) < 2^{m+1}$: $m \sim C \log L^3(r)$. For $n \geq m$, (9) shows that the annulus $\mathcal{A}_{(n)} := \{2^n r \leq |\zeta| < 2^{n+1} r\}$ has $O(L^{1/3}(2^n r))$ arcs of $\Gamma^\sharp$ joining its boundary components, each arc of which is the union of $O(L(2^n L(r)))$ intervals of unit μ-mass. The first estimate (37) gives for each term

$$\left| \int_J \log \left| \frac{\zeta - z}{\zeta + z} \right| - \log \left| \frac{\zeta_J - z}{\zeta_J + z} \right| d\mu(\zeta) \right| \leq Cr \int_J \frac{1}{|\zeta|} d\mu(\zeta),$$

$C > 0$ an absolute constant and $J \subset \{z : |z| > rL^3(r)\}$. The essential condition (7) yields that

$$L(2^n r) = L(r) + o(n).$$

Since $\mu(J) = 1$, (9) and $2^m > CL^3(r)$, we have

$$\sum_{J \subset \{|z| > rL^3(r)\}} r \int_J \frac{1}{|\zeta|} d\mu(\zeta) \leq C \sum_{n \geq m} \frac{L^{1/3}(2^n r) L(2^n r)}{2^n} < C \frac{L^{4/3}(2^m r)}{2^m}$$
$$\leq C \frac{(L(r) + o(m))^{4/3}}{2^m} \leq \frac{C}{L(r)} = o(1) \qquad (r \to \infty)$$

(the ratio of successive terms in the series is $\frac{1}{2} + o(1)$).

Consider now the pairs of intervals in $\mathcal{I}_3$. All these intervals intersect the annulus $\{rL^{-3}(r) < |\zeta| < rL^3(r)\}$, and (39) holds for each of them. These pairs of intervals are apportioned into two groups. Take as $\mathcal{I}_3'$ those pairs such that both intervals are in the core $\mathcal{B} := \{r/2 < |\zeta| < 2r\}$; those pairs remaining are in $\mathcal{I}_3^*$.

First consider the contribution from $\mathcal{I}_3^*$, intervals in annuli $\mathcal{A}_{(n)}$ with $n \le -1$ or $1 \le n < C \log L^3(r) = C \log L(r)$. When $n < -1$ and $J \subset \mathcal{A}_{(n)}$, then $|J| < C2^n r/L(2^n r)$ and $|z-\zeta_J| \wedge |z-\zeta_{J^*}| > (1/2+o(1))r > r/4$, so that

$$\frac{|J|}{|z-\zeta_J| \wedge |z-\zeta_{J^*}|} \le \frac{C2^n r}{rL(2^n r)} = \frac{C2^n}{L(2^n r)}.$$

There are $O(L(2^n r))$ intervals J on each component of $\Gamma^\sharp \cap \mathcal{A}_{(n)}$ with $n \le -1$, and (9) again shows there are most $CL^{1/3}(2^n r)$ branches in $\mathcal{A}_{(n)}$. Thus

$$\begin{aligned}\sum_{n<-1} \sum_{J \subset \mathcal{A}_{(n)}} \left(\frac{|J|}{|z-\zeta_J| \wedge |z-\zeta_{J^*}|} \right)^2 &\le C \sum_{n<-1} \frac{2^{2n}}{L(2^n r)} L^{1/3}(2^n r) \\ &< C \sum_{n<-1} L^{-2/3}(2^n r) 2^{2n} = o(1) \qquad (r \to \infty).\end{aligned}$$

When $1 \le n \le C \log L(r)$ and $J \subset \mathcal{A}_{(n)}$, $|z-\zeta_J| \wedge |z-\zeta_{J^*}| \ge C2^n r$, and so

$$\frac{|J|}{|z-\zeta_J| \wedge |z-\zeta_{J^*}|} \le \frac{C2^n r}{2^n r L(2^n r)} = \frac{C}{L(2^n r)}.$$

There are $O(L(2^n r))$ intervals J in each component of $\mathcal{A}_{(n)} \cap \Gamma^\sharp$ and $O(L^{1/3}(2^n r))$ such components with $n < C \log L(r)$. Hence

$$\begin{aligned}\sum_{n=1}^{C \log L(r)} \sum_{J \subset \mathcal{A}_{(n)}} \left(\frac{|J|}{|z-\zeta_J| \wedge |z-\zeta_{J*}|} \right)^2 &\le C \sum_{n=1}^{C \log L(r)} \frac{L^{1/3}(2^n r)}{L(2^n r)} \\ &\le C \frac{\log L(r)}{L^{2/3}(r)} = o(1) \qquad (r \to \infty).\end{aligned}$$

To complete the proof, we estimate the contribution from pairs of intervals $\mathbf{J} \in \mathcal{I}_3'$; each of those intervals have nonempty intersection with $\mathcal{B}$. We recall (9) once again and divide this annulus into congruent regions (wedges) obtained by intersecting $\mathcal{B}$ with sectors of angular opening $O(L^{-1/3}(r))$, oriented so that z itself lies on the bisector of one of these regions (wedges). As before, the number of intervals of $\Gamma^\sharp$ in each sector is $O(L(r))$. Let $\Omega(z)$ be the wedge which contains z.

If $(J \cup J^*) \cap \Omega(z) = \varnothing$, so z is separated from J and J^* by $1 \le \ell \le O(L^{1/3}(r))$ sectors, then

$$\frac{|J|}{|z-\zeta_J| \wedge |z-\zeta_{J^*}|} \le C \frac{r/L(r)}{\ell r L^{-1/3}(r)} = \frac{C}{\ell L^{2/3}(r)},$$

and each sector contains $O(L(r))$ intervals of $\Gamma^\sharp$. For simplicity write $\mathbf{J} \subset \Omega(z)$ if $\mathbf{J} = (J, J^*)$ and either J or J^* intersects $\Omega(z)$. Then summing for $\mathbf{J} \subset \mathcal{I}_3' \setminus \Omega(z)$, we have

$$\sum_{\mathbf{J} \subset \mathcal{I}_3' \setminus \Omega(z)} \left(\frac{|J|}{|z-\zeta_J| \wedge |z-\zeta_{J^*}|} \right)^2 \le C \frac{L(r)}{L^{4/3}(r)} \sum_1^\infty \frac{1}{\ell^2} = o(1) \qquad (z \to \infty).$$

Next, consider the sum over pairs of intervals such that one member of the pair intersects $\Omega(z)$. Divide $\Omega(z)$ into disjoint subregions $\Omega_\ell(z)$ using circles centered at z of radius $\ell r/L(r)$, $\ell \in \mathbb{N}$. Now since $\mu(J) = 1$ (or $\mu^*(J^*) = 1$) then $|J| = |J^*| =$

$cr/L(r)$ and therefore the number of intervals in each $\Omega_\ell(z)$ is uniformly bounded. Since (39) holds, we have $\ell \geq 2$, and so

$$\sum_{\ell\geq 2}\sum_{\mathbf{J}\subset\Omega_\ell(z)}\left(\frac{|J|}{|z-\zeta_J|\wedge|z-\zeta_{J^*}|}\right)^2 \leq C\sum_{\ell\geq 2}\left(\frac{r/L(r)}{\ell r/L(r)}\right)^2 = C\sum_{\ell\geq 2}\frac{1}{\ell^2}<\infty. \tag{40}$$

where again we write $\mathbf{J}\subset\Omega_\ell(z)$ if $\mathbf{J}=(J,J^*)$ and $(J\cup J^*)\cap\Omega_\ell(z)\neq\varnothing$. □

That the estimate of Lemma 4 is not $o(1)$ is due to the term (40), but if (39) is replaced by the stronger (44) we get the more flexible (43), which is the key to §4.

