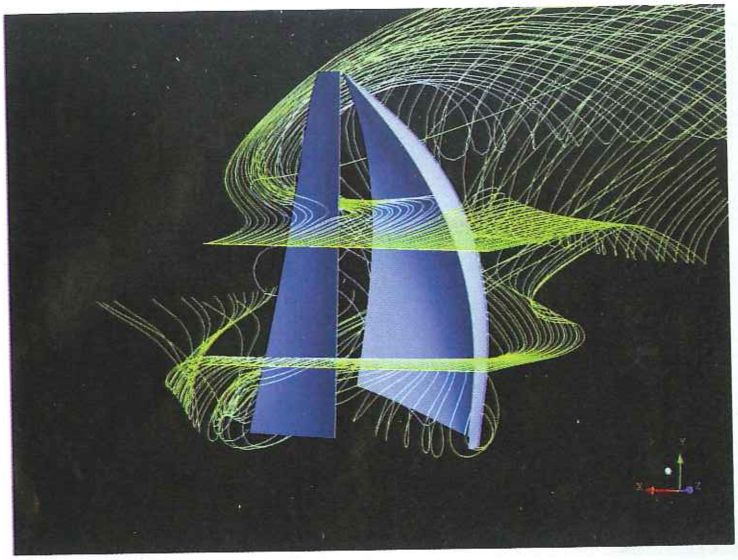

**Figure 9. Streamlines around mainsails and spinnaker in a downwind leg.**

several hundred), it was necessary to build the geometrical model—about 300 splines surfaces are needed to overlay the whole boat—to create the grid on the surface of all the elements of the boat reliable enough to enable the determination of the transition between laminar flow and turbulent flow regions, and consequently to generate the volumetric grid in external domain. The Navier-Stokes equations for incompressible viscous flows must be used to describe both water and wind dynamics and the consequent free surface, which need to be completed by additional equations that allow the computation of turbulent energy and its dissipation rate. These equations cannot be solved exactly to yield explicit solutions in closed form. Their approximate solution requires the introduction of refined discretization methods, which allow an infinite dimensional problem to be transformed into a big but finite dimensional one. The typical calculation, based on finite volume schemes, involved the solution of nonlinear problems with many millions of unknowns. Using parallel algorithms, 24 hours on parallel calculation platforms with 64 processors were necessary to produce a simulation, characterized by more than 160 million unknowns. A further computation is concerned with the simulation of the dynamical interaction between wind and sails by fluid-structure algorithms. These simulations enable the design team to eliminate those solutions that seem innovative and to go on with those that actually guarantee better performance. Moreover, by simulating the effects of aerodynamic interaction between two boats, one can determine the consistency of shadow regions (the areas with less wind because of the position of a boat with respect to the other), the flow perturbation, and the turbulence vorticity generated by the interaction of the air, thus obtaining useful information for the tactician as well. These studies aim to design a boat having an optimal combination of the four features that an America's Cup yacht must have: lightness, speed, resistance, and maneuverability necessary to change the race outcome.

A more in-depth description of the mathematical tools necessary for this kind of investigation is provided in the next section.

## Mathematical Models for America's Cup

The standard approach adopted in the America's Cup design teams to evaluate whether a design change (and all the other design modifications that this change implies) is globally advantageous, is based on the use of a Velocity Prediction Program (VPP), which can be used to estimate the boat speed and attitude for any prescribed wind condition and sailing angle. A numerical prediction of boat speed and attitude can be obtained by modeling the balance between the aerodynamic and hydrodynamic forces acting on the boat.

