Interpolation of a spline developable surface between a curve and two rulings

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Abstract

In this paper we address the problem of interpolating a spline developable patch bounded by a given spline curve and the first and the last rulings of the developable surface. In order to complete the boundary of the patch a second spline curve is to be given. The parametrisation of this curve is determined by nonlinear equations which arise as a consequence of the null gaussian curvature condition imposed on the surface.

Key words: Developable surfaces, Spline surfaces, blossoms.

1. Introduction

Developable surfaces have been used extensively in industry for modelling sheets of steel. These surfaces are plane patches that have been curved by isometric transformations, preserving lengths of curves, angles and areas. They mimic the properties of thin steel plates that are transformed by cutting, rolling or folding, but not by stretching or application of heat, which would raise manufacturing costs.

Their inclusion in the NURBS formalism, however, has not been easy. The condition of developability is a non-linear differential equation which translates into non-linear equations for the vertices of the control net of the surface.

To our knowledge the first reference to NURBS developable surfaces arises in technical reports at General Motors, [1, 2]. One approach has been solving the developability condition for low degrees [3, 4, 5].

Another approach to developable surfaces consists in resorting to projective dual geometry. In this geometry “points” are planes and “planes” are points and this is useful to solve the developability condition [6, 7, 8].

One can also construct surfaces which are approximately developable instead [9, 10, 11, 12, 13, 14]. A nice review may be found in [15]. Applications to ship hull design may be found in [16, 17, 18].

A large family of Bézier developable surfaces was obtained in [19, 20] defining affine transformations between cells of the control net. This result has been extended to spline [21] and Bézier triangular [22] developable patches. A characterisation of Bézier ruled surfaces is found in [23].

In this paper we make use of the latter constructions to find solutions to interpolation problems with developable surfaces.

Following [21], we first review in Section 2 the main features, definitions and the classification of developable surfaces, whereas in Section 3 we deal with the formalism of B-spline curves. In Section 4 a construction of spline developable surfaces grounded on linear relations between vertices of the B-spline net is shown. In Section 5 we use this construction to provide solutions to an interpolation problem between a spline curve and two rulings. Finally, in Section 6 we solve the problem of interpolating a developable patch between a spline curve and segments of the rulings at both ends. This solution is extended to triangular patches in Section 7. A final section of conclusions is included at the end of the paper.

2. Developable surfaces

A *ruled surface* patch fills the space between two parametrised curves $c(u)$, $d(u)$,

$$c(u,v) = (1-v)c(u) + vd(u), \quad u \in [a,b],\ v \in [0,1], \tag{2.1}$$

by linking with segments, named *rulings*, the points on both curves with the same parameter u.

In general, the tangent plane to the ruled surface on a ruling is different for each point on the segment (see Figure 2.1). *Developable surfaces* are the subcase of ruled surfaces for which the tangent plane is constant along each ruling [24, 25]. This feature is profitable in industry, since a steel plate may be combed to a developable surface using a folder machine, which consists on a cylinder which follows the rulings, applying pressure to bend the plate.

Developable surfaces may be also characterised as surfaces with null gaussian curvature, that is, intrinsically flat surfaces. Angles, areas and lengths on the plane are preserved after combing the surface.

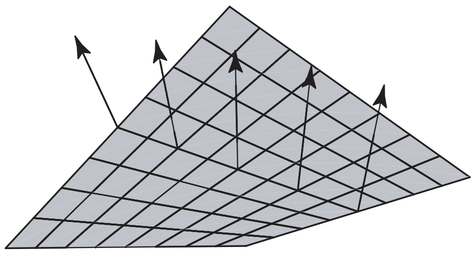

Fig. 2.1. Non-developable ruled surface

Let us compute a normal vector at each point of a ruled surface with the derivatives of the parametrisation (2.1),

$$c_u(u,v) = (1-v)c'(u) + vd'(u), \quad c_v(u,v) = d(u) - c(u),$$

$$(c_u \times c_v)(u,v) = ((1-v)c'(u) + vd'(u)) \times (d(u) - c(u)),$$

which is linear in the parameter v.

If we calculate it on both ends of the rulings,

$$(c_u \times c_v)(u,0) = c'(u) \times (d(u) - c(u)),$$

$$(c_u \times c_v)(u,1) = d'(u) \times (d(u) - c(u)),$$

we learn that the three vectors $c'(u)$, $d'(u)$, $d(u) - c(u)$ are to be coplanary in order to have a constant tangent plane along each ruling of the surface.

Proposition 2.1. *A ruled surface parametrised as (2.1) is developable if and only if the vector* $\mathbf{w}(u) = d(u) - c(u)$*, linking the points* $d(u)$*,* $c(u)$*, and the velocities* $c'(u)$*,* $d'(u)$ *of the curves at these points are coplanary for every value of* u*.*

Therefore, on a developable surface we can write one of these vectors as a linear combination of the other two, since they are coplanary,

$$c'(u) = \lambda(u)\mathbf{w}(u) + \mu(u)\mathbf{w}'(u),$$

and use it to classify developable surfaces in four families (cfr. for instance [24, 25]):

1. Planar surfaces: On the plane all three vectors are trivially coplanary.
2. Cylinders: Surfaces with parallel rulings: For them $\mathbf{w}(u)$ is parallel to $\mathbf{w}'(u)$ and the coplanarity condition is satisfied.
3. Cones: Surfaces with rulings meeting at a point called *vertex*.
4. Tangent surfaces: The surface formed by all tangent lines to a given curve named *edge of regression*.

The last two cases may be obtained by moving the curve $c(u)$ along the rulings to a new curve $\tilde{c}(u) = c(u) - \mu(u)\mathbf{w}(u)$, so that

$$\tilde{c}'(u) = (\lambda(u) - \mu'(u))\,\mathbf{w}(u)\ .$$

Generically, vector $\tilde{c}'(u)$ is parallel to the vector $\mathbf{w}(u)$ along the rulings, which corresponds to the case of a tangent surface to the curve $\tilde{c}(u)$. But in the special case with $\lambda(u) = \mu'(u)$, the velocity $\tilde{c}'(u)$ is zero and the new curve degenerates to a single point, corresponding to the case of a cone.

3. B-spline curves

In this section we review the formalism of B-spline curves and their main properties in order to fix the notation, which follows closely the one in [26].

A spline curve is a piecewise polynomial curve. The name B-spline refers to a particular choice of basis of piecewise polynomials.

We may define a B-spline curve $c(u)$ of degree n and N pieces on an interval $[u_{n-1}, u_{n+N-1}]$, so that the I-th piece of the curve is defined on an interval $[u_{n+I-2}, u_{n+I-1}]$. For this we require an ordered list of values of the parameter u, which are named *knots*, $\{u_0, \ldots, u_{2n+N-2}\}$. The actual knots defining the intervals for each piece are the *inner* knots $\{u_{n-1}, \ldots, u_{n+N-1}\}$. In order to have well defined recursion expressions, $(n-1)$ *auxiliary* knots $\{u_0, \ldots, u_{n-2}\}$ at the beginning of the sequence and $\{u_{n+N}, \ldots, u_{2n+N-2}\}$ at the end. In most cases these auxiliary knots are taken to be equal to the closest inner knot, that is,

$$u_0 = \cdots = u_{n-2} = u_{n-1}, \qquad u_{n+N-1} = u_{n+N} = \cdots = u_{2n+N-2}, \tag{3.1}$$

so that the sequence begins and ends with a knot of *multiplicity* n. These auxiliary knots are introduced in order to have well defined recursion expressions.

The curve $c(u)$ is written then in terms of a family of piecewise polynomial functions of degree n, $N_i^n(u)$, $i = 0, \ldots, n+N-1$, named *nodal* or *B-spline functions of degree* n on the sequence of knots. They are defined recursively, starting with functions which are constant in just one interval,

$$
\begin{aligned}
N_i^n(u) &= \frac{u-u_{i-1}}{u_{i+n-1}-u_{i-1}}\,N_i^{n-1}(u)+\frac{u_{i+n}-u}{u_{i+n}-u_i}\,N_{i+1}^{n-1}(u),\\
N_i^0(u) &= \begin{cases} 1, & u\in[u_{i-1},u_i]\\ 0, & u\notin[u_{i-1},u_i].\end{cases}
\end{aligned}
\tag{3.2}
$$

This set of functions forms a basis for piecewise polynomials of degree n and N pieces defined by the sequence of knots and has the nice properties of being of minimal support ($n+1$ intervals) and of class C^{n-r_i} at the knot u_i, where r_i is the multiplicity of such knot. A B-spline curve may be written hence as

$$c(u)=\sum_{i=0}^{L} c_i N_i^n(u),$$

where the points $\{c_0,\dots,c_{n+N-1}\}$ are named vertices and form the *B-spline polygon* of the parametrised curve $c(u)$.

In the following, we write $L:=n+N-1$, $K:=N+2n-2$ for respectively the last index of vertices and knots.

Instead of defining the B-spline functions, we may encode the same information in a recursive algorithm (De Boor's algorithm) of n iterations for computing points on the curve. The iterations consist on linear interpolations between consecutive vertices,

$$c_i^{1)}(u):=\frac{u_{i+n}-u}{u_{i+n}-u_i}c_i+\frac{u-u_i}{u_{i+n}-u_i}c_{i+1},\qquad i=0,\dots,n-1,$$

and in each iteration we end up with one less vertex,

$$
\begin{aligned}
c_i^{r)}(u) &:= \frac{u_{i+n}-u}{u_{i+n}-u_{i+r-1}}c_i^{r-1)}(u)+\frac{u-u_{i+r-1}}{u_{i+n}-u_{i+r-1}}c_{i+1}^{r-1)}(u),\quad i=0,\dots,n-r,\ r=1,\dots,n,\\
c(u) &= c_0^{n)}(u):=\frac{u_n-u}{u_n-u_{n-1}}c_0^{n-1)}(u)+\frac{u-u_{n-1}}{u_n-u_{n-1}}c_1^{n-1)}(u).
\end{aligned}
\tag{3.3}
$$

A useful construction, named *polarisation* or *blossom* of the parametrisation of the curve, consists on interpolating in each step with a different value v_i of the parameter u,

$$
\begin{aligned}
c_i^{1)}[v_1] &:= \frac{u_{i+n}-v_1}{u_{i+n}-u_i}c_i+\frac{v_1-u_i}{u_{i+n}-u_i}c_{i+1},\qquad i=0,\dots,n-1,\ r=1,\dots,n,\\
c_i^{r)}[v_1,\dots,v_r] &:= \frac{u_{i+n}-v_r}{u_{i+n}-u_{i+r-1}}c_i^{r-1)}[v_1,\dots,v_{r-1}]+\frac{v_r-u_{i+r-1}}{u_{i+n}-u_{i+r-1}}c_{i+1}^{r-1)}[v_1,\dots,v_{r-1}],\\
c[v_1,\dots,v_n] &:= c_0^{n)}[v_1,\dots,v_n]=\frac{u_n-v_n}{u_n-u_{n-1}}c_0^{n-1)}[v_1,\dots,v_{n-1}]\\
&+\ \frac{v_n-u_{n-1}}{u_n-u_{n-1}}c_1^{n-1)}[v_1,\dots,v_{n-1}].
\end{aligned}
\tag{3.4}
$$

With this notation,

$$u^{<i>}=\underbrace{u,\dots,u}_{i\ \text{times}}.$$

we have that $c(u)=c[u^{<n>}]$.