Corollary 1. *For fixed $K\geq 15$ and fixed z_0, with $|z_0|$ so large that*

$$2K < L^{1/3}(|z_0|), \tag{41}$$

let

$$\begin{aligned}\mathsf{D}(z_0) &= \{\zeta : |\zeta - z_0| < 5|z_0|/L(|z_0|)\},\\ \mathsf{D}'(z_0) &= \{\zeta : |\zeta - z_0| < 10K|z_0|/L(|z_0|)\},\end{aligned} \tag{42}$$

and let

$$\mathcal{I}^\star = \{\mathbf{J}=(J,J^*) : \mathsf{D}'(z_0)\cap(J\cup J^*)\neq\varnothing\}.$$

Then, with $h_{\mathbf{J}}$ from (38),

$$\left|\log|g(z)| - U(z) - \sum_{\mathbf{J}\in\mathcal{I}^\star} h_{\mathbf{J}}(z)\right| \leq CK^{-1}+o(1), \qquad z\in\mathsf{D}(z_0). \tag{43}$$

Proof. The only term in Lemma 4 not $o(1)$ is (40), so by increasing the radius of the ball in (39) one gets a better estimate. Concretely, if $|z_0|$ is large enough and $(J,J^*)\notin\mathcal{I}^\star$ then $d(z_0,\zeta_J\cup\zeta_{J^*})\geq 5K|\zeta_J|/L(|\zeta_J|)$ and therefore

$$d(z,\zeta_J\cup\zeta_{J^*}) \geq \frac{5K|\zeta_J|}{2L(|\zeta_J|)} \qquad (z\in\mathsf{D}(z_0)). \tag{44}$$

For $z\in\mathsf{D}(z_0)$, follow the proof of Lemma 4 but now summing over intervals that are not in $\mathcal{I}^\star$. According to (44), $\ell > K$ in (40) so the sum over the $\mathbf{J}\subset\Omega(z)$, where $\Omega(z)$ is the wedge introduced in the lemma, is bounded by $C\sum_{\ell>K}\ell^{-2}=O(K^{-1})$ while the sums over all the intervals not in $\Omega(z)$ remain the same. Therefore

$$\sum_{\mathbf{J}\notin\mathcal{I}^\star}|h_{\mathbf{J}}(z)| \leq CK^{-1}+o(1), \qquad z\in\mathsf{D}(z_0).$$

Since

$$\left|\log|g(z)| - U(z) - \sum_{\mathbf{J}\in\mathcal{I}^\star}h_{\mathbf{J}}(z)\right| \leq \sum_{\mathbf{J}\notin\mathcal{I}^\star}|h_{\mathbf{J}}(z)| + |V_e(z)|,$$

lemma 2 and the estimation above give (43). □

3.5. **Estimates near the exceptional set E: Proof of Theorem 2′(c).** Lemma 5 below complements Lemma 3 when (39) fails. For now we still assume that the component of $E\ni z$ is a single disk, as in hypothesis (c). Together, the two lemmas of this section imply assertion (c) of Theorem 2′.

Lemma 5. *Let $z\in\Omega := B(\zeta_J, 3|\zeta_J|/L(|\zeta_J|))$ where J is an interval of $\Gamma^0\subset\Gamma^\sharp$ of μ-measure one. Let $J^*=-J$ and $\mathbf{J}=(J,J^*)$. Then, with $h_{\mathbf{J}}$ from (38), $h_{\mathbf{J}}(z)<C$, C an absolute constant.*

Equivalently, if $z\in B(\zeta_{J^*}, 3|\zeta_{J^*}|/L(|\zeta_{J^*}|)$, with $J^*\in\Gamma^*\subset\Gamma^\sharp$ let $J=-J^*$ and $\mathbf{J}=(J,J^*)$. Then $h_{\mathbf{J}}(z)>-C$.

Note that the disk $\Omega = \Omega(\zeta_J)$ is somewhat larger than those in E (31); the disks $\Omega(\zeta_J)$ are no longer disjoint.

Proof. We consider only the first assertion, and note that there can only be an upper bound, since $h_{\mathbf{J}}(\zeta_J) = -\infty$.

Let $|z| = r$. It is elementary, from (34) and the fact that $z, \zeta \in \Omega$, that

$$\begin{aligned} h_{\mathbf{J}}(z) &= \log|z-\zeta_J| - \textstyle\int_J \log|z-\zeta|\, d\mu(\zeta) + o(1) \\ &\le \log\frac{r}{L(r)} - \int_J \log|z-\zeta|\, d\mu(\zeta) + O(1) \qquad (z \in \Omega), \end{aligned} \tag{45}$$

and since $\int_J \log|z-\zeta|\, d\mu(\zeta)$ is harmonic in $\Omega \setminus J$, Lemma 3 applies for $z \in \partial\Omega$. By the maximum principle, we need only bound the integral when $z \in J$.

We suppose that $J \subset \mathbb{R}^+$, and let $t \in J$. Set $I = Jr^{-1}$ (where $|z| = r$) and choose $s \in I$ with $s = tr^{-1}$. According to (17), $d\mu = 2r^{-1}L(r)\, dr$ on J, and so

$$\int_J \log|z-\zeta|\, d\mu(\zeta) = \log r + 2\int_I L(rs)\log|s-1|s^{-1}ds + o(1). \tag{46}$$

By (7) and (33), $L(rs) = L(r) + o(1)\log(s/r) = L(r) + o(1)L(r)^{-1}$, the $o(1)$ uniform in $s \in I$. If $I = [1 - c_1/L(r), 1 + c_2/L(r)]$, the condition $\mu(J) = 1$ implies that $c_1 + c_2 = 1/2 + O(L^{-1}(r))$. Since $u \log u$ decreases for $u < e^{-1}$, we have

$$\begin{aligned} 0 \ge \int_I L(rs)\log|s-1|s^{-1}\, ds &= \big(L(r) + o(1)L^{-1}(r)\big)\int_I \log|s-1|s^{-1}\, ds \\ &= \big(L(r) + o(1)L^{-1}(r)\big)\big(1 + O(L^{-1}(r))\big)\int_I \log|s-1|\, ds \\ &= (L(r) + O(1))\int_I \log|s-1|\, ds \\ &= (L(r) + O(1))\,\frac{1}{2}\big(L^{-1}(r)\log(L^{-1}(r)) + O(L^{-1}(r))\big) \\ &= (c_1 + c_2)\log L^{-1}(r) + O(1) = -\frac{1}{2}\log L(r) + O(1), \end{aligned}$$

which we then insert in (46) and then (45). ☐

Finally we consider the situation that $z \in \Omega$, but not too near ζ_J.

Lemma 6. *For $\lambda > 0$, let $z \in \Omega$ as in Lemma 5 with*

$$\lambda\frac{|\zeta_J|}{L(|\zeta_J|)} \le |z-\zeta_J| \wedge |z-\zeta_{J^*}| \le 3\frac{|\zeta_J|}{L(|\zeta_J|)} \tag{47}$$

Then $|h_{\mathbf{J}}(z)| < C = C(\lambda)$.

Proof. Let $|z| = r$ and note that (47) shows that $z \in B(\zeta_J, 10r/L(r))$. Thus

$$\log\frac{r}{L(r)} - C(\lambda) \le \log|z-\zeta_J| \le \log\frac{r}{L(r)} + C,$$

so the proof of Lemma 5 shows the expression in the first line of (45) is uniformly bounded. ☐

3.6. **Statement and proof of Theorem 3: controlling behavior on E.** We exploit the special forms of U and Γ^0 near the inner boundaries of each $\mathcal{A}_k$, as described in §2.5, to give bounds for $V_\Gamma^\sharp$ on E for the situations not settled in Theorem 2. The proofs rely on techniques used in Theorem 2 (Theorem 2′).

Theorem 3. *Let the assumptions and notations of Theorem 2 remain in force, augmented by (25)-(28). Then we also have*

(a) *the components E' of E are either single disks or the union of three disks. In the latter case, E' contains three point masses, one of which is a zero and one a pole of g,*

(b) *if $z \in E'$ where E' is a component of E containing centroids $\zeta_I, \zeta_J, \zeta_K$, atoms of the approximating measure $\sigma - \sigma^*$, with ζ_I a zero of g and ζ_J a pole, then with C_0 the constant of Theorems 2 or 2′*

$$V_{\Gamma^\sharp}(z) \le C_0 \qquad \text{if } |\zeta_I - z| \le |\zeta_J - z| \text{ and } \zeta_K \text{ is a zero of } g,$$
$$V_{\Gamma^\sharp}(z) \ge -C_0 \qquad \text{if } |\zeta_I - z| \le |\zeta_J - z| \text{ and } \zeta_K \text{ is a pole of } g.$$

Proof. Since (28) holds when $z \in \mathcal{K}$ ($\mathcal{K}$ from (24)), if a component of E' consists of more than one disc, it must intersect $\{r_k \le |z| \le \rho_k\}$ for some (large) k. For convenience, let us assume that $E' \subset \mathcal{U}$. According to (17), the centers $\{\zeta_p\}$ of all disks contained in E' have the same modulus, and since (34) holds, (31) shows that disks corresponding to point measures which intersect a single arc $\gamma \cap \mathcal{A}_k$, $\gamma \subset \Gamma^*$, are disjoint. By (27), three branches of $\Gamma^\sharp \cap \mathcal{A}_k^o$ emerge from each bifurcation node $\pm z_k \in S(r_k)$ (since $E' \subset \mathcal{U}$, there are two in Γ^0 and one in Γ^*) and separate uniformly as r increases. Hence components E' associated to these branches consist of one ball or three balls, in the latter case two associated to a zero of g, and the other to a pole. This proves claim (a).

