Isoperimetric inequalities in graphs and surfaces

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Abstract

Let $\mathfrak{M}$ be the set of metric spaces that are either graphs with bounded degree or Riemannian manifolds with bounded geometry. Kanai proved the quasi-isometric stability of several geometric properties (in particular, of isoperimetric inequalities) for the spaces in $\mathfrak{M}$. Kanai proves directly these results for graphs with bounded degree; in order to prove the general case, he uses a graph (an ε-net) associated to a Riemannian manifold with bounded geometry. This paper studies the stability of isoperimetric inequalities under quasi-isometries between non-exceptional Riemann surfaces (endowed with their Poincaré metrics). The present work proves the stability of the linear isoperimetric inequality for planar surfaces (genus zero surfaces) without the condition on bounded geometry. It is also shown the stability of any non-linear isoperimetric inequality.

Keywords: Isoperimetric inequality; linear isoperimetric inequality; quasi-isometry; infinite graphs; Riemann surfaces.

1 Introduction and main results

An interesting problem in the study of geometric properties of graphs and surfaces is to consider their stability under appropriate deformations. Let $\mathfrak{M}$ be the set of metric spaces that are either graphs with bounded degree or Riemannian manifolds with bounded geometry. In the 1985, in [16] M. Kanai proved the quasi-isometric stability (see the definition of quasi-isometry after Theorem 1.1) of several geometric properties (in particular, of isoperimetric inequalities) for the spaces in $\mathfrak{M}$.

We shall be interested not only in his results but in the ideas behind the proofs. Concretely, those relating the manifold with a particular graph (an ε-net of the manifold) in order to study the stability of isoperimetric inequalities by quasi-isometries. Several authors have followed Kanai in studying the stability of some other property, or in proving the equivalence of a manifold with a different associated graph (see, e.g., [1], [6], [12], [15], [17], [18], [21], [22], [23], [24], [25], [26], [28]).

Quasi-isometries play a central role in the theory of Gromov hyperbolic spaces for they preserve hyperbolicity of geodesic metric spaces (see, e.g., [13], [14]).

A *non-exceptional* Riemann surface S will mean a two-dimensional manifold with a complete conformal metric of constant negative curvature -1. In this case, the universal covering space of S is the unit disk $\mathbb{D}$ endowed with its Poincaré metric. The only exceptional Riemann surfaces are the sphere, the plane, the punctured plane and the tori.

A Riemann surface S satisfies the *α-isoperimetric inequality* $(1/2 \le \alpha \le 1)$ if there exists a constant $c_\alpha(S)$ such that

$$A_S(\Omega)^\alpha \le c_\alpha(S) L_S(\partial\Omega) \tag{1}$$

for every relatively compact domain $\Omega \subset S$. Throughout, A_S, L_S and d_S refer to Poincaré area, length and distance of S and LII refers to the 1-isoperimetric inequality also known as the *linear isoperimetric inequality*.

The isoperimetric inequality on a graph G with bounded degree can be defined as follows. For a subset T of $V(G)$, define its *boundary* as

$$\partial T := \{q \in V(G) \setminus T : d_G(q, T) = 1\}.$$

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It is said that G satisfies the *α-isoperimetric inequality* if there exists a constant $c_\alpha(G)$ so that

$$\big(\#T\big)^\alpha \le c_\alpha(G)\,\#\partial T$$

for any non-empty finite subset T of $V(G)$, where $\#$ denotes the cardinal.

There are close connections between LII and some conformal invariants of Riemann surfaces, namely the bottom of the spectrum of the Laplace-Beltrami operator, the exponent of convergence, and the Hausdorff dimensions of the sets of both bounded geodesics and escaping geodesics in the surface (see [3], [4, p.228], [8], [9], [10], [11], [19], [27, p.333]). Isoperimetric inequalities are of interest in pure and applied mathematics (see, e.g., [7], [20]).

The *injectivity radius $\iota(p)$ of $p \in S$* is defined as the supremum of those $r > 0$ such that $B_S(p,r)$ is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at p. The *injectivity radius $\iota(S)$ of S* is the infimum over $p \in S$ of $\iota(p)$.

In this work we consider the stability of isoperimetric inequalities under quasi-isometries between non-exceptional Riemann surfaces. This stability was proved by Kanai in [16] in the very general setting of graphs and Riemannian manifolds in $\mathfrak{M}$ (bounded geometry in a Riemannian manifold M means a lower bound for the Ricci curvature and $\iota(M) > 0$). We have an example showing that the stability fails, even for Riemann surfaces, without the hypothesis $\iota(S) > 0$. Since this example involves non-zero genus surfaces, it is natural to wonder if the stability holds for planar surfaces.