These expressions are valid for B-spline curves with arbitrary number of pieces, replacing the interval $[u_{n-1},u_n]$ of the first piece by the interval of the piece under consideration. For instance, for evaluation on the I-th piece we have to use vertices $\{c_{I-1},\dots,c_{I+n-1}\}$ and knots $\{u_{I-1},\dots,u_{I+2n-2}\}$.

We may summarise the main properties of the De Boor algorithm and the polarisation which are relevant for our purposes:

1. If the auxiliary knots are trivial (3.1), the spline curve begins at the first vertex and ends at the last one,
$$c(u_{n-1})=c_0,\qquad c(u_{n+N-1})=c_L.$$
2. Vertices are recovered from the polarisation as $c_i=c[u_i,\dots,u_{i+n-1}]$, $i=0,\dots,n$.
3. The velocity of the curve is
$$c'(u)=\frac{n}{u_n-u_{n-1}}\left(c_1^{n-1)}(u)-c_0^{n-1)}(u)\right),\qquad u\in[u_{n-1},u_n],\tag{3.5}$$
where the two points that are obtained in the $(n-1)$-th iteration of the algorithm can be written, using the polarisation, as
$$c_0^{n-1)}(u)=c[u^{<n-1>},u_{n-1}],\qquad c_1^{n-1)}(u)=c[u^{<n-1>},u_n].\tag{3.6}$$
4. The polarisation $c[v_1,\dots,v_n]$ of the spline curve $c(u)$, is multiaffine and symmetric. That is, if $\lambda+\mu=1$,
$$c[\lambda v_1+\mu\tilde v_1,\dots,v_n]=\lambda c[v_1,\dots,v_n]+\mu c[\tilde v_1,\dots,v_n].$$

Finally, we review two operations with B-spline curves which we shall need later on:
Insertion of knots: Given a B-spline curve of degree n with vertices $\{c_0,\dots,c_L\}$ and knots $\{u_0,\dots,u_K\}$, we can split into two the piece corresponding to the interval $[u_I,u_{I+1}]$ by inserting a new knot $\tilde u$, $u_I<\tilde u<u_{I+1}$. The new list of knots is then obviously $\{\tilde u_0,\dots,\tilde u_{K+1}\}$,
$$\tilde u_i=u_i,\ i=0,\dots,I,\quad \tilde u_{I+1}=\tilde u,\quad \tilde u_i=u_{i-1},\ i=I+2,\dots,K+1,$$
and, since the curve has not changed, the blossom provides the new sequence of vertices $\{\tilde c_0,\dots,\tilde c_{L+1}\}$,
$$\tilde c_i=c[\tilde u_i,\dots,\tilde u_{i+n-1}],\qquad i=0,\dots,L+1.$$
Degree elevation: Formally we may express a B-spline curve $c(u)$ of degree n as a curve of degree $n+1$, though with no terms u^{n+1}. The blossom c^1 of the degree-elevated parametrised curve is related to the original one in a simple form [26],
$$c^1[v_1,\dots,v_{n+1}]=\frac{1}{n+1}\sum_{i=1}^{n+1}c[v_1,\dots,v_{i-1},v_{i+1},\dots,v_{n+1}],\tag{3.7}$$
and in the list of knots $\{u_0,\dots,u_K\}$ the multiplicity of inner knots, from u_{n-1} to u_{n+N-1}, is increased by one, without modifying the auxiliary knots. For example, if all the inner knots were simple, the new list of knots would be
$$\{u_0,\dots,u_{n-2},u_{n-1},u_{n-1},\dots,u_{n+N-1},u_{n+N-1},u_{n+N},\dots,u_{2n+N-2}\},\tag{3.8}$$
so we end up with $2n+2N$ knots and $n+2N$ vertices in this case.

4. Spline developable surfaces

The developability condition in Proposition 1 may be readily now adapted to spline curves.

For a start, let us consider two B-spline curves of degree n and one segment over a common list of knots $\{u_0,\dots,u_{2n-1}\}$, defined on the interval $[u_{n-1},u_n]$. Their respective B-spline polygons are $\{c_0,\dots,c_n\}$, $\{d_0,\dots,d_n\}$.

We may draw a simple conclusion using the De Boor algorithm. Using (3.5) and the last iteration of (3.3), it is clear that the vectors $c'(u)$, $d'(u)$, $d(u)-c(u)$ are coplanary if and only if the four points $c_0^{n-1)}(u)$, $c_1^{n-1)}(u)$, $d_0^{n-1)}(u)$, $d_1^{n-1)}(u)$ are coplanary (see Figure 4.1).

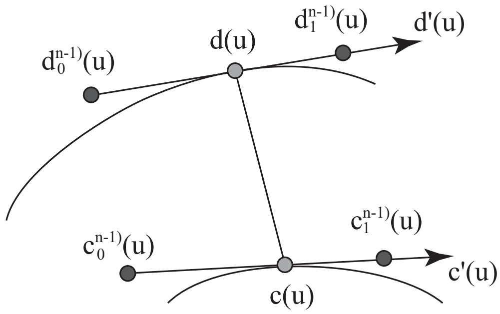

Fig. 4.1. Characterisation of developable surfaces

The developability condition is then equivalent to the possibility of writing one of the points as a barycentric combination of the other ones. For instance,

$$d_1^{n-1)}(u)=\mu_0(u)d_0^{n-1)}(u)+\lambda_0(u)c_0^{n-1)}(u)+\lambda_1(u)c_1^{n-1)}(u),$$

with coefficients $\lambda_0(u)$, $\lambda_1(u)$, $\mu_0(u)=1-\lambda_0(u)-\lambda_1(u)$.

We may rewrite this combination in another form, separating the terms related to each curve, also in a barycentric fashion,

$$(1-\Lambda(u))c_0^{n-1)}(u)+\Lambda(u)c_1^{n-1)}(u)=(1-M(u))d_0^{n-1)}(u)+M(u)d_1^{n-1)}(u), \quad (4.1)$$

$$\Lambda(u)=\frac{\lambda_1(u)}{\lambda_0(u)+\lambda_1(u)},\quad M(u)=\frac{1}{\lambda_0(u)+\lambda_1(u)},$$

which just excludes the case of parallel vectors $d_1^{n-1)}(u)-d_0^{n-1)}(u)$, $c_1^{n-1)}(u)-c_0^{n-1)}(u)$, which corresponds to a cone. In this sense we use the word *generic*, since the following results will be valid for all developable surfaces, but for this type of cone.

Using blossoms and taking into account that these are multiaffine (3.6),

$$\begin{aligned}(1-\Lambda(u))c_0^{n-1)}(u)+\Lambda(u)c_1^{n-1)}(u) &= (1-\Lambda(u))c[u^{<n-1>},u_{n-1}]+\Lambda(u)c[u^{<n-1>},u_n]\\ &= c\left[u^{<n-1>},(1-\Lambda(u))u_{n-1}+\Lambda(u)u_n\right],\end{aligned}$$

the coplanarity condition (4.1) may be written in a more compact expression,

$$c[u^{<n-1>},\Lambda^*(u)]=d[u^{<n-1>},M^*(u)], \quad (4.2)$$

$$\Lambda^*(u)=(1-\Lambda(u))u_{n-1}+\Lambda(u)u_n,\qquad M^*(u)=(1-M(u))u_{n-1}+M(u)u_n.$$

This expression is valid for B-spline curves with arbitrary number of pieces, replacing the interval $[u_{n-1}, u_n]$ of the first piece by the interval of the piece under consideration.

The higher the degree of $\Lambda^*(u)$, $M^*(u)$, the larger the number of conditions imposed by (4.2). Hence, we restrict now to the case with constant Λ^*, M^*, which produces the families of developable surfaces in [19, 21]. In this case expressions on both sides of (4.2) may be viewed as parametrisations of curves of degree $n-1$ and therefore this condition is equivalent to the same one for their blossoms, since a blossom is uniquely determined by its parametrisation:

Theorem 4.1. *Two B-spline curves of degree n and N pieces with the same list of knots $\{u_0, \dots, u_K\}$ define a developable surface on the interval $[u_{n-1}, u_{n+N-1}]$ if their blossoms are related by*

$$c[v_1, \dots, v_{n-1}, \Lambda^*] = d[v_1, \dots, v_{n-1}, M^*],$$

for some values Λ^*, M^*.

We may obtain relations between the B-spline polygons of both curves applying the previous expression to lists of correlative knots, $\{u_{i+1}, \dots, u_{i+n-1}\}$, taking into account that blossoms are multi-affine,

$$\begin{aligned} c[u_{i+1}, \dots, u_{i+n-1}, \Lambda^*] &= c\left[u_{i+1}, \dots, u_{i+n-1}, \frac{u_{i+n} - \Lambda^*}{u_{i+n} - u_i} u_i + \frac{\Lambda^* - u_i}{u_{i+n} - u_i} u_{i+n}\right] \\ &= \frac{u_{i+n} - \Lambda^*}{u_{i+n} - u_i} c\left[u_i, \dots, u_{i+n-1}\right] + \frac{\Lambda^* - u_i}{u_{i+n} - u_i} c\left[u_{i+1}, \dots, u_{i+n}\right] \\ &= \frac{u_{i+n} - \Lambda^*}{u_{i+n} - u_i} c_i + \frac{\Lambda^* - u_i}{u_{i+n} - u_i} c_{i+1}, \end{aligned}$$

since $c_i = c[u_i, \dots, u_{i+n-1}]$.

Corollary 4.1. *Two B-spline curves of degree n with the same list of knots $\{u_0, \dots, u_K\}$ and B-spline polygons $\{c_0, \dots, c_L\}$, $\{d_0, \dots, d_L\}$ define a developable surface if the cells of the B-spline net of the surface are plane and their vertices are related by*

$$(u_{i+n} - \Lambda^*)c_i + (\Lambda^* - u_i)c_{i+1} = (u_{i+n} - M^*)d_i + (M^* - u_i)d_{i+1}, \; i = 0, \dots, L-1, \tag{4.3}$$

for some values Λ^*, M^*.

This family of spline developable surfaces has the advantage of being defined by linear relations between vertices, in spite of the non-linearity of the condition of null gaussian curvature.

The data for this construction are the B-spline polygon $\{c_0, \dots, c_L\}$, the list of knots $\{u_0, \dots, u_K\}$ and, for instance, the first plane cell of the net, given by either d_0, d_1 or d_0 and the parameters Λ^*, M^*.