In considering (b). Let E' be the component of E containing centroids $\zeta_I, \zeta_J, \zeta_K$, where ζ_I is a zero of g (i.e. $I \in \Gamma^0$ where ζ_I centroid of I) and ζ_J a pole (i.e. $J \in \Gamma^*$). Let K be the interval in $\Gamma^\sharp$ with centroid ζ_K, and finally consider the sets of pair of intervals $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ formed by the intervals I, J, K and their negative counterparts $-I$, $-J$, $-K$ ordered as in Lemma 3. When $E' \subset \mathcal{U}$, $g(\zeta_K) = 0$. Using the notation in (38) we show for some absolute constant C that if $z \in E'$ then

$$h_{\mathbf{I}}(z) + h_{\mathbf{J}}(z) + h_{\mathbf{K}}(z) \le C, \quad (z \in E', |\zeta_I - z| \le |\zeta_J - z|) \tag{48}$$

with the opposite estimate when $g(\zeta_K) = \infty$. Once (48) is proved, the estimate in (b) follows from Lemma 4 together with (48) applied to the terms which fail to satisfy (39), as we did at the beginning of §3.5 in Lemma 5.

Let $r = |z|$, γ be the arc of Γ^0 associated to ζ_K, and let $\zeta \in \gamma$, $|\zeta| = t$. Then $S(t)$ meets arcs $\gamma' \subset \Gamma^0 \cap E$ (associated to ζ_I) and $\gamma^* \subset \Gamma^* \cap E$ (corresponding to ζ_J) at ζ', ζ^*, and γ, γ' and γ^* meet at a bifurcation node z_k of $\Gamma^\sharp$. Since we have assumed that $|\zeta_I - z| \le |\zeta_J - z|$, the strict condition (27) near the bifurcation node z_k ensures that

$$\left|\log\left|\frac{z - \zeta_J}{z - \zeta_K}\right|\right| = O(1), \qquad \left|\log\left|\frac{z - \zeta^*}{z - \zeta}\right|\right| = O(1).$$

Hence $|\zeta_I - z| < |\zeta_J - z| (< |\zeta_K - z|)$ and so $h_{\mathbf{I}}(z) + h_{\mathbf{J}}(z) + h_{\mathbf{K}}(z) = h_{\mathbf{I}}(z) + O(1)$. The result now follows from Lemma 5. □

4. On the imaginary parts

4.1. **Two key cases.** To identify the possible asymptotic curves of g, it is clear that more is needed than data on $|g|$. We prove

Theorem 4. *The only possible asymptotic values of $w = g(z)$ are 0 and ∞. Moreover, if η is any asymptotic path for $w = 0$, then there is a curve $\gamma \subset \Gamma^0 \subset \Gamma^\sharp$ on which $g \to 0$, such that for each $\varepsilon > 0$, the set $\{|g(z)| < \varepsilon\}$ contains a component Ω so that η and γ are in $\Omega \cap \{|z| > r'\}$ if r' is sufficiently large. Thus η and γ belong to the same tract corresponding to $w = 0$.*

A similar statement holds with $w = 0$ replaced by $w = \infty$.

Thus consider a (hypothetical) curve η tending to $z = \infty$ on which $g(z) \to a$, so that $|g|$ is nearly constant on η (if $a \neq \infty$). Using the notation from (42), we consider a family of disks $\mathsf{D}'(z_0)$, with $z_0 \in \mathcal{K} \cap \eta$ (recall (24)) through which η would have to pass. Let us denote by $\mathcal{D}_\eta$ such a family. *Comment.* The points z_0 should not be confused with the first node z_0 of the network Γ^0.)

Let K be fixed (and large) with $z_0 \in S(r_0) \cap \mathcal{K}, |z_0| = r_0$ so large that (41) holds (r_0 should not be confused with the inner boundary of $\mathcal{A}_0$ from (6)). Since $z_0 \in \mathcal{K}$, (28) implies that $\mathsf{D}'(z_0)$ intersects at most one curve from $\Gamma^\sharp$. Thus $\mathsf{D}'(z_0)$ meets at most two regions Δ in $\mathbb{C} \setminus \Gamma^\sharp$, and so for each disk $\mathsf{D}'(z_0)$ there are two posibilities:

(a) $\mathsf{D}(z_0) \cap \Gamma^\sharp = \varnothing$ for $\mathsf{D}(z_0) \subset \mathsf{D}'(z_0)$ (see (42)) or,

(b) $\mathsf{D}(z_0)$ contains an arc $\gamma \subset \Gamma^\sharp$,

and then two situations could occur:

(i) There are infinitely many disks in $\mathcal{D}_\eta$ for which posibility (a) holds,

(ii) there are only a finite number of disks in $\mathcal{D}_\eta$ for which (a) holds.

When η is far from $\Gamma^\sharp$ (case (i)) and $z \in \eta$, we may suppose that $\log|g(z)|$ is close to the model function (cf. (20)) on $S(|z|) \cap \mathsf{D}(z_0)$. When η is near $\Gamma^\sharp$ is far more delicate; details are in §4.3. Since the curves of $\Gamma^\sharp$ are asymptotically rays when $z \in \mathcal{K}$, we assume that γ is the positive real axis.

4.2. **Proof of Theorem 4 (start).** Let $g \to a$ on a curve η. If $a = 0, \infty$, we will associate a curve $\gamma \subset \Gamma^*$ 'near' η on which also $g \to a$. To eliminate the possibility $a \neq 0, \infty$ is harder, and for that we need the rest of this section (for case (i)) and the next (case (ii)).

Now let η be an asymptotic curve of g, so that $g(z) \to a$ as $z \to \infty$ on η.

First suppose $a = 0$ or ∞; say $a = 0$. In case (i), choose r_0 large and $z_0 \in \eta \cap S(r_0)$, so that (a) holds for $\mathsf{D}'(z_0)$. We may assume using Theorem 2 or 2′ that if $S(r) \cap \mathsf{D}(z_0) \neq \varnothing$, then in the component $\Omega(z_0)$ of $\mathbb{C} \setminus \Gamma^\sharp$ which contains z_0, $\log|g|$ is close to a model function U of (20), and thus is linear in $\arg z$. Hence we obtain an arc of $S(r_0)$ joining $z \in S(r_0) \cap \eta$ to $\Gamma^\sharp$ with $U(r_0 e^{i\theta})$ having its maximum at z and decreasing on this arc until reaching a minimum at $\Gamma^\sharp$ (outside $\mathsf{D}(z_0)$). It follows that any component of $\{U < -M\}$ which meets η on $S(r_0)$ for large r_0 also intersects some curve $\gamma \subset \Gamma^0$, and so if $g \to 0$ on η, then $g \to 0$ on γ. Analogous comments apply when $a = \infty$.

In case (ii) an even easier argument works, since η is already close to a single arc of Γ^*.

More subtle is that $0, \infty$ are the only possible asymptotic values. Let η be a curve on which $g \to a \neq 0, \infty$.

If case (i) applies, let $\mathsf{D}'(z_0)$ be a disk for which (a) holds, then $\log|g|$ and $\theta = \arg z$ are harmonic in $\mathsf{D}'(z_0)$. We suppose that near z_0, U is given by (20). Thus given $\epsilon > 0$, if K and $|z_0|$ are large ($z_0 \in \eta \cap \mathcal{K}$) then by (43), (7) and (15),

$$
\begin{aligned}
&|(\log|g(z)| - (A + \tau L(r_0)\theta))| := |\epsilon'(z)| < \epsilon \qquad (z \in \mathsf{D}''(z_0)), \\
&|\arg g(z) - (A' - \tau L(r_0)\log r)| := |\epsilon(z)| < \epsilon \qquad (z \in \mathsf{D}''(z_0))
\end{aligned}
\tag{49}
$$

for suitable constants $A, A', \tau \in \{\pm 1\}$ and $\mathsf{D}''(z_0)$ the disk centered at z_0 with radius half of that of $\mathsf{D}(z_0)$ (in fact, the first line holds in the larger $\mathsf{D}(z_0)$). The second line (which restates the first for the conjugate functions) holds in $\mathsf{D}''(z_0)$ since K is large. By hypothesis, $\log|g| = \log|a| + o(1)$ on η and $z_0 = r_0 e^{i\theta_0} \in \eta$. Thus (20) and the first line of (49) show that $|\theta - \theta_0| = O(\epsilon L^{-1}(r_0))$ in $\eta \cap \mathsf{D}''(z_0)$. However, on $\{\arg z = \theta_0\} \cap \mathsf{D}''(z_0)$, the function $\log r$ increases by more than $2/L(r_0)$. The second estimate of (49) with ϵ small and K large but fixed then implies that $\arg g(z)$ varies by at least $\pi/2$ on $\eta \cap \mathsf{D}''(z_0)$. In other words, if $a \neq 0, \infty$, η will contain points in $\mathsf{D}''(z_0)$ whose g-images are well-separated on $\{|w| = a\}$, and so g cannot be uniformly close to a on $\eta \cap \mathsf{D}''(z_0)$.