For example, on the water plane, a steady sailing condition is obtained imposing two force balances in the $x$ direction (aligned with the boat velocity) and the $y$ direction (normal to $x$ on the water plane) and a heeling moment balance around the centerline of the boat:

$$\begin{aligned} D_h + T_a &= 0, \\ S_h + S_a &= 0, \\ M_h + M_a &= 0, \end{aligned} \tag{1}$$

where $D_h$ is the hydrodynamic drag (along the course direction), $T_a$ is the aerodynamic thrust, $S_h$ is the hydrodynamic side force perpendicular to the course, $S_a$ is the aerodynamic side force, $M_h$ and $M_a$ are, respectively, the hydromechanical righting moment and the aerodynamic heeling moment around the boat mean line. The angle $\beta_Y$ between the course direction and the boat centerline is called *yaw* angle. The aerodynamic thrust and side force can be seen as a decomposition in the reference system aligned with the course direction of the aerodynamic lift and drag, which are defined on a reference system aligned with the apparent wind direction. Similar balance equations can be obtained for the other degrees of freedom.

In a VPP program, all the terms in system (1) are modeled as functions of boat speed, heel angle, and yaw angle. Suitable correlations between the degrees of freedom of the system and the different force components can be obtained based on different sources of data: experimental results, theoretical predictions, and numerical simulations.

The role of advanced computational fluid dynamics is to supply accurate estimates of the

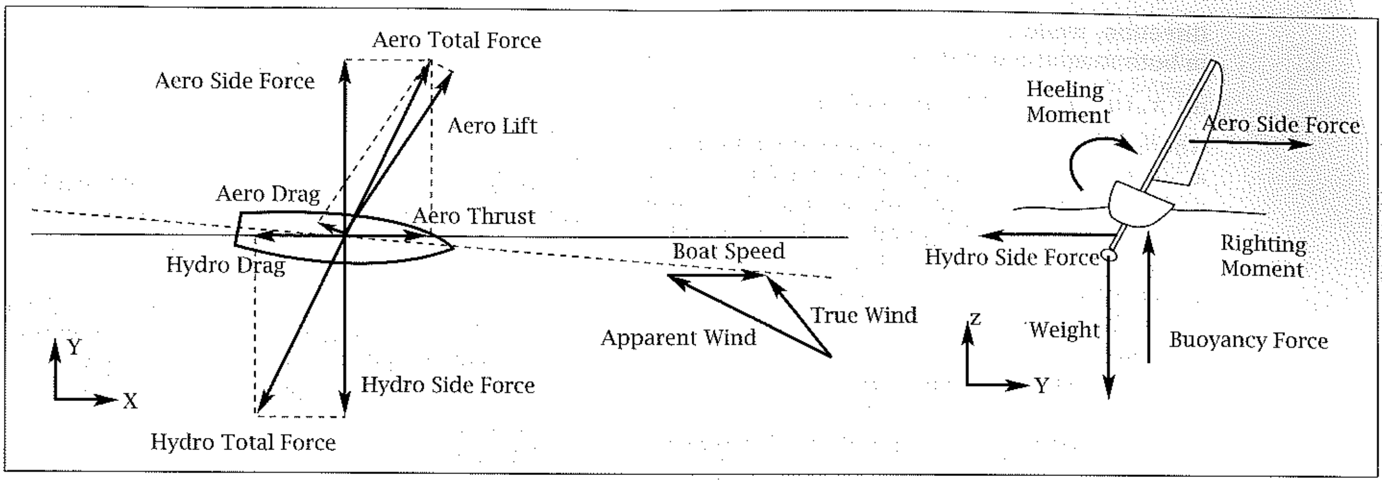


**Figure 10. Forces and moments acting on boat.**

forces acting on the boat in different sailing conditions in order to improve the reliability of the prediction of the overall performance associated with a given design configuration.