Since this construction is based on blossoms of curves, it is compatible with algorithms for B-spline curves grounded on blossoms, such as, for instance, the knot insertion algorithm for subdivision of B-spline curves. That is, if we split into two pieces the interval $[u_I, u_{I+1}]$ by inclusion of a new knot $\tilde{u}$, so that the new list is $\{u_0, \dots, u_I, \tilde{u}, u_{I+1}, \dots, u_K\}$ and we compute the new B-spline polygons $\{\tilde{c}_0, \dots, \tilde{c}_{L+1}\}$ and $\{\tilde{d}_0, \dots, \tilde{d}_{L+1}\}$, these new vertices satisfy (4.3).

However, this construction is not compatible with degree elevation of B-spline curves. The degree-elevated B-spline developable surface through two B-spline curves does not coincide with the B-spline developable surface through the corresponding degree-elevated curves. See, for instance, in Figure 4.3 a developable surface and the control polygons of the degree-elevated

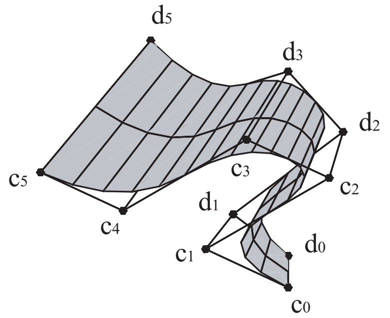

Fig. 4.2. Developable B-spline surface of 4 pieces of degree 2

boundary curves (denoted by tildes): the central cell of the degree-elevated surface is not even planar.

We show it explictly with a simple example:

Example 4.1. *Find a developable surface patch of degree two and just one piece, bounded by two curves, $c(u)$ and $d(u)$, with polygons,*

$$c_0 = (0,0,0),\ c_1 = (3,3,0),\ c_2 = (4,3,0); \quad d_0 = (0,0,2),\ d_1 = (2,2,3),$$

and knots $\{0,0,1,1\}$.

From (4.3) applied to the first cell of the B-spline net, $i = 0$,

$$(u_2 - \Lambda^*)c_0 + (\Lambda^* - u_0)c_1 = (u_2 - M^*)d_0 + (M^* - u_0)d_1,$$

with $n = 2$, $u_0 = 0$, $u_2 = 1$, we get

$$(1 - \Lambda^*)(0,0,0) + \Lambda^*(3,3,0) = (1 - M^*)(0,0,2) + M^*(2,2,3),$$

and hence $\Lambda^* = -4/3$ and $M^* = -2$.

We lack the vertex d_2, but for the second cell of the net,

$$(u_3 - \Lambda^*)c_1 + (\Lambda^* - u_1)c_2 = (u_3 - M^*)d_1 + (M^* - u_1)d_2,$$

$$\frac{7}{3}(3,3,0) - \frac{4}{3}(4,3,0) = 3(2,2,3) - 2d_2,$$

we conclude $d_2 = (13/6, 3/2, 9/2)$.

If we formally elevate the degree of both curves to three, the list of knots extends to $\{0,0,0,1,1,1\}$ and the new polygons obtained with (3.7),

$$\begin{aligned}
\tilde{c}_0 &= \tilde{c}[0,0,0] = c[0,0] = c_0 = (0,0,0) \\
\tilde{c}_1 &= \tilde{c}[0,0,1] = \frac{c[0,0] + 2c[0,1]}{3} = \frac{c_0 + 2c_1}{3} = (2,2,0) \\
\tilde{c}_2 &= \tilde{c}[0,1,1] = \frac{2c[0,1] + c[1,1]}{3} = \frac{2c_1 + c_2}{3} = (10/3,3,0) \\
\tilde{c}_3 &= \tilde{c}[1,1,1] = c[1,1] = c_2 = (4,3,0)
\end{aligned}$$

$$\begin{aligned}
\tilde{d}_0 &= \tilde{d}[0,0,0] = d[0,0] = d_0 = (0,0,2)\\
\tilde{d}_1 &= \tilde{d}[0,0,1] = \frac{d[0,0]+2d[0,1]}{3} = \frac{d_0+2d_1}{3} = (4/3,4/3,8/3)\\
\tilde{d}_2 &= \tilde{d}[0,1,1] = \frac{2d[0,1]+d[1,1]}{3} = \frac{2d_1+d_2}{3} = (37/18,11/6,7/2)\\
\tilde{d}_3 &= \tilde{d}[1,1,1] = d[1,1] = d_2 = (13/6,3/2,9/2)
\end{aligned}$$

correspond to a developable surface with non constant $\Lambda^*(u) = -2 - u/2$, $M^*(u) = -3 - u/2$ and it is easy to check that the four points that form the second cell, $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{d}_1$, $\tilde{d}_2$ do not lie on a plane.

This feature, however, will be shown useful for solving interpolation problems, as it will be noticed in the following sections.

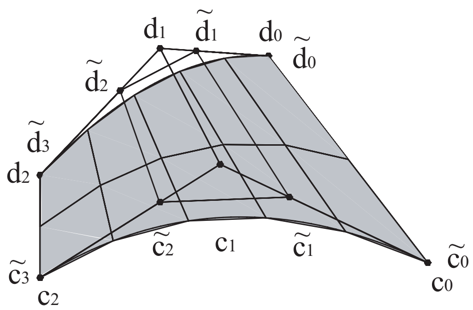

Fig. 4.3. Degree-elevated developable surface of one piece of degree 2

5. Interpolation of B-spline developable surfaces

Let us consider the following interpolation problem:

Problem 1. *Given a spline curve $c(u)$ of degree n, N pieces, B-spline polygon $\{c_0,\dots,c_L\}$ and list of knots $\{u_0,\dots,u_K\}$, $u \in [a,b]$, $a = u_{n-1}$, $b = u_{n+N-1}$, and two straight lines l_a and l_b through the endpoints of $c(u)$ with respective director vectors $\mathbf{v}$, $\mathbf{w}$, find a developable surface $c(u,v)$ such that $c(u,0) = c(u)$ and l_a and l_b are the first and last rulings of the surface, that is, $l_a:\ c(a,v)$, $l_b:\ c(b,v)$.*

The special case of Bézier curves of degree n was solved by Aumann [19], making use of his family of developable surfaces. His solution is extended to spline curves [21], solving the recursion (4.3) for the B-spline net.

We focus on the general case of crossing rulings l_a and l_b, since the particular cases of parallel or intersecting rulings may be solved in a simpler fashion resorting to cylinders and cones respectively.

As in [21], the last ruling of the developable surface can be written in terms of the B-spline net of the curve $c(u)$, the list of knots and the coefficients Λ^*, M^*,

$$d_L - c_L = \prod_{i=0}^{L-1} \frac{M^* - u_{i+n}}{M^* - u_i}(d_0 - c_0) + \frac{\Lambda^* - M^*}{M^* - u_{L-1}}\big(c_L - a(M^*)\big),$$

$$a(M^*) = \frac{M^* - u_{L-1}}{M^* - u_0} \prod_{i=1}^{L-1} \frac{M^* - u_{i+n}}{M^* - u_i} c_0 + \sum_{i=1}^{L-1} \frac{u_{i+n} - u_{i-1}}{M^* - u_{i-1}} \left(\prod_{j=i}^{L-2} \frac{M^* - u_{n+j+1}}{M^* - u_j} \right) c_i. \quad (5.1)$$

From this expression we learn that the vectors along the first and last rulings, $d_0 - c_0 = \sigma \mathbf{v}$, $d_L - c_L = \tau \mathbf{w}$, and the vector, $c_L - a(M^*)$ have to be linearly dependent and this will happen for any solution M_0^* of the algebraic equation

$$\det(a(M^*) - c_L, \mathbf{v}, \mathbf{w}) = 0. \quad (5.2)$$

This allows us to write the linear combination in terms of a basis $\{\mathbf{v}, \mathbf{w}, \mathbf{n}\}$, $\mathbf{n} = \mathbf{v} \times \mathbf{w}$,

$$a(M_0^*) = c_L + \alpha \mathbf{v} + \beta \mathbf{w} + 0\mathbf{n},$$

where the coefficients are readily obtained by Cramer's rule,

$$\alpha = \frac{\det(a(M_0^*) - c_L, \mathbf{w}, \mathbf{n})}{\det(\mathbf{v}, \mathbf{w}, \mathbf{n})}, \quad \beta = \frac{\det(\mathbf{v}, a(M_0^*) - c_L, \mathbf{n})}{\det(\mathbf{v}, \mathbf{w}, \mathbf{n})}.$$

Since M^* is fixed by the coplanarity condition (5.2), if we wish, we can modify the length of the rulings through either σ or τ just with the parameter Λ^*, which remains free so far,

$$\sigma = \alpha \frac{\Lambda^* - M_0^*}{M_0^* - u_{L-1}} \prod_{i=0}^{L-1} \frac{M_0^* - u_i}{M_0^* - u_{i+n}}, \quad \tau = \beta \frac{M_0^* - \Lambda^*}{M_0^* - u_{L-1}}. \quad (5.3)$$

Hence, we have solved the interpolation problem and we can use Λ^* for fixing either d_0 or d_L, but we cannot choose both ends of the rulings. An example of this construction is shown in Figure 5.1

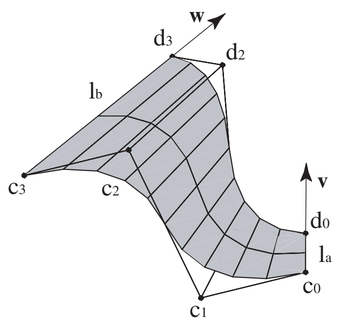

Fig. 5.1. Developable surface of degree 2 and 2 pieces

The procedure is clear:

1. Write the algebraic equation (5.2) with the B-spline polygon for $c(u)$, vectors $\mathbf{v}$, $\mathbf{w}$ and the list of knots and obtain a solution M_0^*. For any value of Λ^* the resulting developable surface will have $c(u)$ as part of the boundary and the first and last rulings will be straight lines with respective directions $\mathbf{v}$, $\mathbf{w}$.

2. Fix Λ_0^* by choosing either d_0 or d_L in (5.3).
3. Use the recursivity relation (4.3) for computing the vertices d_i for $d(u)$.
4. The B-spline polygons $\{c_0, \ldots, c_L\}$, $\{d_0, \ldots, d_L\}$ form the B-spline net for the developable patch complying with the prescription.

We show it with an example:

Example 5.1. *Consider a spline curve of degree three and three pieces with B-spline polygon*

$$c_0 = (0,0,0),\ c_1 = (2,3,0),\ c_2 = (4,3,0),\ c_3 = (5,0,0),\ c_4 = (7,2,1),\ c_5 = (9,-1,3),$$

and list of knots $\{0,0,0,0.3,0.7,1,1,1\}$*, not uniformly spaced. For the first ruling we choose direction* $\mathbf{v} = (0,0,2)$ *and for the last ruling we choose* $\mathbf{w} = (-1,0,1)$*. Find a developable surface patch bounded by* $c(u)$ *and the rulings defined by* $\mathbf{v}$*,* $\mathbf{w}$*.*

We calculate the determinant (5.2),

$$\det(a(M^*) - c_L, \mathbf{v}, \mathbf{w}) = \frac{2({M^*}^4 + 6.2{M^*}^3 - 12.3{M^*}^2 + 9.3M^* - 2.1)}{{M^*}^3(M^* - 0.3)(M^* - 0.7)},$$

and we ensure developability by choosing the parameter M^* as one of the real solutions of

$${M^*}^4 + 6.2{M^*}^3 - 12.3{M^*}^2 + 9.3M^* - 2.1 = 0,$$

which are $M^* = -7.91,\ 0.37$.