4.3. **Case (ii).** This situation is more difficult. Again $g(z) \to a \neq 0, \infty$ on η, but we assume that whenever $z_0 \in \eta \cap \mathcal{K}$ with $|z_0|$ sufficiently large, $\mathsf{D}(z_0) \cap \Gamma^\sharp \neq \varnothing$. By (28) and (41) $\mathsf{D}(z_0) \cap \Gamma^\sharp$ consists of portions of one arc γ; for specificity, take $\gamma \subset \Gamma^0$. Due to (29), $\gamma \cap \mathsf{D}'(z_0)$ is a ray which contains the centroids $\zeta_I \in \mathsf{D}'(z_0)$ (see Figure 3, where γ is shown horizontal).

In contrast to case (i), the geometry of η is not apparent. An insightful example is $w = \sin z$, where $\Gamma^\sharp = \mathbb{R}$. The level-set $\{|\sin z| = 1\} = \{\pi/2 \pm k\pi,\ k \in \mathbb{Z}\}$, is a 'necklace' of topological circles meeting tangentially at the critical points. Thus, by moving alternately in the upper and lower half-planes, we find a curve η which $|\sin z| = 1$, but $\arg(\sin z)$ never varies more than π.

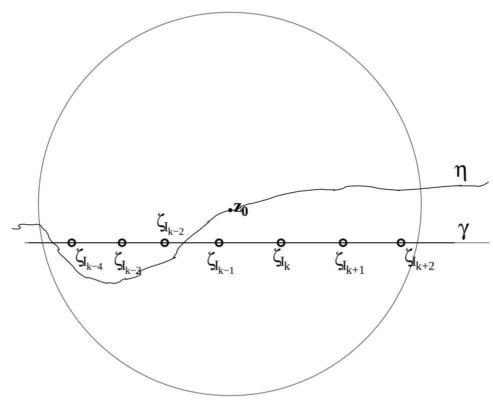

FIGURE 3. Intersection of $\mathsf{D}'(z_0)$ with γ and η.

Write $\gamma \cap \mathsf{D}'(z_0) = \cup_{\mathcal{I}} I$, where each I has mass one (this may require slightly modifying $\partial\mathsf{D}'(z_0)$). Since $d\mu = 2r^{-1}L(r)$ on γ, there are at most $O(K)$ centroids ζ_I with $I \in \mathcal{I}$.

It is also useful to extend γ in both directions to separate $\mathbb{C}$ (as well as $\mathsf{D}'(z_0)$ and $\mathsf{D}(z_0)$) into two components: γ an interval on $\mathbb{R}$.

Let $\mathsf{D}'^{+}(z_0)$, $\mathsf{D}'^{-}(z_0)$ be the two components of $\mathsf{D}'(z_0) \setminus \gamma$, and in each of $\mathsf{D}'^{\pm}(z_0)$ take branches of $\arg(z - \zeta)$ $(\zeta \in \gamma)$, $\arg(z - \zeta_I)$ $(I \in \mathcal{I})$. Similarly let $\mathsf{D}^{\pm}(z_0) \subset$

$\mathsf{D}'^{\pm}(z_0)$ be the components of $\mathsf{D}(z_0) \setminus \gamma$. Using notation from Corollary 1 (§3.4), consider $\mathcal{I}^\star = \{\mathbf{I} = (I, -I) : I \in \mathcal{I}\}$ and write $\mathcal{H}(z) = \sum_{\mathcal{I}^\star} h_{\mathbf{I}}(z)$ for $z \in \mathsf{D}(z_0)$ and $h_{\mathbf{I}}$ from (38). One utility of assumption (29) is that we have explicit expressions for $\mathcal{H}$ and its conjugate $\tilde{\mathcal{H}}$, the latter defined in each component of $\mathsf{D}'(z_0)$:

$$\mathcal{H}(z) = \sum_{\mathcal{I}^\star} h_{\mathbf{I}}(z) = \sum_{\mathcal{I}} \log|1 - z/\zeta_I| - 2\int_{\gamma\cap\mathsf{D}'(z_0)} \log|1 - z/t|\, t^{-1}L(t)\, dt + o(1),$$
$$\tilde{\mathcal{H}}(z) = \sum_{\mathcal{I}^\star} \tilde{h}_{\mathbf{I}}(z) = \sum_{\mathcal{I}} \arg(1 - z/\zeta_I) - 2\int_{\gamma\cap\mathsf{D}'(z_0)} \arg(1 - z/t)\, t^{-1}L(t)\, dt + o(1),$$

where $o(1)$ accounts for the contribution from $-(\gamma \cap \mathsf{D}'(z_0))$, which lies far from z_0.

Lemma 7. *Let $\tilde{\mathcal{H}}^\pm$ be a suitable branch in each component of $\mathsf{D}'(z_0) \setminus \gamma$, and let $p, q \in \gamma \cap \partial\mathsf{D}'(z_0)$. Then*

$$\begin{aligned} &\|\tilde{\mathcal{H}}^\pm\|_\infty \le \pi/2, \\ &\tilde{\mathcal{H}}^+(q + i0) - \tilde{\mathcal{H}}^+(p + i0) = -\left[\tilde{\mathcal{H}}^-(q - i0) - \tilde{\mathcal{H}}^-(p - i0)\right]. \end{aligned} \tag{50}$$

Proof. This is straightforward. Each function $\tilde{\mathcal{H}}^\pm$ is a sum of a finite number of terms $h_{\mathbf{I}}$ from (38). If $\zeta \in \gamma$ (possibly $\zeta = \zeta_I$, $I \in \mathcal{I}$), then $\arg(z - \zeta) = 0$ when $z \in \gamma, z > \zeta$ (using language inherited from viewing $\gamma \subset \{\Re z > 0\}$), while $\arg(z - \zeta) = \pm\pi$ when $z \in \gamma, z < \zeta$; the sign depending on the function $\tilde{\mathcal{H}}^\pm$ under scrutiny. Thus, the boundary values of the conjugate $\tilde{h}_I$ of any single term $\log|z - \zeta_I| - \int_I \log|z - \zeta| d\mu(\zeta)$ are zero for $z \in \gamma \setminus \{I\}$. This remark also justifies the other assertion. □

To adapt (49) to the situation (ii), let K, z_0 be large, $z_0 \in \eta$, subject to (41). The left side of (43) is harmonic in $\mathsf{D}'(z_0)$, since $\mathcal{H}$ cancels the Riesz mass. Hence we make take conjugates, with constant A in $\mathsf{D}'(z_0)$ and constants A' in $\mathsf{D}'^{\pm}(z_0)$:

$$\begin{aligned} &|(\log|g| - (A + \tau L(r_0)|\theta|) + \mathcal{H}(z))| := |\epsilon'(z)| < \epsilon \qquad (z \in \mathsf{D}(z_0)), \\ &\left|\arg g(z) - \left(A' \mp \tau L(r_0)\log r + \tilde{\mathcal{H}}^\pm(z)\right)\right| := |\epsilon(z)| < \epsilon \qquad (z \in \mathsf{D}^\pm(z_0)). \end{aligned} \tag{51}$$

Lemma 8. *Let η be a curve on which $g(z) \to a$ such that η passes through the center z_0 of $\mathsf{D}(z_0)$ with $\Gamma^\sharp \cap \mathsf{D}(z_0) \neq \varnothing$. Then η contains an arc η' on which $\arg g(z)$ varies by at least $\pi/2$. Hence if $a \neq 0, \infty$, g cannot be uniformly close to a on all of η.*

Proof. Recall that we are in case (ii). We consider two possibilities.

First, suppose there is a subarc $\eta_1 \subset \eta$, with $\eta_1 \cap \gamma = \varnothing$ which is not insignificant, in the sense that its extremes are points ζ_1, ζ_2 in $\mathsf{D}(z_0)$ with $\log(|\zeta_2/\zeta_1|) > 4\pi(L(r_0))^{-1}$. We then consider the second estimate of (51) at each $\zeta \in \eta'$ relative to $\mathsf{D}^\pm(z_0)$ as appropriate, using some branch of $\arg g(\zeta_1)$. We reach a contradiction since $L(r_0)\log r$ has changed by at least 3π while (by (50)) $\|\tilde{\mathcal{H}}\|_\infty \le \pi/2$. Once again, $\arg g(z)$ cannot be nearly constant on η'.

