ON SMOOTHNESS OF SYMMETRIC MAPPINGS II

A. CANTÓN

ABSTRACT. If the dilatation of a quasiconformal selfmap of the upper half plane vanishes near the real line as a power of the height, the induced quasisymmetric mapping is Lipschitz with the same exponent. In this note, it is shown that the converse does not hold for any positive exponent. In addition, a sufficient condition is found to have locally a quasiconformal extension with the desired growth in the dilatation.

1. INTRODUCTION AND NOTATION

We will consider quasiconformal selfmappings f, of the upper half plane $\mathbb{H}$, that fix ∞. These maps induce a boundary homeomorphism, h on $\mathbb{R}$, that satisfies the M-condition, namely:

$$\frac{1}{M} \leq \left| \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \right| \leq M$$

for all $t > 0$ and $x \in \mathbb{R}$.

Conversely, if $h : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism for which the M-condition holds (so called quasisymmetric mapping), there exists a quasiconformal extension of h to the upper half plane. This extension is not unique. In spite of the regularity implied by the M-condition, quasisymmetric mappings can be singular with respect to the Lebesgue measure (see [4]).

We shall use the standard notations for the derivatives of f, $f_z = \partial f/\partial z$ and $f_{\overline{z}} = \partial f/\partial \overline{z}$. The complex dilatation of a quasiconformal mapping f at the point z is $\mu_f(z) = f_{\overline{z}}(z)/f_z(z)$. This dilatation is defined almost everywhere and $\|\mu_f\|_\infty < 1$. When $\|\mu_f\|_\infty = 0$, f is a conformal automorphism of $\mathbb{H}$.

We say that f is locally quasiconformal if $\text{ess sup}_{z \in K} |\mu_f(z)| \leq k < 1$ for every compact set in $K \subset \mathbb{H}$, where k depends on K. Let

$$K(y) = \text{ess sup}_{\substack{x \in \mathbb{R}, \\ 0 < t \leq y}} |\mu_f(x + it)|.$$

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If $K(y) \searrow 0$ as $y \searrow 0$ one might expect the corresponding quasisymmetric mapping h to be smoother. Carleson [5], Anderson and Hinkkanen [2], Hamilton [10], Semmes [14], Gardiner and Sullivan [9], Dahlberg [6], Dyn'kin [7], Belinskij and, Nikolaev and Shefel [13] among others have studied this problem.

In the remainder of the paper we will always consider β a positive exponent and $n \in \mathbb{N} \cup \{0\}$ an integer such that $0 < \beta - n \leq 1$.

Since we are interested in the behaviour of quasiconformal mappings near the real line, we are going to introduce the class $\mathcal{Q}_\beta$ that consists of those locally quasiconformal mappings for which

$$\text{ess sup}_{\substack{x\in\mathbb{R},\\ 0<t\leq y}} |\mu_f(x+it)| = \mathcal{O}(y^\beta).$$

Different classes of smoothness of real functions will be used in this note.

1. Let Λ^β stand for the class of Lipschitz functions of exponent β, *i.e.* those functions $h \in C^n(\mathbb{R})$, so that
$$|h^{(n)}(x+t) - h^{(n)}(x)| \leq Ct^{\beta-n}$$
uniformly on compact sets.
2. We say that a function h has n-th Peano derivative at a point x if there exists a polynomial of degree at most n, P_n, such that
$$h(x+t) - P_n(t) = o(t^n), \quad t \to 0.$$
Then the n-th Peano derivative of h at x is $P_n^{(n)}(0)$. See [15] for references. It is easy to see that if h has n-th Peano derivative at x $(n \geq 1)$ then $h'(x)$ exists.
We will denote by $\mathcal{P}^\beta$ the class of functions h that can be approximated by a polynomial of degree at most n, in the following sense,
$$|h(x+t) - P_n(t)| \leq C(x)t^\beta.$$
The constant may depend on x, but it will be uniform in compact subsets of $\mathbb{R}$. Note that if $h \in \mathcal{P}^\beta$, then h has k-th Peano derivative for any integer $k \leq \beta$. Clearly if $h \in \Lambda^\beta$ then $h \in \mathcal{P}^\beta$.
3. Finally, for $\alpha \in (0,1)$ denote by $\mathcal{M}_\alpha$ the collection of quasisymmetric mappings h which satisfy
$$\sup_{|x|\leq N,\ 0<t<N} \left| \frac{h(x+t) + h(x-t) - 2h(x)}{h(x+t) - h(x)} \right| t^{-\alpha} < \infty.$$
This condition was introduced in more generality by Carleson in [5].