We further choose $d_0 = c_0 + \mathbf{v} = (0,0,2)$ along the first ruling, which amounts to choosing $\sigma = 1$ in (5.3), to obtain the respective values of the parameter $\Lambda^* = -6.18,\ 0.61$. We perform the calculations for the first pair of parameters, $\Lambda^* = -6.18$, $M^* = -7.91$.

We may use now Corollary 4.1 to obtain the B-spline polygon of the other boundary curve of the developable patch through $c(u)$ with prescribed rulings,

$$d_{i+1} = \frac{(u_{i+n} - \Lambda^*)c_i + (\Lambda^* - u_i)c_{i+1} + (M^* - u_{i+n})d_i}{M^* - u_i}, \qquad i = 0 \ldots L-1,$$

$$\begin{aligned}
d_1 &= \frac{(u_3 - \Lambda^*)c_0 + (\Lambda^* - u_0)c_1 + (M^* - u_3)d_0}{M^* - u_0} = (1.56, 2.34, 2.08)\\
d_2 &= \frac{(u_4 - \Lambda^*)c_1 + (\Lambda^* - u_1)c_2 + (M^* - u_4)d_1}{M^* - u_1} = (3.09, 2.29, 2.26)\\
d_3 &= \frac{(u_5 - \Lambda^*)c_2 + (\Lambda^* - u_2)c_3 + (M^* - u_5)d_2}{M^* - u_2} = (3.75, -0.15, 2.55)\\
d_4 &= \frac{(u_6 - \Lambda^*)c_3 + (\Lambda^* - u_3)c_4 + (M^* - u_6)d_3}{M^* - u_3} = (5.22, 1.42, 3.55)\\
d_5 &= \frac{(u_7 - \Lambda^*)c_4 + (\Lambda^* - u_4)c_5 + (M^* - u_7)d_4}{M^* - u_4} = (6.76, -1.00, 5.24).
\end{aligned}$$

and check that in fact d_5 lies on the last ruling since

$$d_5 - c_5 = (-2.24, 0.00, 2.24),$$

which is a vector proportional to $\mathbf{w}$. The resulting patch is shown in Figure 5.2.

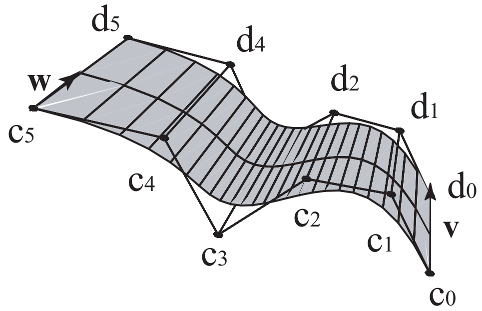

Fig. 5.2. Developable surface of degree 3 and 3 pieces

6. Degree elevation of developable surfaces

As it is pointed out in [20], degree elevation may be used for enlarging a developable patch by modifying the length of the ruling segments of the patch. The idea is simple. We may modify the length of the director vector

$$\mathbf{w}(u) = d(u) - c(u),$$

of each ruling by multiplication by a function $f(u)$,

$$\tilde{\mathbf{w}}(u) = f(u)\mathbf{w}(u) = \tilde{d}(u) - c(u),$$

and as a consequence the boundary of the surface patch changes. For instance the new second curve $\tilde{d}(u)$ starts at $\tilde{d}_0 = c_0 + f(u_{n-1})(d_0 - c_0)$ and ends at $\tilde{d}_L = c_L + f(u_{n+N-1})(d_L - c_L)$.

It is clear that this transformation just changes the patch of the developable surface that is covered by the parametrisation and it allows us to change the endpoints d_0 and d_L of the first and last rulings. The only problem is that the curve $\tilde{d}(u)$ is no longer a spline of degree n. The simplest choice for the factor is an affine function $f(u) = au + b$, and in this case the new surface patch

$$\tilde{c}(u, v) = (1 - u)c(u) + v\tilde{d}(u)$$

will be of degree $(n + 1, 1)$. An example is shown in Figure 6.1.

The next step will be the calculation of the B-spline polygon of the new boundary of the extended surface patch.

First, we obtain the blossom of the new parametrised curve,

$$\tilde{d}(u) = (1 - f(u))\, c(u) + f(u)d(u).$$

The blossom is a $(n + 1)$-affine symmetric form $\tilde{d}[u_0, \ldots, u_n]$ for which

$$\tilde{d}(u) = \tilde{d}[u^{<n+1>}].$$

Since $f(u)$ is an affine function, it is already its own blossom, $f[u] = f(u)$. For the product $h(u) = f(u)d(u)$ it is simple to produce an $(n + 1)$-affine form $\hat{h}$ satisfying $\hat{h}[u^{<n+1>}] = h(u)$,

$$\hat{h}[u_0, \ldots, u_n] = f(u_0)d[u_1, \ldots, u_n],$$

but this form is clearly non-symmetric.

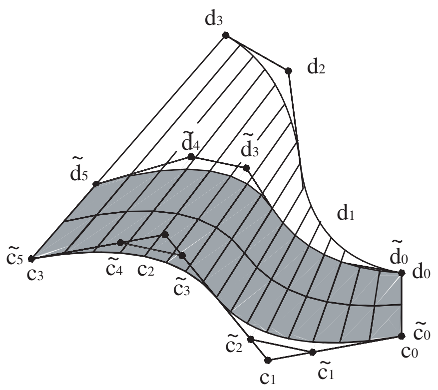

Fig. 6.1. Developable surface of degree 2 and 2 pieces stretched to a patch of degree 3

However, we may obtain a symmetric form just by permuting the argument of the function f,

$$h[u_0,\dots,u_n]=\frac{1}{n+1}\sum_{i=0}^{n}f(u_i)d[u_0,\dots,u_{i-1},u_{i+1},\dots,u_n].$$

This form h is $(n+1)$-affine, symmetric and clearly $h[u^{<n+1>}]=h(u)$. Hence, it is the blossom of the parametrisation $h(u)$.

We may use this result to conclude that the blossom of $\tilde{d}(u)$ is given by

$$\begin{aligned}\tilde{d}[u_0,\dots,u_n] &= \frac{1}{n+1}\sum_{i=0}^{n}\Big(f(u_i)d[u_0,\dots,u_{i-1},u_{i+1},\dots,u_n]\\ &+ \big(1-f(u_i)\big)c[u_0,\dots,u_{i-1},u_{i+1},\dots,u_n]\Big).\end{aligned} \tag{6.1}$$

The degree of the curve $c(u)$ must be formally elevated to $n+1$ in order to complete the B-spline net of the surface patch of degree $(n+1,1)$. It can be computed by taking $f\equiv 1$ in the previous formula for $\tilde{d}$. The degree-elevated blossom for $c(u)$ is

$$\tilde{c}[u_0,\dots,u_n]=\frac{1}{n+1}\sum_{i=0}^{n}c[u_0,\dots,u_{i-1},u_{i+1},\dots,u_n].$$

The list of knots of the degree-elevated curves [26] is also modified by increasing by one the multiplicity of the inner knots $u_{n-1},\dots,u_{n+N-1}$,

$$\{u_0,\dots,u_{n-1},u_{n-1},\dots,u_{n+N-1},u_{n+N-1},\dots,u_K\}. \tag{6.2}$$

Then the new B-spline polygons of the curves $c(u)$ and $\tilde{d}(u)$ will be $\{\tilde{c}_0,\dots,\tilde{c}_{L'}\}$, $\{\tilde{d}_0,\dots,\tilde{d}_{L'}\}$,

$$\tilde{c}_i=\tilde{c}[\tilde{u}_i,\dots,\tilde{u}_{i+n}],\ \tilde{d}_i=\tilde{d}[\tilde{u}_i,\dots,\tilde{u}_{i+n}],\ i=0,\dots,L', \tag{6.3}$$

where the list of knots (6.2) has been renumbered as $\{\tilde{u}_0,\dots,\tilde{u}_{K'}\}$ in order to have correlative indices.

This construction is useful to solve the following interpolation problem:

Problem 2. *Given a spline curve $c(u)$ of degree n, N pieces, B-spline polygon $\{c_0, \dots, c_L\}$ and list of knots $\{u_0, \dots, u_K\}$, $u \in [a, b]$, $a = u_{n-1}$, $b = u_{n+N-1}$, and two points d_0, d_L, find a developable surface $c(u, v)$ such that $c(u, 0) = c(u)$, $c(a, 1) = d_0$, $c(b, 1) = d_L$.*

The procedure for solving this problem is clear:

1. Write the algebraic equation (5.2) with the B-spline polygon for $c(u)$, the list of knots and vectors for the rulings $\overrightarrow{c_0 d_0}$, $\overrightarrow{c_L d_L}$ and obtain a solution M_0^*.
2. Fix Λ_0^* by choosing d_0 in (5.3) ($\sigma = 1$, but $\tau \neq 1$ in general).
3. Use the recursivity relation (4.3) for computing the vertices of $d(u)$.
4. Increase by one the multiplicity of the inner knots of the boundary curves.
5. Formally raise the degree of $c(u)$ and compute the new B-spline vertices $\tilde{c}_i$ with (3.7).
6. Choose $f(u)$ so that $f(a) = 1$, $f(b) = 1/\tau$,
$$f(u) = \frac{b-u}{b-a} + \frac{1}{\tau}\frac{u-a}{b-a}. \tag{6.4}$$
7. Use this function to compute the B-spline vertices $\tilde{d}_i$ for the new boundary curve $\tilde{d}(u)$ with (6.3) and (6.1).
8. The B-spline polygons $\{\tilde{c}_0, \dots, \tilde{c}_{L'}\}$, $\{\tilde{d}_0, \dots, \tilde{d}_{L'}\}$ form the B-spline net for the developable patch complying with the prescription.

We go back now to example 5.1:

Example 6.1. *Consider a spline curve of degree three and three pieces with B-spline polygon*

$$c_0 = (0,0,0),\ c_1 = (2,3,0),\ c_2 = (4,3,0),\ c_3 = (5,0,0),\ c_4 = (7,2,1),\ c_5 = (9,-1,3),$$

and list of knots $\{0, 0, 0, 0.3, 0.7, 1, 1, 1\}$. For the first ruling we choose direction $\mathbf{v} = (0, 0, 2)$ and for the last ruling we choose $\mathbf{w} = (-1, 0, 1)$. Find a developable surface patch bounded by $c(u)$, an unknown curve $\tilde{d}(u)$ and the rulings defined by $\mathbf{v}$, $\mathbf{w}$, such that $\tilde{d}(0) = c_0 + \mathbf{v} = (0, 0, 2)$, $\tilde{d}(1) = c_5 + \mathbf{w} = (8, -1, 4)$.