The more subtle case is when there is no significant subarc of η in any $\mathsf{D}'(z_0) \setminus \gamma$ (as with $w = \sin z$). Let

$$P(\eta) = \eta \bigcap (\gamma \cap \mathsf{D}(z_0)).$$

With $s > 0$ small but fixed, we have that $P(\eta) \cap (\cup_{\mathcal{I}} B(\zeta_I, s)) = \varnothing$, and may assume that $P(\eta)$ is discrete in γ. Suppose η contains a subarc η' having only its endpoints ξ, ξ' ($|\xi'| > |\xi|$) in $P(\eta)$, such that the (closure of the) domain (in one of

$\mathsf{D}^+(z_0)$ or $\mathsf{D}^-(z_0)$) bounded by η' and a subarc $\hat{\gamma} \subset \gamma$ contains at least one ζ_I, say $\{\zeta_I : I \in \mathcal{I}'\}$.

We claim that η' contains a subarc on which $\arg g(\zeta)$ varies by more than a fixed amount. Thus, we compute $\arg g(\xi') - \arg g(\xi)$ in the second formula of (51) in each of $\mathsf{D}^\pm(z_0) \setminus \gamma$, since one of these computations is with the change of $\arg g$ on η'.

Lemma 7 shows that the change, on $[\xi, \xi']$ relative to $\mathsf{D}^+(z_0)$, of the sum

$$-L(r_0)\log r + \tilde{\mathcal{H}}^+$$

in $\mathsf{D}^+(z_0)$ is the negative of that of the sum $+L(r_0)\log r + \tilde{\mathcal{H}}^-$ in $\mathsf{D}^-(z_0)$. But the second line of (51) shows that each of these is (up to $o(1)$) the change of $\arg g(z)$.

Finally, a closed curve consisting of simple arcs from ξ to ξ' in $\mathsf{D}^+(z_0)$ and then $\mathsf{D}^-(z_0)$ form a closed curve on which the change of $\arg g$ is $2\pi\,\mathrm{card}(\mathcal{I}')$. That means that $|\arg g(\xi') - \arg g(\xi)|$, when computed relative to $\mathsf{D}^\pm(z_0)$, is well-defined up to $o(1)$, and is at least $\pi\,\mathrm{card}(\mathcal{I}')$. Thus $\arg g(z)$ cannot be nearly constant on all of η' if $a \neq 0, \infty$.

This completes the proof of Theorem 4. □

4.4. **The asymptotic values.** It is easy to guarantee that $As(g) = \{0, \infty\}$.

Lemma 9. *Suppose there is a curve $\gamma \subset \Gamma^0 \cap \mathcal{U}$, on which $U(z) \to -\infty$. Then $As(g) = \{0, \infty\}$.*

Proof. By Theorem 2′, $\log|g(z)| \leq U(z) + C_0$ if $z \in \gamma \setminus E$ or if $z \in \gamma \cap E$ and the component of E containing z consists of a single ball centered at a zero of g. So by the construction in §2.5 we only need consider the situation that the component E' of E containing z consists of three balls: E' contains two zeros and one pole of g. Let $z_c \in E'$ be the zero of g associated to γ and $z_p \in E'$ a pole. Elementary geometry shows that $|z_c - z| \leq |z_p - z|$ when $z \in \gamma$. Thus by Theorem 3,

$$\log|g(z)| \leq U(z) + C_0, \quad (z \in \gamma),$$

and since $U \to -\infty$ on γ,

$$\log|g(z)| \to -\infty, \quad (z \to \infty,\ z \in \gamma).$$

Now with γ as above, let $\gamma' = -\gamma$ be a second curve on $\Gamma^* \subset \Gamma^\sharp$. Since $U(z) = -U(-z)$ ((4)), $U \to \infty$ on γ', and our argument shows that $\log|g| \to \infty$ on γ'. □

Let Γ be the subnetwork of $\Gamma^0 \cap \mathcal{U}$ on which $U \to -\infty$. In the next chapter, we guarantee that $\Gamma \neq \varnothing$.

5. Compositions with quasiconformal transformations

In this section g will be transformed by means of compositions with quasiconformal mappings to produce a quasiregular function F with asymptotic values precisely A^*. Recall that $A = A^* \setminus \{\infty\}$ (1) is analytic, and until §8.2 $A \subset B(0,2)$.

An analytic set A is obtained from Lusin's operations:

$$A = \bigcup_{\mathbb{N}^\mathbb{N}} \bigcap_{p \geq 1} \mathcal{S}_{n_1,\dots,n_p}, \tag{52}$$

where the sets $\mathcal{S}_{n_1,\dots,n_p}$ are closed (see [3] or [16], p. 207) and $\mathbb{N}^\mathbb{N}$ is the collection of infinite sequences of (positive) natural numbers. Sierpinski calls A the *nucleus* of the system $\mathcal{S}_{n_1,\dots,n_p}$.

We need a very precise description of the sets $\mathcal{S}_{n_1,\dots,n_p}$, and the situation is complicated since different authors often use different definitions. Our formulation uses the ideas of [16, Thm. 112] but our condition 2), which is indispensable here, is slightly different than in [16] and does not appear in [3]. For convenience, we sketch a proof, and refer the reader to [16], §86 for full details.

Let $\mathcal{N}_0$ be the collection of all finite sequences $(n_1,\dots,n_p)$.

Theorem A. *Let $A\subset\mathbb{C}$ be a nonempty analytic set in $\mathbb{C}$, and let a decreasing positive sequence $\{\delta_p\}$, $\delta_p\downarrow 0$ be given. Then we may write A as in (52) where*

1) each $\mathcal{S}_{n_1,\dots,n_p}$ is a closed set,
2) diam $(\mathcal{S}_{n_1,\dots,n_p})<\delta_p$,
3) $\mathcal{S}_{n_1,\dots,n_p,n_{p+1}}\subset\mathcal{S}_{n_1,\dots,n_p}$,
4) $\mathcal{S}_{n_1,\dots,n_p}\neq\varnothing$ *for all* $(n_1\dots,n_p)$ *in* $\mathcal{N}_0$.

Proof. The original definition in §82 of [16] uses only 1) and 3), and avoids 4). However, we are considering only nonempty analytic sets A. Thus for the moment assume that A is as in (52), where only 1) and 3) hold; we call these sets $\mathcal{S}'_{n_1,\dots,n_p}$, and convert them to ones which satisfy 2) and 4) as well (in [16], $\delta_p=1/p$).

To secure 2), let the $\{\delta_p\}$ be given, and introduce for each p a countable covering of $\mathbb{C}$ by closed balls $\{M_n^{(p)}\}$ of diameter $\delta_p/2\le\operatorname{diam}M_n^{(p)}<\delta_p$. Then for each $n\in\mathbb{Z}$, take $\mathcal{S}^o_n=M_n^{(2)}$, and $\mathcal{S}^o_{n_1,n_2}=\mathcal{S}^o_{n_1}=M_{n_1}^{(2)}$. This is augmented for $p>1$ by

$$\mathcal{S}^o_{n_1,n_2,\dots,n_{2p}}=\mathcal{S}^o_{n_1,n_2,\dots,n_{2p-1}}=\mathcal{S}'_{n_2,n_4,\dots,n_{2p-2}}\cap M^{(2p)}_{n_{2p-1}},$$

where $(n_1,n_2,\dots,n_{2p})$ range over $\mathcal{N}_0$. It is clear that the sets $\mathcal{S}^o$ are closed, and easy to check that the nucleus of $\mathcal{S}^o$ coincides with that of $\mathcal{S}'$. Thus 2) is satisfied.

Property 4) may be arranged as in [16], §86. Since $A\neq\varnothing$, choose some fixed $\omega_0\in A$. Then for any combination of k indices, $m(k)\in\mathcal{N}_0$, and any sequence $(n_1,n_2,\dots)$ of natural numbers, let $\mathcal{S}^o_{m(k),n_1,n_2,\dots}=\cap_{p\ge1}\mathcal{S}^o_{m(k),n_1,n_2,\dots,n_p}$, and set

$$\mathcal{S}^*_{m(k)}=\cup_{\mathbb{N}^{\mathbb{N}}}\mathcal{S}^o_{m(k),n_1,n_2,\dots}.$$

Sets of this nature must be included in (52). Whenever $\mathcal{S}^*_{m(k)}\neq\varnothing$, define $\mathcal{S}_{m(k)}=\overline{\mathcal{S}^*_{m(k)}}$. However when $\mathcal{S}^*_{m(1)}=\varnothing$, set $\mathcal{S}_{m(1)}=\omega_0\in A$ (since $A\neq\varnothing$), and if k_0+1 is the least integer with $\mathcal{S}^*_{m(k)}=\varnothing$, set $\mathcal{S}_{m(j)}=\omega_{m(k_0)}\in\mathcal{S}^*_{m(k_0)},\ j>k_0$.