### The flow equations

Let $\Omega$ denote the three-dimensional computational domain in which we solve the flow equations. If $\hat{\Omega}$ is a region surrounding the boat $B$, the computational domain is the complement of $B$ with respect to $\hat{\Omega}$, that is $\Omega = \hat{\Omega} \backslash B$. The equations that govern the flow around $B$ are the density-dependent (or inhomogeneous) incompressible Navier-Stokes equations, which read:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{2}$$

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u) - \nabla \cdot \tau(u, p) = \rho g \tag{3}$$

$$\nabla \cdot u = 0 \tag{4}$$

for $x \in \Omega$ and $0 < t < T$, and where $\rho$ is the (variable) density, $u$ is the velocity field, $p$ is the pressure, $g = (0, 0, g)^T$ is the gravity acceleration, and $\tau(u, p) = \mu(\nabla u + \nabla u^T) - pI$ is the stress tensor with $\mu$ indicating the (variable) viscosity. The above equations have to be complemented with suitable initial conditions and boundary conditions. For the latter we typically consider a given velocity profile at the inflow boundary, with a flat far field free-surface elevation.

In the case we are interested in, the computational domain $\Omega$ is made of two regions, the volume $\Omega_w$ occupied by the water and that $\Omega_a$ occupied by the air. The interface $\Gamma$ separating $\Omega_w$ from $\Omega_a$ is the (unknown) free-surface, which may be a disconnected two-dimensional manifold if wave breaking is accounted for. The unknown density $\rho$ actually takes two constant states, $\rho_w$ (in $\Omega_w$) and $\rho_a$ (in $\Omega_a$). The values of $\rho_w$ and $\rho_a$ depend on the fluid temperatures, which are considered to be constant in the present model. The fluid viscosities $\mu_w$ (in $\Omega_w$) and $\mu_a$ (in $\Omega_a$) are constants that depend on $\rho_w$ and $\rho_a$, respectively.

The set of equations (2)-(4) can therefore be seen as a model for the evolution of a two-phase flow consisting of two immiscible incompressible fluids with constant densities $\rho_w$ and $\rho_a$ and different viscosity coefficients $\mu_w$ and $\mu_a$. In this respect, in view of the numerical simulation, we could regard equation (2) as the candidate for updating the (unknown) interface location $\Gamma$, then treat equations (3)-(4) as a coupled system of Navier-Stokes equations in the two sub-domains $\Omega_w$ and $\Omega_a$:

$$\frac{\partial (\rho_w u_w)}{\partial t} + \nabla \cdot (\rho_w u_w \otimes u_w) - \nabla \cdot \tau_w(u_w, p_w) = \rho_w g,$$

$$\nabla \cdot u_w = 0,$$

in $\Omega_w \times (0, T)$,

$$\frac{\partial (\rho_a u_a)}{\partial t} + \nabla \cdot (\rho_a u_a \otimes u_a) - \nabla \cdot \tau_a(u_a, p_a) = \rho_a g,$$

$$\nabla \cdot u_a = 0,$$

in $\Omega_a \times (0, T)$. We have set $\tau_w(u_w, p_w) = \mu_w(\nabla u_w + \nabla u_w{}^T) - p_w I$, while $\tau_a(u_a, p_a)$ is defined similarly.

The free surface $\Gamma$ is a sharp interface between $\Omega_w$ and $\Omega_a$, on which the normal components of the two velocities $u_a \cdot n$ and $u_w \cdot n$ should agree. Furthermore, the tangential components must match as well since the two flows are incompressible. Thus we have the following kinematic condition

$$u_a = u_w \quad \text{on } \Gamma. \tag{5}$$

Moreover, the forces acting on the fluid at the free-surface are in equilibrium. This is a dynamic condition and means that the normal forces on either side of $\Gamma$ are of equal magnitude and opposed direction, while the tangential forces must agree in both magnitude and direction:

(6) $\tau_a(u_a, p_a) \cdot n = \tau_w(u_w, p_w) \cdot n + \kappa\sigma n \quad \text{on } \Gamma,$

where $\sigma$ is the surface tension coefficient, that is a force per unit length of a free surface element acting tangentially to the free-surface. It is a property of the liquid and depends on the temperature as well as on other factors. The quantity $\kappa$ in (6) is the curvature of the free-surface, $\kappa = R_{t_1}^{-1} + R_{t_2}^{-1}$, where $R_{t_1}$ and $R_{t_2}$ are radii of curvature along the coordinates $(t_1, t_2)$ of the plane tangential to the free-surface (orthogonal to $n$).