We already have obtained that the spline curve with B-spline polygon

$$d_0 = (0,0,2),\ d_1 = (1.56, 2.34, 2.08),\ d_2 = (3.09, 2.29, 2.26),\ d_3 = (3.75, -0.15, 2.55),$$

$$d_4 = (5.22, 1.42, 3.55),\ d_5 = (6.76, -1.00, 5.24),$$

and the same list of knots provides a developable surface patch with the required prescription except that d_5 lies on the final ruling, but it is not $(8, -1, 4)$. In fact, $d_5 = c_5 + \tau\mathbf{w}$ with $\tau = 2.24$.

In order to shorten the surface patch so that the final vertex of the new boundary curve $\tilde{d}(u)$ is $(8, -1, 4)$, we have to raise the degree of the curves from three to four.

Increasing the multiplicity of the inner knots 0, 0.3, 0.7, 1, we get the new list of knots for the degree-elevated curves,

$$\{0, 0, 0, 0, 0.3, 0.3, 0.7, 0.7, 1, 1, 1, 1\}.$$

We calculate first the B-spline polygon for $c(u)$ as a curve of formal degree four (3.7). The auxiliary points are computed in Appendix A.

$$\begin{aligned}
\tilde{c}_0 &= \tilde{c}[0,0,0,0] = c[0,0,0] = (0,0,0)\\
\tilde{c}_1 &= \tilde{c}[0,0,0,0.3] = \frac{c[0,0,0]+3c[0,0,0.3]}{4} = (1.5,2.25,0)\\
\tilde{c}_2 &= \tilde{c}[0,0,0.3,0.3] = \frac{c[0,0,0.3]+c[0,0.3,0.3]}{2} = (2.43,3,0)\\
\tilde{c}_3 &= \tilde{c}[0,0.3,0.3,0.7] = \frac{c[0,0.3,0.3]+2c[0,0.3,0.7]+c[0.3,0.3,0.7]}{4} = (3.79,2.78,0)\\
\tilde{c}_4 &= \tilde{c}[0.3,0.3,0.7,0.7] = \frac{c[0.3,0.3,0.7]+c[0.3,0.7,0.7]}{2} = (4.5,1.5,0)\\
\tilde{c}_5 &= \tilde{c}[0.3,0.7,0.7,1] = \frac{c[0.3,0.7,0.7]+2c[0.3,0.7,1]+c[0.7,0.7,1]}{4} = (5.21,0.51,0.14)\\
\tilde{c}_6 &= \tilde{c}[0.7,0.7,1,1] = \frac{c[0.7,0.7,1]+c[0.7,1,1]}{2} = (6.57,1.57,0.79)\\
\tilde{c}_7 &= \tilde{c}[0.7,1,1,1] = \frac{3c[0.7,1,1]+c[1,1,1]}{4} = (7.5,1.25,1.5)\\
\tilde{c}_8 &= \tilde{c}[1,1,1,1] = c[1,1,1] = (9,-1,3).
\end{aligned}$$

Now we have to move the curve $d(u)$ over the developable surface patch so that the new boundary curve $\tilde{d}(u)$ goes through the endpoints of both rulings, shortening the director vector $\mathbf{w}(u)$ by a factor $f(u)$ (6.4),

$$f(u) = (1-u) + \frac{u}{2.24}.$$

Finally, we use (6.1) to compute the B-spline polygon of the new boundary curve of degree four that goes through the endpoints of both rulings,

$$
\begin{aligned}
\tilde{d}_0 &= \tilde{d}[0,0,0,0] = f(0)d[0,0,0] + (1-f(0))c[0,0,0] = d_0 = (0,0,2)\\
\tilde{d}_1 &= \tilde{d}[0,0,0,0.3] = \frac{f(0.3)d[0,0,0] + 3f(0)d[0,0,0.3] + (1-f(0.3))c[0,0,0] + 3(1-f(0)c[0,0,0.3]}{4}\\
&= (1.17, 1.76, 1.97)\\
\tilde{d}_2 &= \tilde{d}[0,0,0.3,0.3] = \frac{f(0.3)d[0,0,0.3] + f(0)d[0,0.3,0.3] + (1-f(0.3))c[0,0,0.3] + (1-f(0)c[0,0.3,0.3]}{2}\\
&= (1.93, 2.39, 1.94)\\
\tilde{d}_3 &= \tilde{d}[0,0.3,0.3,0.7] = \frac{f(0.7)d[0,0.3,0.3] + 2f(0.3)d[0,0.3,0.7] + f(0)d[0.3,0.3,0.7]}{4}\\
&+ \frac{(1-f(0.7))c[0,0.3,0.3] + 2(1-f(0.3))c[0,0.3,0.7] + (1-f(0))c[0.3,0.3,0.7]}{4} = (3.06, 2.24, 1.86)\\
\tilde{d}_4 &= \tilde{d}[0.3,0.3,0.7,0.7] = \frac{f(0.3)d[0.3,0.7,0.7] + f(0.7)d[0.3,0.3,0.7]}{2}\\
&+ \frac{(1-f(0.3))c[0.3,0.7,0.7] + (1-f(0.7))c[0.3,0.3,0.7]}{2} = (3.71, 1.20, 1.74)\\
\tilde{d}_5 &= \tilde{d}[0.3,0.7,0.7,1] = \frac{f(1)d[0.3,0.7,0.7] + 2f(0.7)d[0.3,0.7,1] + f(0.3)d[0.7,0.7,1]}{4}\\
&+ \frac{(1-f(1))c[0.3,0.7,0.7] + 2(1-f(0.7))c[0.3,0.7,1] + (1-f(0.3))c[0.7,0.7,1]}{4} = (4.38, 0.35, 1.73)\\
\tilde{d}_6 &= \tilde{d}[0.7,0.7,1,1] = \frac{f(1)d[0.7,0.7,1] + f(0.7)d[0.7,1,1]}{2} + \frac{(1-f(1))c[0.7,0.7,1] + (1-f(0.7))c[0.7,1,1]}{2}\\
&= (5.68, 1.30, 2.14)\\
\tilde{d}_7 &= \tilde{d}[0.7,1,1,1] = \frac{3f(1)d[0.7,1,1] + f(0.7)d[1,1,1] + 3(1-f(1))c[0.7,1,1] + (1-f(0.7))c[1,1,1]}{4}\\
&= (6.56, 1.05, 2.70)\\
\tilde{d}_8 &= \tilde{d}[1,1,1,1] = f(1)d[1,1,1] + (1-f(1))c[1,1,1] = (8, -1, 4).
\end{aligned}
$$

The degree-elevated B-spline net for the new surface patch, complying with the requirements of the example can be seen in Figure 6.2.

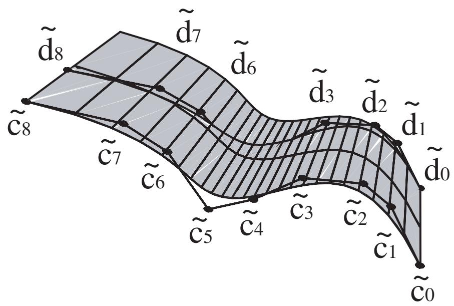

Fig. 6.2. Degree-elevation and restriction of the developable surface patch in Figure 5.2

7. Triangular developable surfaces

We may pose another interpolation problem in which the first ruling collapses to a point, $c(a) = d(a)$,

$$c(u, v) = (1 - v)c(u) + vd(u), \quad u \in [a, b].$$

The resulting developable patch is triangular in the sense that it is bounded by two curves and just one straight segment. Instead of the first point of the unknown curve of the boundary, we may give as datum its initial velocity $d'(a)$.

Problem 3. *Given a spline curve $c(u)$ of degree n, N pieces, B-spline polygon $\{c_0, \dots, c_L\}$ and list of knots $\{u_0, \dots, u_K\}$, $u \in [a, b]$, $a = u_{n-1}$, $b = u_{n+N-1}$, a point d_L and a vector $d'(a)$, find a triangular developable surface $c(u, v)$ through $c(u)$, such that $c(u, 0) = c(u)$, $c(a, v) = c_0$ for all v, $c(b, 1) = d_L$, $c_u(a, 1) = d'(a)$.*

We do not know the first the ruling of the surface, but we may use previous constructions to compute a spline developable patch through the curve $c(u)$ and use d_L to fix the last ruling,

$$c(u, v) = c(u) + v\mathbf{w}(u), \quad \mathbf{w}(u) = d(u) - c(u).$$

In order to collapse the first ruling to a point, we shorten the patch along the rulings,

$$\hat{c}(u, v) = c(u) + vf(u)\mathbf{w}(u), \quad f(u) = \frac{u - a}{b - a}, \tag{7.1}$$

so that $\hat{c}(a, v) = c_0$ for all v.

We compute the velocity,

$$\hat{c}_u(u, v) = c'(u) + \frac{v}{b - a}\mathbf{w}(u) + vf(u)\mathbf{w}'(u),$$

of the boundary curve $d(u)$ at $u = a$, making use of (3.5)

$$\hat{d}'(a) = \hat{c}_u(a, 1) = c'(a) + \frac{\mathbf{w}(a)}{b - a} = n\frac{c_1 - c_0}{u_n - u_{n-1}} + \frac{d_0 - c_0}{b - a}$$

and from this expression we get the vertex d_0 that is necessary for obtaining the velocity $\hat{d}'(a)$,

$$d_0 = c_0 + (b - a)\left(\hat{d}'(a) - n\frac{c_1 - c_0}{u_n - u_{n-1}}\right), \tag{7.2}$$

Since we need to fix both d_0 and d_L to obtain the developable patch $c(u, v)$, the construction from the previous section is required and hence such patch must be of degree $n+1$. Since $c(u)$ is still of degree n, the calculation done in (7.2) is nonetheless valid whereas we keep the original vertices c_0 and c_1. Finally, shortening the surface patch (7.1) with $f(u)$ produces a triangular patch of degree $n + 2$.

Summarising, the solution of this problem is reduced to the one of Problem 2:

1. Calculate the vertex d_0 and $\mathbf{v} = d_0 - c_0$ using (7.2).
2. Write the algebraic equation (5.2) with the B-spline polygon for $c(u)$, the list of knots and vectors for the rulings $\overrightarrow{c_0 d_0}$, $\overrightarrow{c_L d_L}$ and obtain a solution M_0^*.
3. Fix Λ_0^* by choosing d_0 in (5.3) ($\sigma = 1$, but $\tau \neq 1$ in general).