This note is a follow up to [1] where, to answer a question raised in [2], it was shown that there are quasisymmetric Lipschitz maps of exponent $\alpha \in (0,1)$ which are not the restriction to the real line of a quasiconformal function in $\mathcal{Q}_\alpha$. In Theorem 2 we extend this result for any exponent $\beta > 0$ and in Theorem 3 we find a sufficient condition to have locally such an extension.

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2. Dilatation decay and smoothness

In this section there will be stated several known results that relate the rate of the decay of the dilatation of a quasiconformal map in a neighbourhood of a point with the smoothness of the map at such a point.

The first of these results is the well-known Teichmüller-Wittich-Belinskij theorem that roughly asserts that when the dilatation of a quasiconformal map vanishes at a point then the function is conformal at that point (see [11]; p.224-232, for references). Namely,

Theorem A. *Let f be a quasiconformal map of the plane such that $f(0) = 0$, $f(\infty) = \infty$ and*

$$I(r) = \frac{1}{2\pi} \int\int_{|z|<r} \frac{|\mu_f(z)|}{|z|^2} dA(z) < \infty, \quad \text{for } r < R.$$

Then there exists a complex number $w \neq 0$ such that

$$\left| \frac{f(z)}{z} - w \right| \leq |w| \varepsilon(|z|),$$

where $\varepsilon(|z|) \to 0$ as $z \to 0$ and ε depends only on I, R and $\|\mu_f\|_\infty$.

In the same vein, the next theorem due to Nikolaev and Sheffel shows that if the dilatation decays at certain rate in a neighbourhood of a point then the quasiconformal map has Peano derivative at such a point (see [13]).

Theorem B. *Let f be quasiconformal on a disc D, centered at 0 and radius R, and suppose $|\mu_f(z)| \leq C|z|^\beta$ for almost every z in D. Then there exists a polynomial P_{n+1} of degree at most $n+1$, $0 < \beta - n \leq 1$, such that,*

$$|f(z) - P_{n+1}(z)| \leq \tilde{C}|z|^{\beta+1},$$

where $\tilde{C}$ depends on C, β, R and the diameter of $f(D)$.

When f is quasiconformal in $\mathbb{H}$ and the dilatation of f decays near a boundary point, then the corresponding quasisymmetric mapping is smooth at that point. This was Carleson's point of view in [5]. Concretely,

Theorem C. *Let $\alpha \in (0, 1)$, then $f \in \mathcal{Q}_\alpha$ if and only if $h = f|_{\mathbb{R}} \in \mathcal{M}_\alpha$.*

The theorem appears in this form in [1].

For general exponents $\beta > 0$, something else can be said about the smoothness of the quasisymmetric restriction, as the following theorem due to Anderson and Hinkkanen and Dyn'kin shows (see [2] and [7] respectively).

Theorem D. *If $f \in \mathcal{Q}_\beta$ then $h = f|_{\mathbb{R}} \in \Lambda^{\beta+1}$.*

Anderson and Hinkkanen have obtained that $h \in \Lambda^{\gamma}$ for all $\gamma < \beta+1$ (see [2]). Dyn'kin showed the result in the form presented here (see [7], Theorem 5).

3. Quasiconformal extensions

Here and hereafter by a quasiconformal extension we mean a quasiconformal extension to the upper half plane of a quasisymmetric mapping of the real line that fixes ∞.