The main result in this paper is the following.

Theorem 1.1 *Let S and S' be quasi-isometric non-exceptional genus zero Riemann surfaces. Then S' satisfies the linear isoperimetric inequality if and only if S satisfies the linear isoperimetric inequality. Furthermore, if $f: S \longrightarrow S'$ is a c-full (a,b)-quasi-isometry, and $c_1(S') < \infty$ then $c_1(S) \le C$, where C is a universal constant which just depends on a, b, c and $c_1(S')$.*

A function between two metric spaces $f: X \longrightarrow Y$ is said to be an (a,b)-*quasi-isometric embedding* with constants $a \ge 1$, $b \ge 0$, if

$$\frac{1}{a}\, d_X(x_1,x_2) - b \le d_Y(f(x_1), f(x_2)) \le a\, d_X(x_1,x_2) + b\,,$$

for every $x_1, x_2 \in X$. Such a quasi-isometric embedding f is a *quasi-isometry* if, furthermore, there exists a constant $c \ge 0$ such that f is *c-full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \le c$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. It is easy to check that to be quasi-isometric is an equivalence relation on the set of metric spaces.

For surfaces of positive finite genus, the following result shows that the first conclusion of Theorem 1.1 holds:

Theorem 1.2 *Let S and S' be quasi-isometric non-exceptional Riemann surfaces with finite genus. Then S' satisfies the LII if and only if S satisfies the LII.*

However, we have an example showing that the second conclusion of Theorem 1.1 fails in this case of positive finite genus.

The idea behind the proof of Theorem 1.1 is simple: each surface is split into a thin part (with small injectivity radius) and a thick part; a slight modification of the proof of Kanai's theorem applied to the thick part, together with some new arguments to show that the thin part is "essentially" preserved under the quasi-isometry give the theorem. A difficulty is the following: two quasi-isometric surfaces have a similar shape at a large scale (if viewed from sufficiently far), but they can look very different at a small scale (by definition a quasi-isometry may not be continuous). In particular, the image of a continuous loop by a quasi-isometry need not be a continuous curve, and thus the injectivity radii can be very different in two quasi-isometric surfaces. Theorem 1.3 deals with this situation and states that a quasi-isometry between planar surfaces maps points with small injectivity radius to points with small injectivity radius (in a precise quant itative way).

Theorem 1.3 *Let S and S' be non-exceptional genus zero Riemann surfaces and let $f : S \longrightarrow S'$ be a c-full (a, b)-quasi-isometry. For each $\varepsilon' > 0$ there exists $\varepsilon > 0$ which just depends on ε', a, b, c, such that if $\iota(z) < \varepsilon$ then $\iota(f(z)) < \varepsilon'$.*

We show that a very different situation appears when dealing with the α-isoperimetric inequality, $1/2 \le \alpha < 1$.

Theorem 1.4 *Let S and S' be quasi-isometric non-exceptional Riemann surfaces with $\iota(S) > 0$, and $1/2 \le \alpha < 1$. Then S' satisfies the α-isoperimetric inequality if and only if S satisfies the α-isoperimetric inequality and $\iota(S') > 0$.*

Note that here we have no hypothesis on genus.

Hence, the behavior of the α-isoperimetric inequality in Riemann surfaces under quasi-isometries is very different in the cases $\alpha = 1$ and $\alpha < 1$.

One of the main ingredients in the proofs is the relation between a surface S and a graph (an ε-net) associated to S.

2 Sketch of the proof of Theorem 1.1

This section is devoted to present the main ideas in the proof of Theorem 1.1, which follows Kanai's approach. See [5] for details.

In Kanai's results it is essential that both $\iota(S)$ and $\iota(S')$ are positive; these conditions will be avoided due to Theorem 1.3 and the thick-thin decomposition of Riemann surfaces given by Margulis Lemma (see, e.g., [2, p.107]). Concretely, for any $\varepsilon < \sinh^{-1} 1$ ($\sinh^{-1}$ denotes the inverse function of sinh) any Riemann surface, S, can be partitioned into a thick part, $S_\varepsilon := \{z \in S : \iota(z) > \varepsilon\}$, and a thin part, $S \setminus S_\varepsilon$, whose connected components have a simple structure (the fundamental group of each connected component of $S \setminus S_\varepsilon$ is generated by a single element).