4. Use the recursivity relation (4.3) for computing the vertices of $d(u)$.

5. Increase by one the multiplicity of the inner knots of the boundary curves.

6. Formally raise the degree of $c(u)$ and compute the new B-spline vertices $\tilde{c}_i$ with (3.7).

7. Choose $f(u)$ so that $f(a) = 1$, $f(b) = 1/\tau$,

$$f(u) = \frac{b-u}{b-a} + \frac{1}{\tau}\frac{u-a}{b-a}.$$

8. Use this function to compute the B-spline vertices $\tilde{d}_i$ for the new boundary curve $\tilde{d}(u)$ with (6.3) and (6.1).

9. Increase by one the multiplicity of the inner knots of the boundary curves.

10. Formally raise the degree of $\tilde{c}(u)$ and compute the new B-spline vertices $\hat{c}_i$ with (3.7).

11. Use a function $\hat{f}(u) = u$ to shrink the first ruling to a point and compute the B-spline vertices $\hat{d}_i$ for the new boundary curve $\hat{d}(u)$ with (6.3) and (6.1).

12. The B-spline polygons $\{\hat{c}_0, \dots, \hat{c}_{L'}\}$, $\{\hat{d}_0, \dots, \hat{d}_{L'}\}$ form the B-spline net for the triangular developable patch complying with the prescription.

Example 7.1. *Consider a spline curve of degree three and three pieces with B-spline polygon*

$$c_0 = (0,0,0),\ c_1 = (2,3,0),\ c_2 = (4,3,0),\ c_3 = (5,0,0),\ c_4 = (7,2,1),\ c_5 = (9,-1,3),$$

and list of knots $\{0,0,0,0.3,0.7,1,1,1\}$. *For the last ruling we choose direction* $\mathbf{w} = (-1,0,1)$. *Find a triangular developable surface patch bounded by* $c(u)$*, an unknown curve* $\hat{d}(u)$ *and the ruling defined by* $\mathbf{w}$*, such that* $\hat{d}(0) = c_0$*,* $\hat{d}'(0) = (20, 30.5, 2)$*,* $\hat{d}(1) = c_5 + \mathbf{w} = (8,-1,4)$.

First of all, we calculate the first ruling of the developable surface. According to (7.2) we need

$$\mathbf{v} = d_0 - c_0 = \hat{d}'(0) + \frac{3}{0.3}(c_0 - c_1) = (0, 0.5, 2),$$

and we calculate the determinant (5.2),

$$\det(a(M^*) - c_L, \mathbf{v}, \mathbf{w}) = \frac{8M^{*4} + 2.6M^{*3} - 16M^{*2} + 14.5M^* - 3.5}{M^{*3}(M^* - 0.3)(M^* - 0.7)},$$

so that developability is granted by choosing parameter M^* as a real solution of

$$8M^{*4} + 2.6M^{*3} - 16M^{*2} + 14.5M^* - 3.5 = 0,$$

that is $M^* = -1.92,\ 0.38$. The other two solutions are complex.

For having $d_0 = (0, 0.5, 2)$ on the first ruling, we need to take $\sigma = 1$ in (5.3). The respective values of parameter Λ^* are $-1.16,\ 0.59$. We choose the first pair of parameters for our calculations, $\Lambda^*_0 = -1.16$, $M^*_0 = 0.59$. We calculate next the B-spline polygon for the second boundary curve according to Corollary 4.1,

$$d_{i+1} = \frac{(u_{i+n} - \Lambda^*)c_i + (\Lambda^* - u_i)c_{i+1} + (M^* - u_{i+n})d_i}{M^* - u_i}, \qquad i = 0 \dots L-1,$$

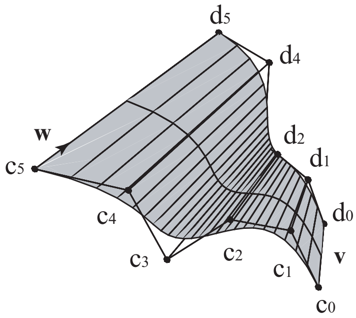

Fig. 7.1. Developable surface of degree 3 and 3 pieces

$$
\begin{aligned}
d_0 &= (0, 0.5, 2) \\
d_1 &= \frac{(u_3 - \Lambda^*)c_0 + (\Lambda^* - u_0)c_1 + (M^* - u_3)d_0}{M^* - u_0} = (1.21, 2.39, 2.31) \\
d_2 &= \frac{(u_4 - \Lambda^*)c_1 + (\Lambda^* - u_1)c_2 + (M^* - u_4)d_1}{M^* - u_1} = (2.13, 2.17, 3.16) \\
d_3 &= \frac{(u_5 - \Lambda^*)c_2 + (\Lambda^* - u_2)c_3 + (M^* - u_5)d_2}{M^* - u_2} = (1.77, -0.07, 4.80) \\
d_4 &= \frac{(u_6 - \Lambda^*)c_3 + (\Lambda^* - u_3)c_4 + (M^* - u_6)d_3}{M^* - u_3} = (2.07, 1.22, 6.97) \\
d_5 &= \frac{(u_7 - \Lambda^*)c_4 + (\Lambda^* - u_4)c_5 + (M^* - u_7)d_4}{M^* - u_4} = (2.92, -1.00, 9.08).
\end{aligned}
$$

On the contrary, $d_5 - c_5 = \tau\mathbf{w}$, with $\tau = 6.08$. We show the surface patch in Figure 7.1.

Next we shorten the surface patch so that the new boundary curve $\hat{d}(u)$ ends up at $(8, -1, 4)$. From the previous example we know that we are to increase the multiplicity of the inner knots by one,

$$\{0, 0, 0, 0, 0.3, 0.3, 0.7, 0.7, 1, 1, 1, 1\},$$

and formally raise the degree of $c(u)$ to four,

$$\tilde{c}_0 = (0, 0, 0),\ \tilde{c}_1 = (1.5, 2.25, 0),\ \tilde{c}_2 = (2.43, 3, 0),\ \tilde{c}_3 = (3.79, 2.78, 0),\ \tilde{c}_4 = (4.5, 1.5, 0),$$

$$\tilde{c}_5 = (5.21, 0.51, 0.14),\ \tilde{c}_6 = (6.57, 1.57, 0.79),\ \tilde{c}_7 = (7.5, 1.25, 1.5),\ \tilde{c}_8 = (9, -1, 3),$$

and shorten the director vector $\mathbf{w}(u)$ by a factor $f(u)$ (6.4),

$$f(u) = (1 - u) + \frac{u}{6.08},$$

so that the new boundary curve $\tilde{d}(u)$ has degree four and B-spline polygon (6.1) given by

$$
\begin{aligned}
\tilde{d}_0 &= \tilde{d}[0,0,0,0] = f(0)d[0,0,0] + (1-f(0))c[0,0,0] = d_0 = (0, 0.5, 2) \\
\tilde{d}_1 &= \tilde{d}[0,0,0,0.3] = \frac{f(0.3)d[0,0,0] + 3f(0)d[0,0,0.3] + (1-f(0.3))c[0,0,0] + 3(1-f(0)c[0,0,0.3]}{4} \\
&= (0.91, 1.89, 2.11) \\
\tilde{d}_2 &= \tilde{d}[0,0,0.3,0.3] = \frac{f(0.3)d[0,0,0.3] + f(0)d[0,0.3,0.3] + (1-f(0.3))c[0,0,0.3] + (1-f(0)c[0,0.3,0.3]}{2} \\
&= (1.51, 2.42, 2.20) \\
\tilde{d}_3 &= \tilde{d}[0,0.3,0.3,0.7] = \frac{f(0.7)d[0,0.3,0.3] + 2f(0.3)d[0,0.3,0.7] + f(0)d[0.3,0.3,0.7]}{4} \\
&+ \frac{(1-f(0.7))c[0,0.3,0.3] + 2(1-f(0.3))c[0,0.3,0.7] + (1-f(0))c[0.3,0.3,0.7]}{4} = (2.39, 2.24, 2.37) \\
\tilde{d}_4 &= \tilde{d}[0.3,0.3,0.7,0.7] = \frac{f(0.3)d[0.3,0.7,0.7] + f(0.7)d[0.3,0.3,0.7]}{2} \\
&+ \frac{(1-f(0.3))c[0.3,0.7,0.7] + (1-f(0.7))c[0.3,0.3,0.7]}{2} = (2.97, 1.26, 2.37) \\
\tilde{d}_5 &= \tilde{d}[0.3,0.7,0.7,1] = \frac{f(1)d[0.3,0.7,0.7] + 2f(0.7)d[0.3,0.7,1] + f(0.3)d[0.7,0.7,1]}{4} \\
&+ \frac{(1-f(1))c[0.3,0.7,0.7] + 2(1-f(0.7))c[0.3,0.7,1] + (1-f(0.3))c[0.7,0.7,1]}{4} = (3.64, 0.39, 2.34) \\
\tilde{d}_6 &= \tilde{d}[0.7,0.7,1,1] = \frac{f(1)d[0.7,0.7,1] + f(0.7)d[0.7,1,1]}{2} + \frac{(1-f(1))c[0.7,0.7,1] + (1-f(0.7))c[0.7,1,1]}{2} \\
&= (5.20, 1.37, 2.48) \\
\tilde{d}_7 &= \tilde{d}[0.7,1,1,1] = \frac{3f(1)d[0.7,1,1] + f(0.7)d[1,1,1] + 3(1-f(1))c[0.7,1,1] + (1-f(0.7))c[1,1,1]}{4} \\
&= (6.26, 1.15, 2.87) \\
\tilde{d}_8 &= \tilde{d}[1,1,1,1] = f(1)d[1,1,1] + (1-f(1))c[1,1,1] = (8, -1, 4),
\end{aligned}
$$

where the auxiliary points are computed with blossoms in Appendix B. The result of this restriction of the surface patch is shown in Figure 7.2.

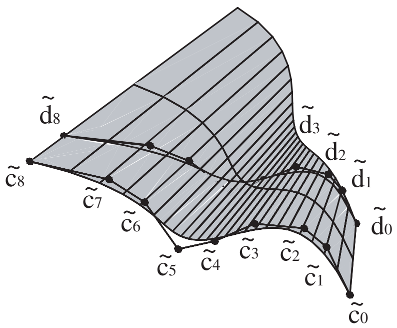

Fig. 7.2. Restriction of the developable surface patch in Figure 7.1

Finally, we further trim (7.1) the surface patch bounded by $c(u)$ and $\tilde{d}(u)$ to shrink the first ruling to the vertex c_0.