□

The set of asymptotic values $\{0,\infty\}$ will be transformed into A^* by successive compositions with quasiconformal transformations. Recall that a homeomorphism φ is said to be K-quasiconformal ($K\ge1$) in $\mathbb{C}$ if it is in the Sobolev space $\varphi\in W^{1,2}_{\text{loc}}(\mathbb{C})$ and its (formal) derivatives satisfy $|\varphi'(z)|^2\le KJ_\varphi(z)$ a.e. $z\in\mathbb{C}$, where J_φ is the Jacobian determinant (see [1] for more properties).

The sequence $\{\delta_p\}$ in Theorem A arises from repeated use of an elementary lemma on quasiconformal mappings (known to Teichmüller and proved in [1], see also [4]). The various choices of $\{R,\delta\}$ depend on the sets (52).

Lemma A. *Let $2>K>1$ and $R>\delta$ be given. Consider the (K,δ,R) problem of finding a quasiconformal self-mapping of $\mathbb{C}$, φ, such that*

1) $\varphi(w)=w$ if $|w|\ge R$,

2) for any given α such that $|\alpha| \leq \delta$, we have $\varphi(w) = w + \alpha$ if $|w| \leq \delta$,
3) φ is K-quasiconformal.

Then, given either R or δ, there are choices of $\delta = \delta(R)$ or $R = R(\delta)$ which solve the problem.

We use this lemma in an iterative way. For a given $K > 1$, take a sequence $K_0 > K_1 > \cdots$ with

$$\prod_j K_j < K \tag{53}$$

(thus $K_j \downarrow 1$ (very) rapidly). Apply Lemma A with $\delta = \delta_0 = 2$ and $K = K_0$, thus obtaining R_0, and for $j \geq 1$ take $K = K_j, R_j = \delta_{j-1}$ to obtain $\delta_j \downarrow 0$. The point α_0 and in general, α_j $(j \geq 0)$ will be specified later in section §6. It is convenient to assume, if necessary by decreasing δ_{j-1} at each appearance, that

$$\log R_j + 10C_0 < \log R_{j-1} \qquad (j \geq 1),$$

where C_0 is from Theorem 2.

As in [5], this lemma will produce a large collection of quasiconformal mappings, all applied to $g(z)$ from Theorems 2 and 3. At each point z, the final quasiconformal mapping Ψ will have at $w = g(z)$ the form

$$\Psi(w) = \cdots \varphi_j \circ \cdots \varphi_1 \circ \varphi_0(w) \tag{54}$$

where φ_j is K_j-quasiconformal mapping of $\mathbb{C}$, so that

$$F(z) = \Psi \circ g(z) \tag{55}$$

is a continuous K-quasiregular mapping (which unlike a K-quasiconformal mapping it need not to be a homeormorphism, see [15]). The functions $\{\varphi_j\}$ are related to the desired behavior of F on a given branch $\gamma \subset \Gamma \subset \Gamma^0 \subset \Gamma^\sharp \cap \mathcal{U}$ (recall Figure 2), with Γ introduced at the end of §4.4. (We are simplifying notation, since in principle there should be different subscripts corresponding to each group of mappings in (54) associated to different paths γ. However the data $\{\delta_j, R_j, K_j\}$ is the same for each choice of φ_j.)

Thus let $\gamma \subset \Gamma \subset \Gamma^0 \subset \mathcal{U}$ be a path on which $z \to \infty$ and $g(z) \to 0$. We arrange the $\{\varphi_j\}$ and $a_n \to a, a_n \in A$, so that the orbit of $w = 0$ under F as z passes through γ will be

$$\begin{array}{ccccccc} 0 & \to & \varphi_0(0) & \to & \varphi_1(\varphi_0(0)) & \to & \cdots \\ 0 & \to & a_0 & \to & a_1 & \to & \cdots, \end{array} \tag{56}$$

leading to $F(z) \to a = \lim a_n$ as $z \to \infty, z \in \gamma$.

There is a natural way to correspond each path $\gamma \subset \Gamma$ to a point of the set A of (1), where $\Gamma \subset (\Gamma^0 \cap \mathcal{U})$ has been introduced at the end of §4.4. Each node of $\Gamma^0 \cap \mathcal{U}$ will be associated to a specific point $a \in A$ using Theorem A. Since $\Gamma^0 \cap \mathcal{U}$ is combinatorially a dyadic tree, its nodes correspond in a natural way to finite sequences of 0's and 1's with first entry 0. Let $\mathcal{B}$ be the countable collection of all such sequences. For each m, $\mathcal{B}$ has 2^m elements having m entries after the first 0. In turn, each such b has two *successors* b' and b'' with $m+1$ entries after the first 0: their first m entries coincide with those of b, and the final entry is 0 or 1. This leads to the standard binary graph $\mathcal{G}$ associated with $\mathcal{B}$. Following [16], we associate a finite sequence $(n_1, n_2 \ldots, n_p) \in \mathcal{N}_0$ to each $b \in \mathcal{B} \setminus \{0\}$, so that each node in a dyadic tree corresponds either to 0 or to a (unique) finite sequence of natural numbers. Let $b \in \mathcal{B}$. Then $b = 0$ and $b = 0.0\ldots0$ correspond to the number 0. Otherwise, $b = 0.\xi_1 \cdots \xi_j$, where $\xi_i \in \{0,1\}$, $1 \le i \le j$, and at least one $\xi_i \ne 0$, corresponds to $(n_1, \ldots, n_k) \in \mathcal{N}_0$, where

$$\sum_{i=1}^{j} \frac{\xi_i}{2^i} = \sum_{\ell=1}^{k} \frac{1}{2^{n_1+\cdots+n_\ell}}. \tag{57}$$

This correspondence is coherent in the sense that if b' is has the same binary expansion as b through the first ℓ appearances of 1, then the first ℓ digits of $\{n_1, \ldots, n_k\}$ and $\{n_1, \ldots, n_{k'}\}$ coincide.

In this way, every node of a dyadic tree is associated with a finite sequence of natural numbers or zero, and conversely, any finite sequence of natural numbers is associated to countably many nodes in a dyadic tree.

Once we have this correspondence, it is natural to exhaust $\mathcal{N}_0$ in the order induced by the tree structure of $\mathcal{B}$:

$$0;\ 0.0, 0.1;\ 0.00, 0.01, 0.10, 0.11;\ 0.000, 0.001, \ldots, \tag{58}$$

which produces the ξ_i in (57). Thus if $k \ge 1$, the k-th bifurcation node $z_k \in \Gamma^\sharp \cap \mathcal{U}$ (see §2 and Figure 2) corresponds to the k-th new element in this display of $\mathcal{B}$; this is a number from (57) with 1 as final entry. In turn, (57) associates this node to a set $S_{n_1,\ldots,n_p}$ in the system (52). The specific mappings $\{\varphi_j\}$ chosen below reflect the data (52) as well as $\{R_j\}$ from Lemma A and C_0 from Theorems 2 and 3.

The connection between (58) and the evolution of $\Gamma^\sharp$ through bifurcations can be made concrete, in that at a bifurcation node $z_k \in S(r_k) \cap \mathcal{U}$ the new branch of Γ^0 (which corresponds to an element of $\mathcal{B}$ with last digit one), originating at z_k is the arc of Γ^0 having larger argument. The curves $\gamma \in \Gamma \subset (\Gamma^0 \cap \mathcal{U})$ on which U tends to $-\infty$ will be paths which have infinitely many segments corresponding to elements in $\mathcal{B}$ of (58) having terminal digit one (see (61), which then applies for infinitely many p). On these curves on which $g \to 0$ (see proof of Lemma 9) are where F (and later f) attains asymptotic values $a \in A$.

We now define U at the $\{z_k\}$ and use the procedure (16) to extend U to the arcs of Γ^0 and then (3) and (4) to define U on all of $\mathbb{C}$.

Start with $z_0 = ir_0$ (recall (10)), the first node corresponding to $0 \in \mathcal{B}$ and define

$$U(z_0) = \log R_0 + 4C_0. \tag{59}$$

Note that other nodes z_p that correspond to $0.0\ldots0 \in \mathcal{B}$ will appear on each $S(r_k)$, $k \ge 1$ as in (58).