In the middle 50's Beurling and Ahlfors showed that,

$$f(x+iy) = \frac{1}{2}\left(\int_0^1 h(x+ty) + h(x-ty)dt\right) + \frac{i}{2}\left(\int_0^1 h(x+ty) - h(x-ty)dt\right)$$

is a quasiconformal extension of h whenever h is a quasisymmetric mapping.

Many variants of the Beurling-Ahlfors construction have been studied where the real part of the extension consists on a convolution of h with a positive even kernel K_e that integrates to 1 and whose imaginary part is a convolution with an odd kernel, K_o such that $K_o(x) \geq 0$ for $x \geq 0$. In order to obtain a quasiconformal extension, decay conditions on the kernels as $|x| \to \infty$ are needed, see for example [8], p.115.

In the same line we can consider extensions defined as follows. Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism, $h \in C^1(\mathbb{R})$. We say that $h \in \mathcal{A}B_k$ if there exist $k+1$ distinct real numbers $0 < a_0 < ... < a_k < 1$ and $M \geq 1$ so that

$$\frac{1}{M} \leq \frac{\sum_{j=0}^{k} h'(x+a_j y)}{\sum_{j=0}^{k} h'(x-a_j y)} \leq M, \tag{1}$$

for every $x \in \mathbb{R}$ and $y \geq 0$.

Theorem 1. *$h \in \mathcal{A}B_k$ if and only if*

$$\begin{aligned} f(x+iy) &= \tfrac{1}{2}\left(\textstyle\sum_{j=0}^{k} p_j(h(x+a_j y) + h(x-a_j y))\right) \\ &\quad + \tfrac{i}{2}\left(\textstyle\sum_{j=0}^{k} p_j(h(x+a_j y) - h(x-a_j y))\right) \end{aligned}$$

is a quasiconformal extension of h, where $a_j \in (0,1)$ as given above and p_j are positive numbers $\sum_{j=0}^{k} p_j = 1$.

Here we are considering $\frac{1}{2}\sum_{j=0}^{k} p_j(\delta_{a_j} + \delta_{-a_j})$ as the even kernel of the extension and $\frac{1}{2}\sum_{j=0}^{k} p_j(\delta_{a_j} - \delta_{-a_j})$ as the odd kernel.

Remarks

1. If the condition $\mathcal{A}B_k$ is defined locally (*i.e.* the quotient is locally bounded) then the extension is locally quasiconformal.
2. $h \in \mathcal{A}B_0$ if and only if h is a $C^1(\mathbb{R})$ homeomorphism such that $\log h'$ is bounded. As a consequence h is a $C^1(\mathbb{R})$ increasing bilipschitz homeomorphism.

3. If $h \in \mathcal{A}B_k$ then h is quasisymmetric.
4. Choosing a_j so that $\frac{1}{2k} < |a_j - a_{j-1}| \leq \frac{2}{k}$ and taking limits in the extension one gets Beurling-Ahlfors extension.

Proof. Clearly f is an extension of h, *i.e.* $f|_{\mathbb{R}} = h$. Indeed, $f(x) = \frac{1}{2}\sum_{j=0}^{k} p_j 2h(x) = h(x)\sum_{j=0}^{k} p_j = h(x)$.

Notice also that f is a proper function, that is, $\lim_{|z|\to\infty} |f(z)| = \infty$.

The modulus of f is given by,

$$2|f|^2 = \left(\sum_{j=0}^{k} p_j h(x + a_j y)\right)^2 + \left(\sum_{j=0}^{k} p_j h(x - a_j y)\right)^2$$

Let $z = x + iy$, then, if $|z| \to \infty$, either $x + a_j y \to \infty$ for all j, or $x - a_j y \to -\infty$ for all j. Since every p_j is positive, $|f| \to \infty$; *i.e.* f is proper.

Next, it will be shown that $f : \mathbb{H} \to \mathbb{H}$. Since h is an increasing homeomorphism and the a_j's and the p_j's are all positive, we get

$$\sum_j p_j h(x + a_j y) > \sum_j p_j h(x - a_j y),$$

and therefore $\mathrm{Im}(f) > 0$. This shows $f : \mathbb{H} \to \mathbb{H}$.

