SEMICLASSICAL ANALYSIS

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Course duration: 16 hours (8 sessions of 2 hours each).

Program

1. **The fundamental equations of motion and existence of dynamics.**
   (a) Classical Mechanics. Hamiltonian systems, classical completeness.
   (b) Quantum Mechanics. The Schrödinger equation. Essential self-adjointness and quamtum completeness.
   (c) Basic notions on the Fourier transform. The Schwartz class and tempered distributions. Fourier transform of quadratic exponentials. The fundamental solution to the Schrödinger equation in the constant coefficient case.
2. **The Quantum-Classical correspondence principle**.
   (a) Weak convergence of probability measures.
   (b) Semiclassical propagation of wave-packets. Statement of the problem.
   (c) The case $V \equiv 0$.
   (d) The W.K.B. method.
3. **Quantization and semiclassical pseudo-differential operators.**
   (a) Motivations.
   (b) The Weyl quantization.
   (c) L^2-boundedness.
   (d) The product formula.
4. **The Semiclassical Limit.**
   (a) The Wigner distribution.
   (b) Semiclassical measures.
   (c) Wigner and Liouville equations.

(d) Semiclassical measures on a Riemannian manifold.

Comments on the bibliography. A general reference for the course is the book [3]. Other textbooks on the subject that develop a different approach are [2] and [10]. The basic results on essential sel-adjointness and a proof of Stone's theorem can be found in [11]. A comprehensive account on essential self-adjointness and quamtum completeness is [12]. This reference also provides with an introduction to the Fourier transform. Other well-written books that discuss harmonic analysis are [4] and [8]. This last reference also provides a proof of the Riesz-Markov theorem. For a discussion on Hamilton-Jacobi equations and a proof of the Schwartz kernel theorem the reader may consult [14]. Besides the aforementioned texts, a good introduction to quantization and the Wigner transform is [5], other good references on pseudo-differential operators are [1], [13], and [15]. Semiclassical measures were introduced in [6], [9]. The first two section of [7] form a self-contained presentation of the theory and present an extension of the results in $\mathbb{R}^d$ to smooth manifolds.

References

[1] Serge Alinhac and Patrick Gérard. *Pseudo-differential operators and the Nash-Moser theorem.* Translated from the 1991 French original by Stephen S. Wilson. Graduate Studies in Mathematics, 82. American Mathematical Society, Providence, RI, 2007.

[2] Mouez Dimassi and Johannes Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.

[3] Lawrence C. Evans and Maciej Zworski. *Lectures on semiclassical analysis.* 2010. Avalaible at: http://www.math.berkeley.edu/ zworski/semiclassical.pdf.

[4] H. Dym and H. P. McKean. *Fourier series and integrals.* Probability and Mathematical Statistics, No. 14. Academic Press, New York-London, 1972.

[5] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[6] P. Gérard. Mesures semi-classiques et ondes de Bloch. (French) [Semiclassical measures and Bloch waves] Séminaire sur les Équations aux Dérivées Partielles, 1990–1991, Exp. No. XVI, 19 pp., École Polytech., Palaiseau, 1991.

[7] Patrick Gérard and Éric Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.* **71** (1993), no. 2, 559–607.

[8] Elliott H. Lieb and Michael Loss. *Analysis*. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001

[9] Pierre-Louis Lions and Thierry Paul. Sur les mesures de Wigner. *Rev. Mat. Iberoamericana*, **9**(3):553–618, 1993.

[10] André Martinez. *An introduction to semiclassical and microlocal analysis.* Universitext. Springer-Verlag, New York, 2002.

[11] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis.* Second edition. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980.

[12] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[13] Michael E. Taylor. *Pseudodifferential operators and nonlinear PDE*. Progress in Mathematics, 100. Birkhäuser Boston, Inc., Boston, MA, 1991.

[14] Michael E. Taylor. *Partial differential equations. I. Basic theory.* Applied Mathematical Sciences, 115. Springer-Verlag, New York, 1996.

[15] Michael E. Taylor. *Partial differential equations. II. Qualitative studies of linear equations.* Applied Mathematical Sciences, 116. Springer-Verlag, New York, 1996.