In fact once an arc of $\Gamma^\sharp \cap \mathcal{U}$ is assigned to Γ^*, the locus of local maxima, it never is subject to bifurcation as $|z| \to \infty$. To complete the definition of U on these 'free arcs' $\gamma^\sharp \subset \Gamma^* \cap \mathcal{U}$, we observe that its initial point lies at some bifurcation node $z_k\,(k > 0)$, where U will be defined in a moment (see (61)). As we follow along $\gamma^\sharp$ and encounter $z_{k\ell} = \gamma^\sharp \cap S(r_{k+\ell})\,(\ell \ge 1)$, we require that

$$U(z_{k\ell}) \ge U(z_k) + \ell, \tag{60}$$

and so we obtain infinitely many curves $\gamma^\sharp \subset \mathcal{U}$ on which $U \to \infty$.

In general, if the node z_k $(k \ge 1)$ corresponds to $b_k \in \mathcal{B}$ and $n(p) = (n_1, \ldots, n_p)$ is the sequence of natural numbers associated to b_k by (57), we define

$$U(z_k) = \log R_p + 2C_0, \tag{61}$$

which we copy at any successor z', which corresponds to b' (itself a successor of b_k) associated to the same sequence $n(p) = (n_1, \dots, n_p) \in \mathcal{N}_0$. In this way we also obtain countably many curves in Γ^0 on which $|U|$ does not have ∞ as an asymptotic value; on these g will have no asymptotic value, as suggested at the end of §2.1. On the other hand, the curves for which (61) for an increasing sequence of infinitely many p's are the ones that conform Γ, where $U \to -\infty$.

6. The families of quasiconformal mappings.

It follows from (52) that to any $a \in A$ corresponds a sequence $(n_1, n_2, n_3, \dots)$ so that $a = \cap_{p=1}^{\infty} \mathcal{S}_{n_1,\dots,n_p}$. We have already selected the $\{R_p, \delta_p\}$ in Lemma A.

Next, we identify the specific quasiconformal compositions $\{\varphi_j\}$ and the domains in which they act, all of which are in $\mathcal{U}$.

For each n consider Ω_n, the unbounded components of $\{|g(z)| < R_n\}$ that intersect $\Gamma^\sharp \cap \mathcal{U}$. Then each path $\gamma \subset \Gamma$ (on which $g \to 0$) passes through components $\mathcal{D}_n$ of Ω_n for each $n \geq 0$. Theorems 2 and 3 with (59) show that $|g(z)| > R_0$ on the arcs of Γ^* contained in $\{|z| > r_0\}$ which meet at the node z_0: thus $\Omega_0 \subset \mathcal{U}$, and Ω_0 is separated from $\partial\mathcal{U}$ by arcs of Γ^*.

Similar considerations show that two components $\mathcal{D}_n^i$ and $\mathcal{D}_n^j$ are separated by arcs of $\Gamma^\sharp$.

In general each component $\mathcal{D}_n^m$ of Ω_n will contain countably many components $\mathcal{D}_{n+1}^j$ of Ω_{n+1}, each of which will contain countably many disjoint components $\mathcal{D}_{n+2}^\ell$ of Ω_{n+2}, ..., imitating the process (52).

It is in these domains $\mathcal{D}_n$ that we introduce the mappings φ_n. Consider a nested chain of sets

$$\mathcal{D}_0^{n_1} \supset \mathcal{D}_1^{n_2} \supset \mathcal{D}_2^{n_3} \supset \cdots ;$$

these sets will then contain the asymptotic path γ at which the asymptotic value $a = \cap_{k=1}^{\infty} \mathcal{S}_{n_1,\dots,n_k}$ will be attained.

The quasiregular mapping F is defined on each chain $\mathcal{D}_p^{n_{p+1}}$ inductively in the domains $\overline{\Omega_p} \setminus \Omega_{p+1}$. First, take $F(z) = g(z)$ if $z \in \mathbb{C} \setminus \Omega_0$, observing from (59) and the role of C_0 in Theorem 2 that since $|g(z_0)| > \log R_0 + 2C_0$, z_0 is not in Ω_0. Fix $n \in \mathbb{N}$ and consider a domain $\mathcal{D}_0^n \subset \Omega_0$. We set

$$F(z) = \varphi_0 \circ g(z), \quad z \in \overline{\mathcal{D}_0^n} \setminus \Omega_1,$$

where φ_0 is the K_0-quasiconformal map given by Lemma A (which produced the original R_0), so that $\varphi_0(w) = w$ if $|w| \geq R_0$ and $\varphi_0(w) = w + a_1$ if $|w| \leq \delta_0$, where $a_1 \in A \cap \mathcal{S}_n$ as in (52), with (cf. Theorem A) diam $\mathcal{S}_n < \delta_1$. Thus, if $z \in \partial\Omega_1 \subset \mathcal{D}_0^n$ then $|g(z)| = R_1 = \delta_0$ and therefore $F(z) = g(z) + a_1$ (so by means of φ_0, $g(z)$ has been translated to $g(z) + a_1$, the first step of the chain (56)). More important, since $a_1 \in A \cap \mathcal{S}_n$, properties 2) and 3) of Theorem A ensure that when φ_1 is introduced as in (54), all possible choices $a_2 \in \mathcal{S}_{n,m} \subset \mathcal{S}_n$ satisfy

$$|a_2 - a_1| < \delta_1.$$

Notice that F is well-defined and continuous in $\mathbb{C} \setminus \Omega_1$. Indeed, for $m \neq n$, let $\mathcal{D}_0^m$ be another component of Ω_0. Then

$$F(z) = \varphi_0' \circ g(z), \quad z \in \overline{\mathcal{D}_0^m} \setminus \Omega_1,$$

where φ_0' is the K_0-quasiconformal map given by Lemma A with $\varphi_0'(w) = w$ if $|w| \geq R_0$ and $\varphi_0(w) = w + a_1'$ if $|w| \leq \delta_0$, where $a_1' \in A \cap \mathcal{S}_m$. We have already checked that $\mathcal{D}_0^m \cap \mathcal{D}_0^n = \varnothing$. Thus F is well-defined in $\mathbb{C} \setminus \Omega_1$ and continuous on $\partial\mathcal{D}_0^n$ since if $z \in \partial\mathcal{D}_0^n$ then $|g(z)| = R_0$ and therefore $F(z) = g(z)$.

Suppose that F is defined in $\mathbb{C} \setminus \Omega_p$, now we show how to extend it to $\mathbb{C} \setminus \Omega_{p+1}$. Let $\mathcal{D}_{p-1}^{n_p} \subset \Omega_{p-1}$ be given and consider a domain $\mathcal{D}_p^n \subset \mathcal{D}_{p-1}^{n_p}$. Assume that

$$F(z) = \Phi_{p-1} \circ g(z), \quad z \in \overline{\mathcal{D}_{p-1}^{n_p}} \setminus \Omega_p,$$

where $\Phi_{p-1} = \varphi_{p-1} \circ \cdots \circ \varphi_0$ is K_{p-1}-quasiconformal mapping chosen from Lemma A with data R_{p-1}, δ_{p-1}, such that in particular

$$F(z) = g(z) + a_p, \quad z \in \partial\Omega_p \subset \mathcal{D}_{p-1}^{n_p}, \tag{62}$$

and a_p is in $\mathcal{S}_{n_1,\ldots,n_p}$, a set of diameter less than δ_p. In $\mathcal{D}_p^n$ define

$$F(z) = \Phi_p \circ g(z), \quad z \in \overline{\mathcal{D}_p^n} \setminus \Omega_{p+1},$$

where Φ_p is a quasiconformal mapping defined by $\Phi_p = \varphi_p \circ \Phi_{p-1}$ and φ_p is a function given by Lemma A with $K = K_p$,

$$\varphi_p(w) = w, \quad \text{when} \quad |w - a_p| \geq R_p,$$

and

$$\varphi_p(w) = w + a_{p+1} - a_p, \quad \text{when} \quad |w - a_p| \leq \delta_p,$$

$a_{p+1} \in \mathcal{S}_{n_1,\ldots,n_p,n}$ and φ_p is well defined since a_{p+1} and a_p lie in $\mathcal{S}_{n_1,\ldots,n_p}$, a set of diameter less than δ_p.

The function F is well-defined in $\mathbb{C} \setminus \Omega_{p+1}$ since the domains $\mathcal{D}_p^k$ and $\mathcal{D}_p^n$ $k \neq n$ are disjoint, again using an appeal to (60).