Since f is proper to show that f is a homeomorphism it suffices to show that f is injective. For $k = 0$, this follows easily from the injectivity of h. Suppose $k \geq 1$.

Observe that f is injective in the upper half plane if and only if

$$\sum_{j=0}^{k} p_j h(x+a_j y) = \sum_{j=0}^{k} p_j h(x'+a_j y') \Rightarrow \sum_{j=0}^{k} p_j h(x-a_j y) \neq \sum_{j=0}^{k} p_j h(x'-a_j y'),$$

for $x + iy \neq x' + iy'$ and $y, y' > 0$.

Suppose

$$\sum_{j=0}^{k} p_j h(x + a_j y) = \sum_{j=0}^{k} p_j h(x' + a_j y').$$

If $h(x + a_j y) = h(x' + a_j y')$ for every j we would have $x = x'$ and $y = y'$, since h is a homeomorphism, and $k \geq 1$. Thus, there exist l and m such that,

$$h(x + a_l y) < h(x' + a_l y'), \quad \text{and} \quad h(x + a_m y) > h(x' + a_m y').$$

Assume $a_l < a_m$. Since h is increasing the points are ordered as,

$$x + a_l y < x' + a_l y' < x' + a_m y' < x + a_m y,$$

which implies that $x < x'$ and $y > y' > 0$ and so, $x - a_j y < x' - a_j y'$ for all j. Using again that h is monotone and that each p_j is positive,

$$\sum_{j=0}^{k} p_j h(x - a_j y) < \sum_{j=0}^{k} p_j h(x' - a_j y').$$

So f is injective.

Finally, we shall see that the dilatation of f is bounded away from 1, that is, $|\mu_f(z)| \leq \kappa < 1$. Doing some computations we get,

$$\begin{aligned}
4f_z(x,y) &= \textstyle\sum_{j=0}^{k} p_j((1+a_j)+i(1-a_j))h'(x+a_jy) \\
&+ \textstyle\sum_{j=0}^{k} p_j((1+a_j)-i(1-a_j))h'(x-a_jy), \\
4f_{\overline{z}}(x,y) &= \textstyle\sum_{j=0}^{k} p_j((1-a_j)+i(1+a_j))h'(x+a_jy) \\
&+ \textstyle\sum_{j=0}^{k} p_j((1-a_j)-i(1+a_j))h'(x-a_jy).
\end{aligned}$$

Now setting,

$$A = \textstyle\sum_{j=0}^{k} p_j(1+a_j)h'(x+a_jy); \quad B = \sum_{j=0}^{k} p_j(1+a_j)h'(x-a_jy)$$

$$C = \textstyle\sum_{j=0}^{k} p_j(1-a_j)h'(x+a_jy); \quad D = \sum_{j=0}^{k} p_j(1-a_j)h'(x-a_jy).$$

We have that, $4f_z = (A+B)+i(C-D)$ and $4f_{\overline{z}} = (C+D)+i(A-B)$ and then the expression for the dilatation of f becomes,

$$\mu_f = \frac{(C+D)+i(A-B)}{(A+B)+i(C-D)}.$$

Now if $h \in \mathcal{A}B_k$ then,

$$\frac{1}{\tilde{M}} \leq \frac{A}{B} \leq \tilde{M}.$$

Observe that $\tilde{M}$ could depend on p_j and a_j.

Clearly $A > C$ and $B > D$, so there exists a $\lambda < 1$ (depending only on a_j, $j = 0, \ldots, n$) such that $\lambda A > C$ and $\lambda B > D$. Therefore

$$AB - CD > \delta(A^2 + B^2 + C^2 + D^2)$$

where δ depends on M and λ. And we conclude that,

$$|\mu_f(z)|^2 = \frac{A^2+B^2+C^2+D^2-2(AB-CD)}{A^2+B^2+C^2+D^2+2(AB-CD)} \leq \kappa^2 < 1,$$

for some κ that depends on M and λ.