### Coupling with a 6-DOF rigid body dynamical system

The attitude of the boat advancing in calm water or wavy sea is strictly correlated with its performance. For this reason, a state-of-the-art numerical tool for yacht design predictions should be able to account for the boat motion.

Following the approach adopted in [2, 3], two orthogonal cartesian reference systems are considered: an inertial reference system $(O, X, Y, Z)$, which moves forward with the mean boat speed, and a body-fixed reference system $(G, x, y, z)$, whose origin is the boat center of mass $G$, which translates and rotates with the boat. The $XY$ plane in the inertial reference system is parallel to the undisturbed water surface, and the $Z$-axis points upward. The body-fixed $x$-axis is directed from bow to stern, $y$ positive starboard, and $z$ upwards.

The dynamics of the boat in the 6 degrees of freedom are determined by integrating the equations of variation of linear and angular momentum in the inertial reference system, as follows

(7) $$m\ddot{X}_G = F$$

(8) $$\bar{\bar{T}}\bar{\bar{I}}\bar{\bar{T}}^{-1}\dot{\Omega} + \Omega \times \bar{\bar{T}}\bar{\bar{I}}\bar{\bar{T}}^{-1}\Omega = M_G$$

where $m$ is the boat mass, $\ddot{X}_G$ is the linear acceleration of the center of mass, $F$ is the force acting on the boat, $\dot{\Omega}$ and $\Omega$ are the angular acceleration and velocity, respectively, $M_G$ is the moment with respect to $G$ acting on the boat, $\bar{\bar{I}}$ is the tensor of inertia of the boat about the body-fixed reference system axes, and $\bar{\bar{T}}$ is the transformation matrix between the body-fixed and the inertial reference system (see [2] for details).

The forces and moments acting on the boat are given by

$$\begin{aligned} F &= F_{\text{Flow}} + mg + F_{\text{Ext}} \\ M_G &= M_{\text{Flow}} + (X_{\text{Ext}} - X_G) \times F_{\text{Ext}} \end{aligned}$$

where $F_{\text{Flow}}$ and $M_{\text{Flow}}$ are the force and moment, respectively, due to the interaction with the flow and $F_{\text{Ext}}$ is an external forcing term (which may model, e.g., the wind force on sails) while $X_{\text{Ext}}$ is its application point.

### The equations for wind-sails interaction

The sail deformation is due to the fluid motion: the aerodynamic pressure field deforms the sail surfaces and this, in its turn, modifies the flow field around the sails.

From a mathematical viewpoint, this yields a coupled system that comprises the incompressible Navier-Stokes equations with constant density $\rho = \rho_{air}$ (3-4) and a second order elastodynamic equation that models the sail deformation as that of a membrane. More specifically, the evolution of the considered elastic structure is governed by the classical conservation laws for continuum mechanics.

Considering a Lagrangian framework, if $\hat{\Omega}_s$ is the reference 2D domain occupied by the sails, the governing equation can be written as follows:

(9) $$\rho_s \frac{\partial^2 d}{\partial t^2} = \nabla \cdot \sigma_s(d) + f_s \quad in \quad \hat{\Omega}_s \times (0, T],$$

where $\rho_s$ is the material density, the displacement $d$ is a function of the space coordinates $x \in \hat{\Omega}_s$ and of the time $t \in [0; T]$, $\sigma_s$ are the internal stresses while $f_s$ are the external loads acting on the sails (these are indeed the normal stresses $\tau(u, p) \cdot n$ on the sail surface exerted by the flowfield). In fact, $\hat{\Omega}_s$ represents a wider (bounded and disconnected) domain that includes also the mast and the yarns as parts of the structural model. The boundary of $\hat{\Omega}_s$ is denoted by $\partial\hat{\Omega}_s$ and $[0; T] \subset \mathbb{R}^+$ is the time interval of our analysis. For suitable initial and boundary conditions and an assignment of an appropriate constitutive equation for the considered materials (defining $\sigma_s(d)$), the displacement field $d$ is computed by solving (9) in its weak form:

(10) $$\begin{aligned} \int_{\hat{\Omega}_s} \rho_s \frac{\partial^2 d_i}{\partial t^2} (\delta d_i) dx + \int_{\hat{\Omega}_s} \sigma^{II}{}_{ik} (\delta \epsilon_{ki}) dx \\ = \int_{\hat{\Omega}_s} f_{s\,i} (\delta d_i) dx, \end{aligned}$$

where $\sigma^{II}$ is the second Piola-Kirchoff stress tensor, $\epsilon$ is the Green-Lagrange strain tensor, and $\delta d$ are the test functions expressing the virtual deformation. The second coupling condition enforces the continuity of the two velocity fields, $u$ and $\frac{\partial d}{\partial t}$, on the sail surface.

### Fluid-structural coupling algorithm

As previously introduced, the coupling procedure iteratively loops between the fluid solver (passing sail velocities and getting pressure fields) and the structural solver (passing pressures and getting velocities and structural deformations) until the structure undergoes no more deformations because a perfect balance of forces and moments is reached. When dealing with transient simulations, this must be true for each time step, and the sail geometry evolves over time as a sequence of converged states. On the other hand, a steady simulation can be thought of as a transient one with an infinite time step, such that "steady" means a sort of average of the true (unsteady) solution over time. More formally, we can define two operators called `Fluid` and `Struct` that represent the fluid and structural solvers, respectively. In particular, `Fluid` can be any procedure that can solve the incompressible Navier-Stokes equations while `Struct` should solve a membrane-like problem, possibly embedding suitable nonlinear models to take into account complex phenomena such as, for example, the structural reactions due to a fabric wrinkle.

The fixed-point problem can be reformulated with the new operators as follows:

$$\texttt{Fluid}(\texttt{Struct}(p)) = p. \tag{11}$$

A resolving algorithm can be devised as follows. At a given iteration the pressure field on sails $p$ is passed to the structural solver (`Struct`), which returns the new sail geometries and the new sail velocity fields. Afterwards, these quantities are passed to the fluid solver (`Fluid`) which returns the same pressure field $p$ on sails. Clearly, the "equal" sign holds only at convergence. The resulting fixed-point iteration can be rewritten more explicitly as follows: Given a pressure field on sails $p_k$, do:

$$\begin{aligned} (G_{k+1}, \mathbf{U}_{k+1}) &= \texttt{Struct}(p_k) \\ \tilde{p}_{k+1} &= \texttt{Fluid}(G_{k+1}, \mathbf{U}_{k+1}) \\ p_{k+1} &= (1-\theta_k)p_k + \theta_k \tilde{p}_{k+1} \end{aligned} \tag{12}$$

where $G_{k+1}$ and $\mathbf{U}_{k+1}$ are the sail geometry and the sail velocity field at step $k+1$, respectively, while $\theta_k$ is a suitable acceleration parameter.

Even though the final goal is to run an unsteady simulation, the fluid-structure procedure has to run some preliminary steady couplings to provide a suitable initial condition. The steady run iterates until a converged sail shape and flow field are obtained, where converged means that there does exist a value of $k_c$ such that (11) is satisfied for every $k > k_c$ (within given tolerances on forces and/or displacements). When running steady simulations the velocity of the sails is required to be null at each coupling, thus somehow enforcing the convergence condition (which prescribes null velocities at convergence). This explains why convergence is slightly faster when running steady simulations with respect to transient ones (clearly only when such a solution reflects a steady state physical solution).

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