Since we are raising the degree of the curves from four to five, we have to increase the multiplicity of the inner knots by one,

$$\{0, 0, 0, 0, 0, 0.3, 0.3, 0.3, 0.7, 0.7, 0.7, 1, 1, 1, 1, 1\}.$$

The curve $c(u)$ becomes formally of degree five (3.7) with B-spline polygon,

$$\begin{aligned}
\hat{c}_0 &= \hat{c}[0,0,0,0,0] = \tilde{c}[0,0,0,0] = (0,0,0) \\
\hat{c}_1 &= \hat{c}[0,0,0,0,0.3] = \frac{\tilde{c}[0,0,0,0] + 4\tilde{c}[0,0,0,0.3]}{5} = (1.20, 1.80, 0.0) \\
\hat{c}_2 &= \hat{c}[0,0,0,0.3,0.3] = \frac{2\tilde{c}[0,0,0,0.3] + 3\tilde{c}[0,0,0.3,0.3]}{5} = (2.06, 2.70, 0.0) \\
\hat{c}_3 &= \hat{c}[0,0,0.3,0.3,0.3] = \frac{3\tilde{c}[0,0,0.3,0.3] + 2\tilde{c}[0,0.3,0.3,0.3]}{5} = (2.66, 2.96, 0.0) \\
\hat{c}_4 &= \hat{c}[0,0.3,0.3,0.3,0.7] = \frac{\tilde{c}[0,0.3,0.3,0.3] + 3\tilde{c}[0,0.3,0.3,0.7] + \tilde{c}[0.3,0.3,0.3,0.7]}{5} = (3.69, 2.69, 0.0) \\
\hat{c}_5 &= \hat{c}[0.3,0.3,0.3,0.7,0.7] = \frac{2\tilde{c}[0.3,0.3,0.3,0.7] + 3\tilde{c}[0.3,0.3,0.7,0.7]}{5} = (4.34, 1.79, 0.0) \\
\hat{c}_6 &= \hat{c}[0.3,0.3,0.7,0.7,0.7] = \frac{3\tilde{c}[0.3,0.3,0.7,0.7] + 2\tilde{c}[0.3,0.7,0.7,0.7]}{5} = (4.66, 1.27, 0.03) \\
\hat{c}_7 &= \hat{c}[0.3,0.7,0.7,0.7,1] = \frac{\tilde{c}[0.3,0.7,0.7,0.7] + 3\tilde{c}[0.3,0.7,0.7,1] + \tilde{c}[0.7,0.7,0.7,1]}{5} = (5.31, 0.72, 0.20) \\
\hat{c}_8 &= \hat{c}[0.7,0.7,0.7,1,1] = \frac{2\tilde{c}[0.7,0.7,0.7,1] + 3\tilde{c}[0.7,0.7,1,1]}{5} = (6.34, 1.39, 0.68) \\
\hat{c}_9 &= \hat{c}[0.7,0.7,1,1,1] = \frac{3\tilde{c}[0.7,0.7,1,1] + 2\tilde{c}[0.7,1,1,1]}{5} = (6.94, 1.44, 1.07) \\
\hat{c}_{10} &= \hat{c}[0.7,1,1,1,1] = \frac{4\tilde{c}[0.7,1,1,1] + 2\tilde{c}[1,1,1,1]}{5} = (7.80, 0.80, 1.80) \\
\hat{c}_{11} &= \hat{c}[1,1,1,1,1] = \tilde{c}[1,1,1,1] = (9, -1, 3),
\end{aligned}$$

and we shrink the rulings (7.1) with a factor $\hat{f}(u) = u$. The auxiliary points are computed using the multiaffinity property of blossoms in Appendix C.

Making use of (6.1), we obtain the B-spline polygon of the final boundary curve $\hat{d}(u)$ of degree five,

$$
\begin{aligned}
\hat{d}_0 &= \hat{d}[0,0,0,0,0] = \hat{f}(0)\tilde{d}[0,0,0,0] + (1-\hat{f}(0))\tilde{c}[0,0,0,0] = \tilde{c}_0 = (0,0,0)\\
\hat{d}_1 &= \hat{d}[0,0,0,0,0.3] = \frac{\hat{f}(0.3)\tilde{d}[0,0,0,0] + 4\hat{f}(0)\tilde{d}[0,0,0,0.3] + (1-\hat{f}(0.3))\tilde{c}[0,0,0,0] + 4(1-\hat{f}(0)\tilde{c}[0,0,0,0.3]}{5}\\
&= (1.20, 1.83, 0.12)\\
\hat{d}_2 &= \hat{d}[0,0,0,0.3,0.3] = \frac{2\hat{f}(0.3)\tilde{d}[0,0,0,0.3] + 3\hat{f}(0)\tilde{d}[0,0,0.3,0.3]}{5}\\
&+ \frac{2(1-\hat{f}(0.3))\tilde{c}[0,0,0,0.3] + 3(1-\hat{f}(0)\tilde{c}[0,0,0.3,0.3]}{5} = (1.99, 2.66, 0.25)\\
\hat{d}_3 &= \hat{d}[0,0,0.3,0.3,0.3] = \frac{3\hat{f}(0.3)\tilde{d}[0,0,0.3,0.3] + 2\hat{f}(0)\tilde{d}[0,0.3,0.3,0.3]}{5}\\
&+ \frac{3(1-\hat{f}(0.3))\tilde{c}[0,0,0.3,0.3] + 2(1-\hat{f}(0)\tilde{c}[0,0.3,0.3,0.3]}{5} = (2.50, 2.86, 0.40)\\
\hat{d}_4 &= \hat{d}[0,0.3,0.3,0.3,0.7] = \frac{\hat{f}(0.7)\tilde{d}[0,0.3,0.3,0.3] + 3\hat{f}(0.3)\tilde{d}[0,0.3,0.3,0.7] + \hat{f}(0)\tilde{d}[0.3,0.3,0.3,0.7]}{5}\\
&+ \frac{(1-\hat{f}(0.7))\tilde{c}[0,0.3,0.3,0.3] + 3(1-\hat{f}(0.3))\tilde{c}[0,0.3,0.3,0.7] + (1-\hat{f}(0))\tilde{c}[0.3,0.3,0.3,0.7]}{5} = (3.29, 2.52, 0.75)\\
\hat{d}_5 &= \hat{d}[0.3,0.3,0.3,0.7,0.7] = \frac{2\hat{f}(0.7)\tilde{d}[0.3,0.3,0.3,0.7] + 3\hat{f}(0.3)\tilde{d}[0.3,0.3,0.7,0.7]}{5}\\
&+ \frac{2(1-\hat{f}(0.7))\tilde{c}[0.3,0.3,0.3,0.7] + 3(1-\hat{f}(0.3))\tilde{c}[0.3,0.3,0.7,0.7]}{5} = (3.65, 1.64, 1.09)\\
\hat{d}_6 &= \hat{d}[0.3,0.3,0.7,0.7,0.7] = \frac{3\hat{f}(0.7)\tilde{d}[0.3,0.3,0.7,0.7] + 2\hat{f}(0.3)\tilde{d}[0.3,0.7,0.7,0.7]}{5}\\
&+ \frac{3(1-\hat{f}(0.7))\tilde{c}[0.3,0.3,0.7,0.7] + 2(1-\hat{f}(0.3))\tilde{c}[0.3,0.7,0.7,0.7]}{5} = (3.83, 1.15, 1.30)\\
\hat{d}_7 &= \hat{d}[0.3,0.7,0.7,0.7,1] = \frac{\hat{f}(1)\tilde{d}[0.3,0.7,0.7,0.7] + 3\hat{f}(0.7)\tilde{d}[0.3,0.7,0.7,1] + \hat{f}(0.3)\tilde{d}[0.7,0.7,0.7,1]}{5}\\
&+ \frac{(1-\hat{f}(1))\tilde{c}[0.3,0.7,0.7,0.7] + 3(1-\hat{f}(0.7))\tilde{c}[0.3,0.7,0.7,1] + (1-\hat{f}(0.3))\tilde{c}[0.7,0.7,0.7,1]}{5} = (4.25, 0.62, 1.70)\\
\hat{d}_8 &= \hat{d}[0.7,0.7,0.7,1,1] = \frac{2\hat{f}(1)\tilde{d}[0.7,0.7,0.7,1] + 3\hat{f}(0.7)\tilde{d}[0.7,0.7,1,1]}{5}\\
&+ \frac{2(1-\hat{f}(1))\tilde{c}[0.7,0.7,0.7,1] + 3(1-\hat{f}(0.7))\tilde{c}[0.7,0.7,1,1]}{5} = (5.18, 1.24, 2.15)\\
\hat{d}_9 &= \hat{d}[0.7,0.7,1,1,1] = \frac{3\hat{f}(1)\tilde{d}[0.7,0.7,1,1] + 2\hat{f}(0.7)\tilde{d}[0.7,1,1,1]}{5}\\
&+ \frac{3(1-\hat{f}(1))\tilde{c}[0.7,0.7,1,1] + 2(1-\hat{f}(0.7))\tilde{c}[0.7,1,1,1]}{5} = (5.77, 1.30, 2.47)\\
\hat{d}_{10} &= \hat{d}[0.7,1,1,1,1] = \frac{4\hat{f}(1)\tilde{d}[0.7,1,1,1] + \hat{f}(0.7)\tilde{d}[1,1,1,1] + 4(1-\hat{f}(1))\tilde{c}[0.7,1,1,1] + (1-\hat{f}(0.7))\tilde{c}[1,1,1,1]}{5}\\
&= (6.67, 0.72, 3.03)\\
\hat{d}_{11} &= \hat{d}[1,1,1,1,1] = \hat{f}(1)\tilde{d}[1,1,1,1] + (1-\hat{f}(1))\tilde{c}[1,1,1,1] = \tilde{d}_{11} = (8,-1,4).
\end{aligned}
$$

The triangular B-spline net for the surface patch which satisfies the requirements of the example is shown in Figure 7.3.

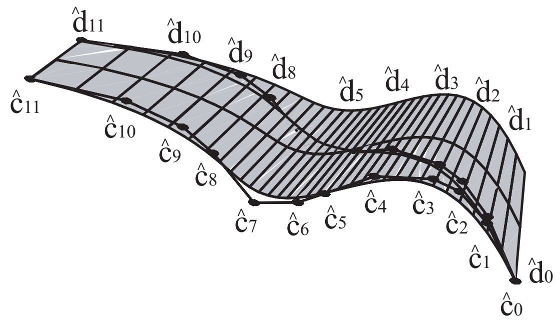

Fig. 7.3. Restriction to a triangular patch of the developable surface patch in Figure 7.2

We check that in fact the velocity of the boundary curve $\hat{d}(u)$ of degree $n = 5$ is as prescribed,

$$\hat{d}'(0) = n\frac{\hat{d}_1 - \hat{d}_0}{\hat{u}_n - \hat{u}_{n-1}} = \frac{5}{0.3}(1.20, 1.83, 0.12) = (20.00, 30.50, 2.00).$$

8. Conclusions

We have made use of a procedure of degree elevation for obtaining spline developable surfaces from which we know the segments of the first and last rulings and one of the curves of the boundary. It consists in first solving the problem with free endpoints of the rulings and then move the resulting boundary curve along the rulings to match the endpoints and increase the degree of the curves by one. This solution is also used to solve the problem of finding a triangular spline developable patch from which we know the last ruling, one of the curves of the boundary and the initial velocity of the other curve.