In order to prove Theorem 1.1, it will be shown that it suffices to consider the thick parts of S and S' for some particular choices of ε and ε', so that Kanai's insight can be brought to S_ε and $S'_{\varepsilon'}$ if we avoid the (possible) contribution to the LII given by ∂S_ε and $\partial S'_{\varepsilon'}$.

Let us consider $H > 0$, a metric space X, and a subset $Y \subseteq X$. The set $V_H(Y) := \{x \in X : d(x, Y) \le H\}$ is called the *H-neighborhood* of Y in X.

We will need the following technical results.

Lemma 2.1 *Let S and S' be non-exceptional genus zero Riemann surfaces, and $f : S \longrightarrow S'$ be a c-full (a,b)-quasi-isometry. Then, given $0 < \varepsilon, \varepsilon_1 < \sinh^{-1} 1$, there exist $0 < \varepsilon', \tilde{\varepsilon} < \varepsilon_1$, which just depend on $\varepsilon, \varepsilon_1, a, b, c$, so that*

$$f(S_\varepsilon) \subset S'_{\varepsilon'} \subset V_c(f(S_{\tilde{\varepsilon}})).$$

As a first goal it is going to be proved the LII intrinsic to a bordered surface, S_ε contained in S; note that S_ε is not necessarily connected. To this end, we define below the "thick" boundary of a subset of S as its intrinsic boundary in S_ε, and the "intrinsic" LII that will refered to as LII_ε.

Definition 2.2 Given a non-exceptional Riemann surface S, $\varepsilon > 0$ and a domain Ω in S_ε, define

$$\partial_\varepsilon \Omega := \partial \Omega \cap S_\varepsilon = \partial \Omega \setminus \partial S_\varepsilon.$$

Definition 2.3 S_ε is said to satisfy the *ε-linear isoperimetric inequality*, LII_ε, if there exists a positive constant c, such that if Ω is a relatively compact domain in S_ε with smooth boundary, then

$$A_S(\Omega) \le c\, L_S(\partial_\varepsilon \Omega). \tag{2}$$

A reduction is that it suffices to prove LII_ε for intrinsic geodesic domains in S_ε. A domain $\Omega \subset S$ is said to be a *geodesic domain* if $\partial\Omega$ is a finite number of simple closed geodesics, and $A_S(\Omega)$ is finite. An *intrinsic geodesic domain* is a geodesic domain intrinsic to S_ε, i.e., the intersection of a geodesic domain in S with S_ε.

Let us denote by $c_1(S_\varepsilon)$ the sharp ε-linear isoperimetric constant of S_ε and by $c_{1,g}(S_\varepsilon)$ the sharp ε-linear isoperimetric constant of S_ε for intrinsic geodesic domains.

Lemma 2.4 *Let S be a non-exceptional Riemann surface and $\varepsilon \geq 0$ so that $\varepsilon < \sinh^{-1} 1$. Then,*

$$S_\varepsilon \text{ has } LII_\varepsilon \iff S_\varepsilon \text{ has } LII_\varepsilon \text{ for intrinsic geodesic domains in } S_\varepsilon.$$

In fact, $c_{1,g}(S_\varepsilon) \leq c_1(S_\varepsilon) \leq c_{1,g}(S_\varepsilon) + 2.$

Following Kanai's procedure, the LII will be transferred from bordered surfaces to nets and viceversa. To this end, a subset G of S is said to be δ-separated for $\delta > 0$, if $d_S(p,q) > \delta$ whenever p and q are distinct points of G. It is called *maximal* if it is maximal with respect to the order relation of inclusion.

Consider the distance d_G in G induced by the distance d_S of S. Concretely, given $p_1, p_2 \in G$, $d_G(p_1,p_2) = M$ if and only if $M \geq 0$ is the only natural number such that

$$\delta M \leq d_S(p_1,p_2) < \delta(M+1). \tag{3}$$

The set of neighbors of p in G is defined as $N(p) = \{q \in G : d_G(p,q) = 1\}$ and this gives a graph structure to the set G. Such graph will be referred to as δ-net.

Let S be a Riemann surface and $0 < \varepsilon < \sinh^{-1} 1$. We have that $\iota\left(V_\varepsilon(S_\varepsilon)\right) \geq c(\varepsilon)$, where $c(\varepsilon) := \sinh^{-1}\left(e^{-\varepsilon}\sinh\varepsilon\right)$. The pair (G,δ) will denote a δ-net *associated* to the pair (S,ε) as follows: Set $\delta \leq \frac{1}{2}\,\iota(V_\varepsilon(S_\varepsilon))$, and choose a maximal δ-net G on S_ε so that

$$A_S(S_\varepsilon \cap B_S(p,\delta)) > \frac{1}{2} A_S(B_S(p,\delta)), \tag{4}$$

for all $p \in G$; such choice of G is possible due to Collar Lemma. Note also that G does not need to be connected.

