SEMICLASSICAL ANALYSIS

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EXERCISES

1. Let $e^{i\frac{t}{h}\widehat{H}_h}$ denote the propagator for the semiclassical Schrödinger equation corresponding to $\widehat{H}_h := -\frac{h^2}{2}\Delta$. Show that for every $u \in \mathcal{S}\left(\mathbb{R}^d\right)$ the following holds:
$$\lim_{t\to\infty}\left\| e^{i\frac{t}{h}\widehat{H}_h}u - \frac{1}{(iht)^{d/2}}e^{i\frac{|x|^2}{2ht}}\widehat{u}\left(\frac{x}{ht}\right)\right\|_{L^2\left(\mathbb{R}^d\right)} = 0.$$

2. Let Q be a symmetric, nonsingular $d\times d$ matrix with real entries. Show that the Fourier transform of
$$e^{\frac{i}{2}Qx\cdot x}$$
is
$$\left(\frac{2\pi}{|\det Q|}\right)^{d/2} e^{i\frac{\pi}{4}\operatorname{sgn} Q} e^{-\frac{i}{2}Q^{-1}\xi\cdot\xi},$$
where $\operatorname{sgn} Q$ stands for the signature of Q.

3. Let $u(x) := \rho(x)\, e^{i\frac{\xi_0}{h}\cdot x}$ with $\rho\in L^2\left(\mathbb{R}^d\right)$ and $\xi_0\in\mathbb{R}^d$. Show that:
$$e^{ith\Delta/2}u(x) = e^{i\frac{\xi_0}{h}\cdot x}e^{it\frac{|\xi_0|^2}{2h}}\left(e^{ith\Delta/2}\rho\right)(x - t\xi_0).$$

4. Let $u(x) := \left(\pi\sqrt{h}\right)^{-d/2} e^{-\frac{|x-x_0|^2}{2h}} e^{i\frac{\xi_0}{h}\cdot x}$ with $(x_0,\xi_0)\in\mathbb{R}^d\times\mathbb{R}^d$. Show that:
$$\left|e^{ith\Delta/2}u\right|^2(x) = \frac{1}{\left[\pi\sqrt{h\left(1+t^2\right)}\right]^d}\exp\left(-\frac{|x - x_0 - t\xi_0|^2}{h\left(1+t^2\right)}\right).$$

5. Let $v\in L^2\left(\mathbb{R}^d\right)$ and set $\mathcal{F}_h v(\xi) := (2\pi h)^{d/2}\,\widehat{v}(\xi/h)$ for $h>0$.

   (a) Show that $|\mathcal{F}_h v|^2\, d\xi$ is a probability measure whenevere $|v|^2\, dx$ is.

   (b) Let $u(x) := \frac{1}{h^{d/4}}\rho\left(\frac{x-x_0}{\sqrt{h}}\right) e^{i\frac{\xi_0}{h}\cdot x}$ for $(x_0,\xi_0)\in\mathbb{R}^d\times\mathbb{R}^d$ and $\rho\in L^2\left(\mathbb{R}^d\right)$. Show that
$$|\mathcal{F}_h u|^2(\xi) = \frac{1}{\left(2\pi\sqrt{h}\right)^{d/2}}\left|\widehat{\rho}\left(\frac{\xi-\xi_0}{\sqrt{h}}\right)\right|^2.$$

(c) Show that for every $v \in L^2\left(\mathbb{R}^d\right)$,

$$\left|\mathcal{F}_h e^{ith\Delta/2} v\right|^2 = \left|\mathcal{F}_h v\right|^2 .$$

6. Let $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$. Let $a(x,\xi) := \varphi(\xi)$ for $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$. Show that:

$$\operatorname{Op}_h(a) = \varphi(hD_x).$$

7. Suppose that $p \in C^\infty\left(\mathbb{R}^d \times \mathbb{R}^d\right)$ is a polynomial of degree at most two. Show that for every $a \in \mathcal{S}\left(\mathbb{R}^d \times \mathbb{R}^d\right)$ there exist $r \in \mathcal{S}\left(\mathbb{R}^d \times \mathbb{R}^d\right)$ such that:

$$\operatorname{Op}_h(p)\operatorname{Op}_h(a) = \operatorname{Op}_h(pa) + \frac{h}{2i}\operatorname{Op}_h(\{p,a\}) + h^2\operatorname{Op}_h(r),$$

$$\operatorname{Op}_h(a)\operatorname{Op}_h(p) = \operatorname{Op}_h(pa) - \frac{h}{2i}\operatorname{Op}_h(\{p,a\}) + h^2\operatorname{Op}_h(r),$$

conclude that:

$$\left[\operatorname{Op}_h(p), \operatorname{Op}_h(a)\right] = \frac{h}{i}\operatorname{Op}_h(\{p,a\}).$$

8. Compute the semiclassical measures with respect to the scale h of the following sequences (u_h) in $L^2\left(\mathbb{R}^d\right)$.

   (a) $u_h(x) = \rho(x)\, e^{i\frac{\xi_0}{h}\cdot x}$, $\rho \in L^2\left(\mathbb{R}^d\right)$.

   (b) $u_h(x) = \frac{1}{h^{d/2}}\rho\left(\frac{x-x_0}{h}\right)$, $\rho \in L^2\left(\mathbb{R}^d\right)$.