Conversely, if

$$|\mu_f(z)| \leq \kappa < 1,$$

there exists $\delta > 0$ so that,

$$AB - CD > \delta(A^2 + B^2 + C^2 + D^2).$$

And since $\lambda A > C$ and $\lambda B > D$ for some $\lambda < 1$, we get,

$$\frac{1}{M} \leq \frac{A}{B} \leq M,$$

where M depends on δ and λ. Therefore $h \in \mathcal{A}B_k$. $\square$

Observe that if we write $f(z) = u(z) + iv(z)$ then $AB - CD > \delta(A^2 + B^2 + C^2 + D^2)$ is equivalent to $u_x(z)v_y(z) - v_x(z)u_y(z) > \delta(u_x(z)^2 + v_x(z)^2 + u_y(z)^2 + v_y(z)^2)$.

4. Converse results

In [1], it was shown by a complicated argument that Theorem D does not have a converse for $\beta < 1$. Next we will show that the result is true for any positive β.

Theorem 2. *For each $\beta > 0$ there exists a quasisymmetric homeomorphism, $h : \mathbb{R} \to \mathbb{R}$ in Λ^β that cannot be extended to a quasiconformal map in $\mathcal{Q}_\gamma$, for any $\gamma > 0$.*

To prove Theorem 2 it suffices to find a quasisymmetric map, h, which is in Λ^β for $\beta > 0$ and is not in $\mathcal{M}_\alpha$ for any $\alpha \in (0,1)$. Theorem C implies that there does not exist an extension of h in $\mathcal{Q}_\gamma$ for any $\gamma > 0$ which would conclude the proof of Theorem 2. So it is enough to show the following lemma.

Lemma 1. *For $n > 1$, $h(x) = x|x|^{n-1}$ is a quasisymmetric homeomorphism that belongs to Λ^n but it is not in $\mathcal{M}_\alpha$ for any $\alpha \in (0,1)$.*

Proof. First, $f(z) = z|z|^{n-1}$ is a quasiconformal extension of h to $\mathbb{H}$, therefore h is quasisymmetric.

Now choose $x = t$, then

$$\left|\frac{h(x+t) + h(x-t) - 2h(x)}{h(x+t) - h(x)}\right| = \left|\frac{2t|2t|^{n-1} - 2t|t|^{n-1}}{2t|2t|^{n-1} - t|t|^{n-1}}\right| = \frac{2^n - 2}{2^n - 1},$$

and therefore, $h \notin \mathcal{M}_\alpha$ for any $\alpha \in (0,1)$.

Finally, $h \in \Lambda^n$, since $h^{(n-1)}(x) = n!x$ if n is odd and $h^{(n-1)}(x) = n!|x|$ if n is even. □

Although Theorem D does not admit a converse, with some different conditions a certain converse can be proved.

Theorem 3. *Let $f \in \mathcal{Q}_\beta$ and $h = f|_{\mathbb{R}}$, then $\log h'$ is continuous on $\mathbb{R}$ and $h \in \mathcal{P}^{\beta+1}$.*

Conversely, let h be an increasing homeomorphism such that $h' \in \mathcal{P}^\beta$ and $\log h'$ is locally bounded. Then for every $x \in \mathbb{R}$ there exists a neighbourhood of x, $U_x \subset \mathbb{R} \cup \mathbb{H}$ and an extension of h in $\mathbb{H}$, f, which is quasiconformal in U_x and

$$\operatorname{ess\,sup}_{\substack{\xi \in U_x \cap \mathbb{R}, \\ 0 < t \leq y}} |\mu_f(\xi + it)| = \mathcal{O}(y^\beta), \quad \text{as } y \to 0$$

for y so that $x + iy \in U_x$.

Notice that the direct part of this theorem is a consequence of Theorem D. Nevertheless, the proof presented here uses different techniques.

Proof. The direct part of Theorem 3 is a corollary of Theorems A and B of section §1.