The strategy of the proof of Theorem 1.1 is as follows: Consider S and S' Riemann surfaces and $f : S \longrightarrow S'$ a quasi-isometry, (G,δ) and (G',δ')

nets in (S,ε) and (S',ε'). It will be assumed that S' satisfies the LII that will be transferred to the net (G',δ'). Then it will be shown that (G,δ) and (G',δ') are quasi-isometric and so (G,δ) also satisfies the LII. Finally, this LII will be transferred to S. The next two results deal with transferring the LII between surfaces and nets. A direct application of [16, Lemma 4.5] is the following result:

Lemma 2.5 *There exists a universal constant ε_0 with the following property: Let S' be any non-exceptional Riemann surface satisfying LII and $0<\varepsilon'<\min\left\{\varepsilon_0,(12c_1(S'))^{-1}\right\}$. Let (G',δ') be a δ'-net associated to (S',ε'). Then,*

$$(G',\delta') \text{ also satisfies the } LII \text{ and } c_1(G') \leq \frac{12\sinh\delta'}{\cosh(\delta'/2)-1}\, c_1(S').$$

Lemma 2.6 *Let (G,δ) be a δ-net associated to (S,ε). Then*

$$(G,\delta) \text{ has } LII \Longrightarrow S_\varepsilon \text{ has } LII_\varepsilon. \tag{5}$$

Moreover, $c_1(S_\varepsilon) \leq 2\,m\,c_{1,l}(S_\varepsilon)\max\left\{1,\ 2\,c_1(G)\left(\dfrac{\sinh(9\delta/4)}{\sinh(\delta/4)}\right)^2\right\}+2,$ *where $c_{1,l}(S_\varepsilon)$ is the constant in the local LII and $m =: \sup_{z\in S}\#\{p\in G : z\in B_S(p,\delta)\}<\infty$.*

As a last step, it will be constructed a quasi-isometry between the two nets (G,δ) and (G',δ') associated to (S,ε) and (S,ε') respectively with $0<\varepsilon<\sinh^{-1}1$ and $0<\varepsilon',\tilde{\varepsilon}<\varepsilon$ given by Lemma 2.1.

Proposition 2.7 *The nets (G,δ) and (G',δ') are quasi-isometric. More precisely, there is a C'-full (A,B)-quasi-isometry $g: G\longrightarrow G'$, with $A = a\max\left\{\dfrac{\delta'}{\delta},\dfrac{\delta}{\delta'}\right\}$, $B=5+\dfrac{a\delta}{\delta'}+\dfrac{b}{\delta'}$ and $C'=2+\dfrac{a(2\delta+C(\varepsilon,\tilde{\varepsilon}))+2b+c}{\delta'}$ where $C(\varepsilon,\tilde{\varepsilon})$ is the maximum diameter of the connected components of $S_{\tilde{\varepsilon}}\setminus S_\varepsilon$ where $\tilde{\varepsilon}$ is given by Lemma 2.1.*

Moreover, for any $X\subset G$, $\#X\leq\mu\#g(X)$ where $\mu\leq 13^{\frac{a(2\delta'+b)}{\delta}}$.

In [16, Lemma 4.2] Kanai proves that the LII on graphs is preserved by quasi-isometries; thus an immediate consequence is:

Corollary 2.8 *For (G,δ) and (G',δ') as above,*

$$(G,\delta) \text{ satisfies the } LII \iff (G',\delta') \text{ satisfies the } LII.$$

Moreover, $c_1(G)\leq\mu 12^{A(B+2C-1)+C-2}c_1(G')$, with μ as in Proposition 2.7.

Finally, the combination of all previous results will give the proof of Theorem 1.1.

Proof of Theorem 1.1. Assume that S' has LII. If ε_0 is the constant in Lemma 2.5, let us fix $0 < \varepsilon < \varepsilon_0$ and let $0 < \varepsilon', \tilde{\varepsilon} < \min\left\{\varepsilon_0, (12c_1(S'))^{-1}\right\}$ given by Lemma 2.1. Let (G', δ') be a net associated to (S', ε'). Since S' has LII, by Lemma 2.5, G' has LII. If (G, δ) is a net associated to (S, ε), then Proposition 2.7 gives that (G, δ) and (G', δ') are quasi-isometric, and Corollary 2.8 concludes that (G, δ) has LII. Lemma 2.6 states that S_ε has LII_ε and, since $0 < \varepsilon < \varepsilon_0$, S has LII.