Moreover, F is continuous on $\partial\Omega_p$. Let $z \in \partial\mathcal{D}_p^n \subset \mathcal{D}_{p-1}^{n_p}$, then $|g(z)| = R_p = \delta_{p-1}$ and by (62) and the definition of the function φ_p,

$$F(z) = \Phi_{p-1}(g(z)) = g(z) + a_p = \Phi_p(g(z)).$$

Finally, to verify (62) in these domains, consider a domain $\mathcal{D}_{p+1}^{\ell}$ contained in $\mathcal{D}_p^n$ and let $z \in \partial\mathcal{D}_{p+1}^{\ell} \subset \mathcal{D}_p^n$. Then $|g(z)| = R_{p+1} = \delta_p$ and

$$F(z) = \varphi_p(g(z) + a_p) = g(z) + a_{p+1}.$$

Thus $a = \cap_{p\geq 1}\mathcal{S}_{n_1,\ldots,n_p}$ is an asymptotic value of F, obtained on the path γ passing through the domains $\mathcal{D}_0^{n_1} \supset \mathcal{D}_1^{n_2} \supset \mathcal{D}_2^{n_3} \supset \cdots$.

7. Solution of Theorem 1

7.1. Nevanlinna Characteristic of g.

Theorem 5. *The meromorphic function g has order zero. Indeed,*

$$T(r, g) = o\left(\psi(r)\log^2 r\right) \quad (r \to \infty).$$

The Nevanlinna theory for subharmonic functions is discussed in [10] and adapts readily to δ-subharmonic functions. We first estimate the counting-function for the 'poles' in $B(r)$, $n(r, u)$. Formula (17) shows that the number of poles on any branch of $\Gamma^\sharp \cap B(r)$ is at most $L(r)\log r$, and (9) asserts that the number of branches in $B(r)$ is $O(L^{1/3}(r))$. This means that

$$n(r, g) = O(L^{4/3}(r)\log r).$$

Since L increases and E has density zero, we may integrate:

$$N(r, g) = O(L^{4/3}(r)\log^2 r).$$

To estimate $T(r,g) = m(r,g) + N(r,g)$ we consider the proximity function,

$$m(r,g) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |g(re^{i\theta})| d\theta.$$

By Theorem 2, $\log^+ |g(z)| = O(L(|z|))$ when $|z| \to \infty$ and $z \notin E$, so it is enough to check the contribution to $m(r,g)$ from integration over the exceptional set E.

However the estimate is then routine given the representation (38) of each $h_{\mathbf{J}}$, since we may perform an explicit integration over each of the of disks of $E \cap S(r)$ for each r with $S(r) \cap E \neq \varnothing : m(r,g) = O(L(r))$. Thus (on recalling (8))

$$T(r) = (1 + o(1))N(r,g) = O(L^{4/3}(r) \log^2 r) = o(\psi(r) \log^2 r) \quad (r \to \infty).$$

(Alternatively, since $T(r) = o(\psi(r) \log^2 r)$ when $S(r) \cap E = \varnothing$, we obtain it for the remaining r since T increases.)

7.2. **Asymptotic values of F.** We return to the function F which was obtained in §5. Recall that we still assume that $A = A^* \setminus \{\infty\}$ and $A \subset B(0,2)$.

Lemma 10. *The asymptotic values of F are $w = 0, w = \infty$ and values a which are limits of $g(z)$ on curves $\gamma \subset \Gamma^0 \cap \mathcal{U}$. In particular, $As(F) = A \cup \{\infty\} = A^*$.*

Proof. This depends on the form of the compositions (54) and (55) along with Theorem 4. Note that $\{0, \infty\} \subset As(F)$ since there are many curves in $\Gamma^\sharp$ in the lower half-plane on which $F \to 0, \infty$, with no other asymptotic values.

We first show that only asymptotic values associated by the procedure of §5 are asymptotic values of F. Let $F(z) \to a$ on η. Once we show that $g(z)$ itself has a limit a' on η, Theorem 4 shows that $a' = 0$ or $a' = \infty$. Since all compositions Ψ are the identity outside $B(R_0)$, we certainly have $a' = \infty$ when $a = \infty$.

Thus suppose $|a| < R_0$. Given $\delta > 0$, choose $r' > 0$ so that $|F(z) - a| < \delta$ for $z \in \eta(r')$ with $\eta(r')$ the unbounded component of $\eta \cap \{|z| > r'\}$.

The family of K-quasiconformal homeomorphisms of the sphere which fix $B(R_0)$ are uniformly Hölder continuous. Hence if Ψ is any fixed function of the class (54), any Ψ^{-1} image of $B(a,\delta)$ is contained in $B(\Psi^{-1}(a), C'\delta^\alpha)$, with $\alpha = \alpha(K)$.

It follows that if Ψ' is a choice of Ψ at $g(z')$, with $z' \in \eta \cap S(r')$, then $g(z) \in B(\Psi'^{-1}(F(z')), C'\delta^\alpha)$. Since $\delta \to 0$ as $r' \to \infty$ and the family of functions $\{\Psi\}$ is normal, g itself must have a limit on η. As we showed in §4.2, this means that η is contained in a tract on which $g \to 0$, so this tract also contains a curve $\gamma \subset \Gamma \subset \Gamma^0$. If $\gamma \subset \Gamma^0 \cap \mathcal{U}$ then the choice of compositions in (54) was made so that $F(z) \to a \in A$ in γ, and so in η. If, on the other hand, $\gamma \subset \Gamma^0$ in the lower half plane, then $F(z) = g(z)$ on γ and so $F(z) \to 0 \in A$ in γ, and therefore on η.

□

7.3. **Construction of f; completion of proof.** Let F be from (55). The meromorphic function f of Theorem 1 is obtained using standard techniques. Let $\sigma(z) = (F_{\bar{z}}/F_z)(z)$ and $f := F \circ \tau^{-1}$ where τ is the homeomorphic solution to

$$\tau_{\bar{z}}(z) = \sigma(z)\tau_z(z) \qquad \text{(Beltrami equation)},$$

normalized to fix 0, 1 and ∞. Then f is meromorphic in the plane.

Obviously $As(f) = A^*$, so we need only check (2).

We may avoid delicate distortion theorems on solutions to the Beltrami equation, since g is of slow growth (cf. (2)). A standard distortion theorem [1] (Hölder continuity) gives that if $w = \tau(z)$ satisfies this equation with $\|\sigma\|_\infty < \kappa < 1$, then there are $A = A(\kappa), M = M(\kappa)$ with

$$|\tau(z)| < A|z|^M \qquad (z \in \mathbb{C}). \tag{63}$$

Lemma 11. *The characteristic of the meromorphic function f satisfies (2).*

Proof. Since all quasiconformal compositions used in the previous section fix a neighborhood of $w = \infty$, we have $n(r, \infty, F) \equiv n(r, \infty, g)$ $(r > 0)$. We may suppose that K in (53) has been taken so that (63) holds with $M \leq 20$, and so $n(r, f) \leq n(Cr^{20}, g) = O(CL^{4/3}(r^{20})) \log^2 r$. Using (8), we find

$$N(r, f) := \int^r t^{-1} n(t, f)\, dt \leq N(21CL^{4/3}(r^{21})) = o(\psi(r) \log^2 r).$$

Similarly, $m(r, f) = O(L(Ar^M g))$, and since $T(r, f) = m(r, f) + N(r, f)$, a final appeal to (8) gives (2). □

8. Concluding remarks

In this section, we settle some loose ends.

8.1. **Functions of given order λ.** To construct functions of order $\lambda \neq 0$ requires a simple trick (we thank A. Eremenko for this suggestion). Let g be the meromorphic function (of order zero) just constructed, with $As(g) = \{0, \infty\}$. Let W be an unbounded open set with $d(W, \Gamma^\sharp) > 1$. Choose a sequence $\{w_n\}$ tending to ∞ in W whose exponent of convergence is λ (for example, let the number of w_n in $B(r)$ be asymptotic to r^λ).

Next, for each w_n choose b_n with $w_n - b_n$ tending so rapidly to zero that

$$\Pi(z) = \frac{1 - z/w_n}{1 - z/b_n}$$

is so close to one outside W that if $g_1(z) = g(z)\Pi(z)$, then $g - g_1 = o(1)$ and $\arg g(z) - \arg g_1(z) = o(1)$ as $z \to \infty$ in a neighborhood of $\Gamma^\sharp$. Then g_1 has order λ, and we may perform the compositions of §5 on g_1, yielding f_1 of order λ with A^* its asymptotic set.

8.2. **General analytic sets A^*.** To remove the assumption that the set A of (1) be contained in $B(0, 2)$, we construct a 'forest' of trees in $\mathcal{U}$. Thus, instead of $\Gamma^0 \subset \Gamma^\sharp$ being a single tree beginning on the positive imaginary axis (cf §2.3), there will be a countable collection of trees $\Gamma_{m,n} \subset \mathcal{U}$, $\Gamma_{m,n} \subset \Gamma^\sharp$ with asymptotic values being those of $A \cap B(m + ni, 2)$.

Since the each of the compositions in (54) operates in disjoint regions of the plane, the proofs of Lemma 10 and Lemma 11 apply as before.

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