High Frequency Wave Propagation in Discrete Media

Fabricio Macià

Abstract: We present an explicit computation of limits of energy densities associated to sequences of solutions to semi-discrete wave equations. It is proved that the propagation law (at length scale h) of these asymptotic energy densities differs considerably from that corresponding to the continuous wave equation. This difference is present even in the constant-coefficient case, where **Propagation Failure** may take place: we show the existence of solutions of the semi-discrete wave equation whose energy concentrates at a *fixed* point in space *for every time* t. Finally, some formal analogies between wave propagation in discrete and highly-oscillating media are presented. The main tool used is the theory of **Wigner Measures** associated to discrete functions introduced in [4].

1. Introduction and statement of the problem

In recent years, much progress has been made in the understanding of the mechanisms of propagation of concentration and oscillation effects in conservative systems. Explicit computations of homogenization limits for energy densities (quadratic functions of solutions) have been performed for very general classes of conservative (pseudo)differential systems. This analysis has been possible thanks to the use of **Wigner measures** (see [2] and the references therein).

In the context of the **wave equation**:

$$\begin{cases} \rho(x)\,\partial_t^2 u(t,x) - \Delta_x u(t,x) = 0 \text{ for } t \in \mathbf{R} \text{ and } x \in \mathbf{R}^d, \\ u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1, \end{cases} \tag{1}$$

with $\rho \in C^2\left(\mathbf{R}^d\right)$ satisfying $0 < \alpha \leq \rho(x) \leq \beta$ for every $x \in \mathbf{R}^d$, a typical (and illustrating) result in this direction is:

Proposition 1.1 *Consider the sequence* $\left(u^h\right)_{h>0}$ *of solutions of (1) corresponding to the initial data:*

$$u^h|_{t=0} = \frac{1}{h^{d/4-1}}\phi\left(\frac{x-x_0}{\sqrt{h}}\right)e^{ix\cdot\xi_0/h},\ \partial_t u^h|_{t=0} = 0,$$

for some $\phi \in H^1\left(\mathbf{R}^d\right)$ *and* $x_0, \xi_0 \in \mathbf{R}^d$, $\xi_0 \neq 0$. *Then the energy of* $u^h(t,\cdot)$ *concentrates near the points* $x(\pm t)$, *corresponding to the ray of (1) issued from the initial data* (x_0, ξ_0). *In other words:*

$$\lim_{h\to 0}\int_{|x-x(\pm t)|>\delta}\left[\rho(x)\left|\partial_t u^h(t,x)\right|^2 + \left|\nabla_x u^h(t,x)\right|^2\right]dx = 0 \text{ for every } t\in\mathbf{R} \text{ and every } \delta > 0,$$

where $x(t)$ *solves*

$$\begin{cases} \dot{x}(t) = \rho(x(t))^{-1}\xi(t), \\ \dot{\xi}(t) = 1/2\rho(x(t))^{-2}|\xi(t)|^2\nabla_x\rho(x(t)), \end{cases} \tag{2}$$

with $(x(0), \xi(0)) = (x_0, \xi_0)$.

This is a very special case of Theorem 6.1 in [2]. When particularized to the wave equation, this Theorem states that the asymptotic energy density of any energy-bounded sequence of solutions to (1) propagates along **rays** (i.e. solutions to (2)).

Here we shall perform a similar analysis for a **semi-discrete** approximation of the wave equation (1):

$$\begin{cases} \rho(hn)\,\partial_t^2 U_n^h(t) - L_h U_n^h(t) = 0 \text{ for } t\in\mathbf{R} \text{ and } n\in\mathbf{Z}^d, \\ U^h(0) = U_0^h,\ \partial_t U^h(0) = U_1^h; \end{cases} \tag{3}$$

we have denoted by L_h the following discretization of Δ_x:

$$L_h U_n^h := \frac{1}{h^2}\sum_{j=1}^{d}\left[U_{n+e_j}^h + U_{n-e_j}^h - 2U_n^h\right],$$

the vectors e_1,..., e_d being those of the canonical basis of $\mathbf{Z}^d$. The **energy** of a solution U^h is now defined as

$$E^h\left(U^h\right) := \frac{h^d}{2}\sum_{n\in\mathbf{Z}^d}\left[\rho(hn)\left|\partial_t U_n^h\right|^2 + \sum_{j=1}^{d}\left|\frac{U_{n+e_j}^h - U_n^h}{h}\right|^2\right].$$