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A. Appendix

We perform here calculations of auxiliary points for the curve $c(u)$ over the list of knots $\{0,0,0,0.3,0.7,1,1,1\}$ which are needed for Example 6.1, taking into account that blossoms are multiaffine (3.7):

$$\begin{aligned}
c[0,0,0] &= c_0 = (0,0,0)\\
c[0,0,0.3] &= c_1 = (2,3,0)\\
c[0,0.3,0.3] &= \frac{0.7-0.3}{0.7-0}c[0,0,0.3] + \frac{0.3-0}{0.7-0}c[0,0.7,0.3] = \frac{0.4c_1+0.3c_2}{0.7} = (2.86,3,0)\\
c[0,0.3,0.7] &= c_2 = (4,3,0)\\
c[0.3,0.3,0.7] &= \frac{1-0.3}{1-0}c[0,0.3,0.7] + \frac{0.3-0}{1-0}c[1,0.3,0.7] = 0.7c_2+0.3c_3 = (4.3,2.1,0)\\
c[0.3,0.7,0.7] &= \frac{1-0.7}{1-0}c[0,0.3,0.7] + \frac{0.7-0}{1-0}c[1,0.3,0.7] = 0.3c_2+0.7c_3 = (4.7,0.9,0)\\
c[0.3,0.7,1] &= c_3 = (5,0,0)\\
c[0.7,0.7,1] &= \frac{1-0.7}{1-0.3}c[0.3,0.7,1] + \frac{0.7-0.3}{1-0.3}c[1,0.7,1] = \frac{0.3c_3+0.4c_4}{0.7} = (6.14,1.14,0.57)\\
c[0.7,1,1] &= c_4 = (7,2,1)\\
c[1,1,1] &= c_5 = (9,-1,3).
\end{aligned}$$

And similarly for $d(u)$,

$$\begin{aligned}
d[0,0,0] &= d_0 = (0,0,2)\\
d[0,0,0.3] &= d_1 = (1.56,2.34,2.08)\\
d[0,0.3,0.3] &= \frac{0.7-0.3}{0.7-0}d[0,0,0.3] + \frac{0.3-0}{0.7-0}d[0,0.7,0.3] = \frac{0.4d_1+0.3d_2}{0.7} = (2.21,2.32,2.15)\\
d[0,0.3,0.7] &= d_2 = (3.09,2.29,2.26)\\
d[0.3,0.3,0.7] &= \frac{1-0.3}{1-0}d[0,0.3,0.7] + \frac{0.3-0}{1-0}d[1,0.3,0.7] = 0.7d_2+0.3d_3 = (3.29,1.56,2.35)\\
d[0.3,0.7,0.7] &= \frac{1-0.7}{1-0}d[0,0.3,0.7] + \frac{0.7-0}{1-0}d[1,0.3,0.7] = 0.3d_2+0.7d_3 = (3.55,0.58,2.46)\\
d[0.3,0.7,1] &= d_3 = (3.75,-0.15,2.55)\\
d[0.7,0.7,1] &= \frac{1-0.7}{1-0.3}d[0.3,0.7,1] + \frac{0.7-0.3}{1-0.3}d[1,0.7,1] = \frac{0.3d_3+0.4d_4}{0.7} = (4.59,0.75,3.12)\\
d[0.7,1,1] &= d_4 = (5.22,1.42,3.55)\\
d[1,1,1] &= d_5 = (6.76,-1.00,5.24).
\end{aligned}$$

B. Appendix

We compute here auxiliary points for the curve $d(u)$ over the list of knots $\{0, 0, 0, 0.3, 0.7, 1, 1, 1\}$ which are needed for Example 7.1, using the property of multiaffinity (3.7) for blossoms:

$$\begin{aligned}
d[0,0,0] &= d_0 = (0, 0.5, 2) \\
d[0,0,0.3] &= d_1 = (1.21, 2.39, 2.31) \\
d[0,0.3,0.3] &= \frac{0.7-0.3}{0.7-0} d[0,0,0.3] + \frac{0.3-0}{0.7-0} d[0,0.7,0.3] = \frac{0.4d_1 + 0.3d_2}{0.7} = (1.61, 2.30, 2.67) \\
d[0,0.3,0.7] &= d_2 = (2.13, 2.17, 3.16) \\
d[0.3,0.3,0.7] &= \frac{1-0.3}{1-0} d[0,0.3,0.7] + \frac{0.3-0}{1-0} d[1,0.3,0.7] = 0.7d_2 + 0.3d_3 = (2.02, 1.50, 3.65) \\
d[0.3,0.7,0.7] &= \frac{1-0.7}{1-0} d[0,0.3,0.7] + \frac{0.7-0}{1-0} d[1,0.3,0.7] = 0.3d_2 + 0.7d_3 = (1.88, 0.60, 4.31) \\
d[0.3,0.7,1] &= d_3 = (1.77, -0.07, 4.80) \\
d[0.7,0.7,1] &= \frac{1-0.7}{1-0.3} d[0.3,0.7,1] + \frac{0.7-0.3}{1-0.3} d[1,0.7,1] = \frac{0.3d_3 + 0.4d_4}{0.7} = (1.94, 0.67, 6.04) \\
d[0.7,1,1] &= d_4 = (2.07, 1.22, 6.97) \\
d[1,1,1] &= d_5 = (2.92, -1.00, 9.08).
\end{aligned}$$

C. Appendix

Finally we calculate the auxiliary points which are necessary to formally raise the degree of the curve $\tilde{c}(u)$ with list of knots $\{0, 0, 0, 0, 0.3, 0.3, 0.7, 0.7, 1, 1, 1, 1\}$ from four to five using the property of multiaffinity (3.7) for blossoms:

$$\begin{aligned}
\tilde{c}[0,0,0,0] &= \tilde{c}_0 = (0, 0, 0) \\
\tilde{c}[0,0,0,0.3] &= \tilde{c}_1 = (1.50, 2.25, 0.00) \\
\tilde{c}[0,0,0.3,0.3] &= \tilde{c}_2 = (2.43, 3.00, 0.00) \\
\tilde{c}[0,0.3,0.3,0.3] &= \frac{0.7-0.3}{0.7-0} \tilde{c}[0,0,0.3,0.3] + \frac{0.3-0}{0.7-0} \tilde{c}[0,0.7,0.3,0.3] = \frac{0.4\tilde{c}_2 + 0.3\tilde{c}_3}{0.7} = (3.01, 2.90, 0.00) \\
\tilde{c}[0,0.3,0.3,0.7] &= \tilde{c}_3 = (3.79, 2.78, 0.00) \\
\tilde{c}[0.3,0.3,0.3,0.7] &= \frac{0.7-0.3}{0.7-0} \tilde{c}[0,0.3,0.3,0.7] + \frac{0.3-0}{0.7-0} \tilde{c}[0.7,0.3,0.3,0.7] = \frac{0.4\tilde{c}_3 + 0.3\tilde{c}_4}{0.7} = (4.09, 2.23, 0.00) \\
\tilde{c}[0.3,0.3,0.7,0.7] &= \tilde{c}_4 = (4.50, 1.50, 0.00) \\
\tilde{c}[0.3,0.7,0.7,0.7] &= \frac{1-0.7}{1-0.3} \tilde{c}[0.3,0.3,0.7,0.7] + \frac{0.7-0.3}{1-0.3} \tilde{c}[1,0.3,0.7,0.7] = \frac{0.3\tilde{c}_4 + 0.4\tilde{c}_5}{0.7} = (4.91, 0.93, 0.08) \\
\tilde{c}[0.3,0.7,0.7,1] &= \tilde{c}_5 = (5.21, 0.51, 0.14) \\
\tilde{c}[0.7,0.7,0.7,1] &= \frac{1-0.7}{1-0.3} \tilde{c}[0.3,0.7,0.7,1] + \frac{0.7-0.3}{1-0.3} \tilde{c}[1,0.7,0.7,1] = \frac{0.3\tilde{c}_5 + 0.4\tilde{c}_6}{0.7} = (5.99, 1.12, 0.51) \\
\tilde{c}[0.7,0.7,1,1] &= \tilde{c}_6 = (6.57, 1.57, 0.79) \\
\tilde{c}[0.7,1,1,1] &= \tilde{c}_7 = (7.50, 1.25, 1.50) \\
\tilde{c}[1,1,1,1] &= \tilde{c}_8 = (9, -1, 3).
\end{aligned}$$

And similarly for $\tilde{d}(u)$,

$$
\begin{aligned}
\tilde{d}[0,0,0,0] &= \tilde{d}_0 = (0, 0.5, 2)\\
\tilde{d}[0,0,0,0.3] &= \tilde{d}_1 = (0.91, 1.89, 2.11)\\
\tilde{d}[0,0,0.3,0.3] &= \tilde{d}_2 = (1.51, 2.42, 2.20)\\
\tilde{d}[0,0.3,0.3,0.3] &= \frac{0.7-0.3}{0.7-0}\tilde{d}[0,0,0.3,0.3] + \frac{0.3-0}{0.7-0}\tilde{d}[0,0.7,0.3,0.3] = \frac{0.4\tilde{d}_2 + 0.3\tilde{d}_3}{0.7} = (1.89, 2.35, 2.28)\\
\tilde{d}[0,0.3,0.3,0.7] &= \tilde{d}_3 = (2.39, 2.24, 2.37)\\
\tilde{d}[0.3,0.3,0.3,0.7] &= \frac{0.7-0.3}{0.7-0}\tilde{d}[0,0.3,0.3,0.7] + \frac{0.3-0}{0.7-0}\tilde{d}[0.7,0.3,0.3,0.7] = \frac{0.4\tilde{d}_3 + 0.3\tilde{d}_4}{0.7} = (2.64, 1.82, 2.37)\\
\tilde{d}[0.3,0.3,0.7,0.7] &= \tilde{d}_4 = (2.97, 1.26, 2.37)\\
\tilde{d}[0.3,0.7,0.7,0.7] &= \frac{1-0.7}{1-0.3}\tilde{d}[0.3,0.3,0.7,0.7] + \frac{0.7-0.3}{1-0.3}\tilde{d}[1,0.3,0.7,0.7] = \frac{0.3\tilde{d}_4 + 0.4\tilde{d}_5}{0.7} = (3.35, 0.77, 2.35)\\
\tilde{d}[0.3,0.7,0.7,1] &= \tilde{d}_5 = (3.64, 0.39, 2.34)\\
\tilde{d}[0.7,0.7,0.7,1] &= \frac{1-0.7}{1-0.3}\tilde{d}[0.3,0.7,0.7,1] + \frac{0.7-0.3}{1-0.3}\tilde{d}[1,0.7,0.7,1] = \frac{0.3\tilde{d}_5 + 0.4\tilde{d}_6}{0.7} = (4.53, 0.95, 2.42)\\
\tilde{d}[0.7,0.7,1,1] &= \tilde{d}_6 = (5.20, 1.37, 2.48)\\
\tilde{d}[0.7,1,1,1] &= \tilde{d}_7 = (6.26, 1.15, 2.87)\\
\tilde{d}[1,1,1,1] &= \tilde{d}_8 = (8, -1, 4).
\end{aligned}
$$