Moreover, the isoperimetric constant obtained $c_1(S) < \infty$ depends just on $\varepsilon, a, b, c, c_1(S')$. In order to avoid the dependence on ε, it suffices to take $\varepsilon = \varepsilon_0/2$, since ε_0 is a universal constant. $\square$

References

[1] Alvarez, V., Rodríguez, J.M., Yakubovich, V.A., Subadditivity of p-harmonic "measure" on graphs, *Michigan Mathematical Journal* **49** (2001), 47-64.

[2] Ballman, W., Gromov, M., Schroeder, V., Manifolds of Non-positive Curvature, Birkhauser, Boston, 1985.

[3] Bishop, C. J., Jones, P. W., Hausdorff dimension and Kleinian groups, *Acta Math.* **179** (1997), 1-39.

[4] Buser, P., Geometry and Spectra of Compact Riemann Surfaces. Birkhäuser, Boston, 1992.

[5] Cantón, A., Granados, A., Portilla, A., Rodríguez, J. M., Quasi-isometries and isoperimetric inequalities in planar domains. To appear in *J. Math. Soc. Japan*.

[6] Cantón, A., Fernández, J. L., Pestana, D., Rodríguez, J. M., On harmonic functions on trees, *Potential Analysis* **15** (2001), 199-244.

[7] Chavel, I., Isoperimetric inequalities: differential geometric and analytic perspectives. Cambridge University Press, Cambridge, 2001.

[8] Cheeger, J., A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in Analysis*, Princeton University Press, Princeton (1970), 195-199.

[9] Fernández, J. L., Melián, M. V., Bounded geodesics of Riemann surfaces and hyperbolic manifolds, *Trans. Amer. Math. Soc.* **347** (1995), 3533-3549.

[10] Fernández, J. L., Melián, M. V., Escaping geodesics of Riemannian surfaces, *Acta Math.* **187** (2001), 213-236.

[11] Fernández, J. L., Rodríguez, J. M., The exponent of convergence of Riemann surfaces. Bass Riemann surfaces. *Annales Acad. Sci. Fenn. A. I.* **15** (1990), 165-183.

[12] Fernández, J. L., Rodríguez, J. M., Area growth and Green's function of Riemann surfaces, *Arkiv för matematik* **30** (1992), 83-92.

[13] Ghys, E., de la Harpe, P., *Sur les Groupes Hyperboliques d'après Mikhael Gromov.* Progress in Mathematics, Volume 83, Birkhäuser, 1990.

[14] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75-263.

[15] Holopainen, I., Soardi, P. M., p-harmonic functions on graphs and manifolds, *Manuscripta Math.* **94** (1997), 95-110.

[16] Kanai, M., Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan* **37** (1985), 391-413.

[17] Kanai, M., Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan* **38** (1986), 227-238.

[18] Kanai, M., Analytic inequalities and rough isometries between non-compact Riemannian manifolds. Curvature and Topology of Riemannian manifolds (Katata, 1985). Lecture Notes in Math. **1201**. Springer (1986), 122-137.

[19] Melián, M. V., Rodríguez, J. M., Tourís, E., Escaping geodesics in Riemannian surfaces with pinched variable negative curvature. Submitted.

[20] Pólya, G., Isoperimetric inequalities in mathematical physics. Princeton University Press, 1951.

[21] Portilla, A., Rodríguez, J. M., Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.* **14** (2004), 123-149.

[22] Portilla, A., Tourís, E., A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, *Publ. Mat.* **53** (2009), 83-110.

[23] Rodríguez, J. M., Isoperimetric inequalities and Dirichlet functions of Riemann surfaces, *Publ. Mat.* **38** (1994), 243-253.

[24] Rodríguez, J. M., Two remarks on Riemann surfaces, *Publ. Mat.* **38** (1994), 463-477.

[25] Rodríguez, J. M., Tourís, E., Gromov hyperbolicity of Riemann surfaces, *Acta Math. Sinica* **23** (2007), 209-228.

[26] Soardi, P. M., Rough isometries and Dirichlet finite harmonic functions on graphs, *Proc. Amer. Math. Soc.* **119** (1993), 1239-1248.

[27] Sullivan, D., Related aspects of positivity in Riemannian geometry, *J. Diff. Geom.* **25** (1987), 327-351.

[28] Tourís, E., Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, *J. Math. Anal. Appl.* **380** (2011), 865-881.