As we mentioned above, we are interested in the problem of **propagation** of **concentration** and **oscillation** effects in sequences of solutions of this semi-discrete wave equation. These are described by means of the notion of **asymptotic energy density**: given a sequence $\left(U^h\right)_{h>0}$ of solutions of (3) with $E^h\left(U^h\right) \leq C$, we consider the following **energy density** measures:

$$e_t^h\left[U^h\right] := \frac{h^d}{2}\sum_{n\in\mathbf{Z}^d}\left[\rho(hn)\left|\partial_t U_n^h(t)\right|^2 + \sum_{j=1}^{d}\left|\frac{U_{n+e_j}^h(t) - U_n^h(t)}{h}\right|^2\right]\delta_{hn},$$

where $t \in \mathbf{R}$ and δ_{hn} is the Dirac mass centered at hn. Remark that $e_t^h\left[U^h\right]\left(\mathbf{R}^d\right) = E^h\left(U^h\right)$; combining the uniform boundedness of the energies of the U^h and a simple equicontinuity argument we can conclude that, modulo a subsequence, the $e_t^h\left[U^h\right]$ converge weakly to some measure ν_t for every $t \in \mathbf{R}$. We shall refer to ν_t as the **asymptotic energy density** of the sequence $\left(U^h\right)_{h>0}$.

We shall show that the propagation law of the ν_t differs considerably from that of the continuous wave equation. In particular, we shall prove that **propagation failure** may take place.

2. Wigner measures and propagation of the asymptotic energy density

In order to compute ν_t from a quantity that solely depends on the initial data $\left(U^h(0), \partial_t U^h(0)\right)$ it will be useful to deal with **Wigner measures** associated to a sequence of discrete functions (see [4]). Let $V^h \in L^2\left(h\mathbf{Z}^d\right)$; we define

$$M^h\left[V^h\right](x,\xi) := \frac{h^d}{(2\pi)^d} \sum_{m,n\in\mathbf{Z}^d} \overline{V_m^h} V_n^h e^{i(m-n)\cdot\xi} \delta_{hm}(x), \text{ for } x, \xi \in \mathbf{R}^d.$$

Now let $\left(U^h\right)_{h>0}$ be a sequence of solutions to (3); we define its **microlocal energy density** $\varepsilon_t^h\left[U^h\right]$ by

$$\varepsilon_t^h\left[U^h\right] := M^h\left[\partial_t U^h(t)\right] + \sum_{j=1}^d M^h\left[\frac{U_{n+e_j}^h(t) - U_n^h(t)}{h}\right].$$

Remark that $\varepsilon_t^h\left[U^h\right]$ is quadratic in U^h and $\int_{[-\pi,\pi]^d} \varepsilon_t^h\left[U^h\right](\cdot,\xi)\,d\xi = e_t^h\left[U^h\right]$. This relation passes to the limit: in [4] it is proved that, modulo a subsequence, $\varepsilon_t^h\left[U^h\right]$ converges in the sense of distributions to a positive measure $\mu_t(x,\xi)$ which is $2\pi\mathbf{Z}^d-$periodic in ξ and satisfies $\int_{[-\pi,\pi]^d} \mu_t(\cdot, d\xi) = \nu_t$, for every $t \in \mathbf{R}$. The measure μ_t is called the **Wigner measure** of the sequence $\left(U^h(t)\right)_{h>0}$.

The presence of the dual variable ξ allows us to keep track of the direction of propagation of the oscillations of the U^h, and thus to obtain a transport law for μ_t. For the sake of simplicity, we sate it for the case of initial data with vanishing velocity (see [4] for a complete result):

Proposition 2.1 *Let $\left(U^h\right)_{h>0}$ be a energy-bounded sequence of solutions to (3) such that $\partial_t U^h(0) = 0$; let μ_t be the Wigner measure of the U^h. Then μ_t is completely determined by μ_0 through the formula:*

$$\int_{\mathbf{R}^d\times\mathbf{R}^d} a(x,\xi)\,d\mu_t = \frac{1}{2}\int_{\mathbf{R}^d\times\mathbf{R}^d} \left[a\left(\Phi_t(x,\xi)\right) + a\left(\Phi_{-t}(x,\xi)\right)\right] d\mu_0,$$

for $a \in C_c^\infty\left(\mathbf{R}^d \times \mathbf{R}^d\right)$, where $\Phi_t(x,\xi) = (x(t), \xi(t))$ is the solution of the system

$$\left\{\begin{array}{l} \dot{x}_j(t) = \rho(x(t))^{-1} \sin\xi_j(t), \\ \dot{\xi}_j(t) = 2\rho(x(t))^{-2}\,\partial_{x_j}\rho(x(t)) \sum_{k=1}^d \sin^2(\xi_k(t)/2), \end{array}\right. \tag{4}$$

which satisfies $(x(t), \xi(t)) = (x, \xi)$.