   (c) $u_h(x) = \frac{1}{h^{d/4}}\rho\left(\frac{x-x_0}{\sqrt{h}}\right) e^{i\frac{\xi_0}{h}\cdot x}$, $\rho \in L^2\left(\mathbb{R}^d\right)$.

   (d) $u_h(x) = \rho(x)\, e^{i\frac{S(x)}{h}}$, $\rho \in L^2\left(\mathbb{R}^d\right)$, $S \in C^1\left(\mathbb{R}^d\right)$.

   (e) (u_h) strongly convergent in $L^2\left(\mathbb{R}^d\right)$: $\|u_h - u\|_{L^2\left(\mathbb{R}^d\right)} \to 0$ as $h \to 0^+$ for some $u \in L^2\left(\mathbb{R}^d\right)$.

9. Let (ε_n) be a sequence of positive reals such that $\lim_{n\to\infty}\varepsilon_n = 0$. Determine when the sequences defined in the previous exercise are ε_n-oscillating.

10. Let $a \in \mathcal{S}\left(\mathbb{R}^d \times \mathbb{R}^d\right)$. Let $\operatorname{Op}_h^{\mathrm{WP}}(a)$ be the operator defined from a by wave-packet quantization. Show:

$$\lim_{h\to 0^+}\left\|\operatorname{Op}_h^{WP}(a) - \operatorname{Op}_h(a)\right\|_{L^2\left(\mathbb{R}^d\right)\to L^2\left(\mathbb{R}^d\right)} = 0.$$

11. Let (u_n) be a bounded, h_n-oscillating sequence in $L^2\left(\mathbb{R}^d\right)$ that is compact at infinity. Suppose $u_n \rightharpoonup u$ weakly in $L^2\left(\mathbb{R}^d\right)$ and let μ be the semiclassical measure of $(u_n - u)$. Show that $\mu = 0$ if and only if $u_n \to u$ strongly in $L^2\left(\mathbb{R}^d\right)$. Show that the conclusion may fail if (u_n) is not h_n-oscillating.

12. A bounded sequence (u_n) in $L^2(\mathbb{R}^d)$ is called strictly h_n-oscillating if it is h_n-oscillating and moreover, for every $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$\limsup_{n\to\infty} \int_{|\xi|<\delta/h_n} \left|\widehat{\phi u_n}(\xi)\right|^2 d\xi \to 0, \quad \text{as } \delta \to 0^+.$$

   (a) Show that if (u_n) is strictly h_n-oscillating then every semiclassical measure μ of the sequence (u_n) (at the scale h_n) satisfies: $\mu(\{\xi = 0\}) = 0$.

   (b) Give an example of a sequence (u_n) such that there is no (h_n) such that (u_n) is strictly h_n-oscillating.

   (c) Show that given a sequence (u_n) that is bounded in $L^2(\mathbb{R}^d)$ it is possible to find (h_n) such that (u_n) is h_n-oscillating.

13. Suppose (u_n) and (v_n) are two sequences having as semiclassical measures: μ_1 and μ_2 respectively. Suppose that μ_1 and μ_2 are mutually singular (that is $\mu_1(\Omega)\mu_2(\Omega) = 0$ for every Borel set $\Omega \subset \mathbb{R}^{2d}$)

   (a) Show that for every $a \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ one has:

$$\lim_{n\to\infty} (\mathrm{Op}_h(a) u_n | v_n)_{L^2(\mathbb{R}^d)} = 0.$$

   (b) Deduce from the statement above that the semiclassical measure of $(u_n + v_n)$ is $\mu_1 + \mu_2$.

   (c) Suppose moreover that $\widehat{u_n}$ and $\widehat{v_n}$ are supported on a ball of radius $1/h_n$ centered at the origin. Show that:

$$u_n \overline{v_n} \rightharpoonup 0 \quad \text{as } n \to \infty \text{ in } \mathcal{D}'(\mathbb{R}^d).$$

14. Consider the operator $\widehat{H}_h := -h^2\Delta + |x|^2$ on $L^2(\mathbb{R}^d)$ corresponding to the Hamiltonian $H(x,\xi) = |\xi|^2 + |x|^2$. The spectrum of $\widehat{H}_h$ consists only of the eigenvalues $E_k^h = (2k+d)h$ with $k = 0, 1, \ldots$. Let $h_n \to 0$ and $k_n \to \infty$. Denote by ψ_n an eigenfuncton of $\widehat{H}_h$ corresponding to the eigenvalue $E_{k_n}^{h_n}$ with $\|\psi_n\|_{L^2(\mathbb{R}^d)} = 1$. Suppose $E_{k_n}^{h_n} \to E$ when $n \to \infty$ and that μ is a semiclassical measure of the sequence (ψ_n) with respect to the scale h_n.

   (a) Show that μ is supported on $H^{-1}(E)$.

   (b) Show that μ is invariant by the classical Hamiltonian flow ϕ_t of H.

   (c) Conclude that for $d = 1$

$$\int_{\mathbb{R}\times\mathbb{R}} a(x,\xi)\,\mu(dx, d\xi) = \frac{1}{2\pi\sqrt{E}} \int_0^\pi a\left(\sqrt{E}(\cos 2t, -\sin 2t)\right) dt.$$