In particular, taking $U_n^h(0) := u^h(0, hn)$ where u^h is the function defined in Proposition 1.1 (for a sufficiently smooth ϕ) we obtain the discrete analog of the propagation result of section 1:

Corollary 2.2 *Given $x_0, \xi_0 \in \mathbf{R}^d$, $\xi_0 \neq 0$ there exists a sequence of solutions $\left(U^h\right)_{h>0}$ of the semi-discrete wave equation such that its energy concentrates near the points $x(\pm t)$, the curve $x(t)$ being the ($x-$component) of the solution of (4) corresponding to the initial data (x_0, ξ_0). More precisely:*

$$\lim_{h\to 0} e_t^h\left[U^h\right] = \frac{1}{2}\left[\delta(x - x(t)) + \delta(x - x(-t))\right] \text{ for every } t \in \mathbf{R}.$$

Systems (2) and (4) may considerably differ if ξ_0 is not close to the origin; this leads to completely different propagation laws for the energy density in the continuous and semi-discrete case. A shocking difference already occurs for $\rho \equiv \rho_0 > 0$ constant.

3. Propagation failure

Indeed, in the constant-coefficient case the rays $(x(t), \xi(t))$ associated to the semi-discrete system (3) are straight-line segments in the $x-$component:

$$x_j(t) = t\rho_0^{-1}\sin\xi_{0,j} + x_0,\ \xi(t) = \xi_0.$$

Taking as ξ_0 the vector all of whose components are π we obtain $x(t) = x_0$ for all $t \in \mathbf{R}$. Corollary 2.2 then gives:

Theorem 3.1 *Suppose $\rho \equiv \rho_0 > 0$ is constant. Then, for every $x_0 \in \mathbf{R}^d$ there exists a sequence $\left(U^h\right)_{h>0}$ of solutions concentrated around x_0 for every time t:*

$$\lim_{h\to 0}\frac{h^d}{2}\sum_{|hn-x_0|>\delta}\left[\rho_0\left|\partial_t U_n^h(t)\right|^2 + \sum_{j=1}^d\left|\frac{U_{n+e_j}^h(t) - U_n^h(t)}{h}\right|^2\right] = 0$$

for every $t \in \mathbf{R}$ and every $\delta > 0$.

Similarly, one can construct sequences of solutions of the semi-discrete equation that propagate arbitrarily slowly. This singular behavior is not possible in the framework of the continuous wave equation, since rays in this case propagate at a constant speed independent of ξ_0. This propagation failure result is relevant in the context of the numerical approximation of controls for the wave equation (see [4] and the references therein for more details on this).

However, we would like to emphasize that propagation failure may occur for a wave equation with a highly oscillating density ρ:

$$\begin{cases} \rho(x/h)\,\partial_t^2 u^h(t,x) - \Delta_x u^h(t,x) = 0 \text{ for } t \in \mathbf{R} \text{ and } x \in \mathbf{R}^d, \\ u^h|_{t=0} = u_0^h,\ \partial_t u^h|_{t=0} = u_1^h, \end{cases} \tag{5}$$

with $\rho \in L^\infty\left(\mathbf{R}^d\right)$, $\mathbf{Z}^d-$periodic and satisfying $0 < \alpha \le \rho(y) \le \beta$. The following holds:

Theorem 3.2 (P. Gérard [1]) *For generic ρ in a neighborhood of a positive constant and for every $x_0 \in \mathbf{R}^d$ there exists a sequence $\left(u^h\right)_{h>0}$ of solutions of (5) such that*

$$\lim_{h\to 0}\int_{|x-x_0|>\delta}\left[\rho(x/h)\left|\partial_t u^h(t,x)\right|^2 + \left|\nabla_x u^h(t,x)\right|^2\right]dx = 0 \text{ for every } t \in \mathbf{R} \text{ and every } \delta > 0.$$

Other aspects of the (formal) analogy between the propagation theories in discrete and highly-oscillating media at the critical length scale h can be found in [3].

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Fabricio Macià
Departamento de Matemática Aplicada,
Universidad Complutense de Madrid,
Fac. Matemáticas, Avda. Complutense s/n, 28040 Madrid,
Spain.
e-mail: fabricio_macia@mat.ucm.es