First we will see that $\log h'$ is continuous as a consequence of Theorem A. Indeed, by Carleson's result (see [5] p.1), h' is a continuous function on $\mathbb{R}$, so there remains only to show that $h'(x) \neq 0$ for all $x \in \mathbb{R}$.

Fix $x \in \mathbb{R}$ and define,

$$F_x(z) = \begin{cases} f(z+x) - h(x), & z \in \mathbb{H} \cup \mathbb{R} \\ \overline{f}(\overline{z}+x) - h(x), & z \in \mathbb{H}^-, \end{cases}$$

where $\mathbb{H}^- = \{z : \operatorname{Im}(z) < 0\}$. Then the mapping $F_x(z)$ is locally quasiconformal and it fixes 0 and ∞. It has complex dilatation,

$$\mu_{F_x}(z) = \begin{cases} \mu_f(x+z), & z \in \mathbb{H} \cup \mathbb{R} \\ \mu_{\overline{f}}(x+\overline{z}), & z \in \mathbb{H}^-, \end{cases}$$

so $|\mu_{F_x}(z)| \leq C|z|^\beta$, in a neighbourhood of zero (since $f \in \mathcal{Q}_\beta$).

Therefore $F_x(z)$ satisfies the hypothesis of Theorem A so there exists a complex number, $w_x \neq 0$, such that

$$\left| \frac{F_x(z)}{z} - w_x \right| \leq |w_x| \varepsilon(|z|).$$

Now, for $t \in \mathbb{R}$,

$$\lim_{t \to 0} \frac{F_x(t)}{t} = \lim_{t \to 0} \frac{h(x+t) - h(x)}{t} = h'(x),$$

and so $h'(x) = w_x \neq 0$.

Using Theorem B we will find a polynomial that approximates h. Again, fix $x \in \mathbb{R}$, and define $G_x(z) = F_x(z)/h'(x)$. Since $h'(x) \neq 0$, $G_x(z)$ is a local quasiconformal mapping in $\mathbb{C}$ fixing 0 and ∞ and $\frac{\partial G_x}{\partial z}(0) = 1$. Moreover $G_x(z)$ has complex dilatation, $|\mu_{G_x}(z)| \leq C|z|^\beta$ in some disc $D(0, R)$ where R is uniform on compact sets since $f \in \mathcal{Q}_\beta$.

By Theorem A, we get

$$|G_x(z)| \leq |z|(1 + \varepsilon(|z|)),$$

where ε depends on I,R and $\|\mu_{G_x}\|_\infty$. Since $\|\mu_{G_x}\|_\infty = \|\mu_f\|_\infty$ and R is uniform on compact sets there exists a uniform $\rho > 0$, small enough so that,

$$|G_x(z)| \leq C|z|, \qquad \text{for } |z| < \rho.$$

Then,

$$\operatorname{diam}(G_x(D(0, \rho/2)) \leq C,$$

C uniform on compact sets. Note that neither ρ nor C depend on x since the decay condition on μ_f is uniform on compact subsets of the real line.

By Theorem B, there exists a polynomial $Q_x^{n+1}(z)$ such that,

$$|G_x(z) - Q_x^{n+1}(z)| \leq \tilde{C}|z|^{\beta+1},$$

where $\tilde{C} = \tilde{C}(C, \beta)$ does not depend on x, and therefore it is uniform on compact sets. In particular, for $z = t \in \mathbb{R}^+$, t small enough,

$$|h(x+t) - P_x^{n+1}(t)| \leq \tilde{C}|h'(x)|t^{\beta+1},$$

uniformly on compact sets, where $P_x^{n+1}(t) = h(x) + h'(x)Q_x^{n+1}(t)$.

To prove the converse direction, as usual, let n be so that $0 < \beta - n \leq 1$ and consider the function,

$$A(x+iy) = \sum_{j=0}^{n+1} p_j h(x + a_j y),$$

where $a_j \in (0,1)$ for $j \in \{0, \ldots, n+1\}$. Choose p_j, $j \in \{0, \ldots, n+1\}$ so that,

$$\sum_{j=0}^{n+1} p_j a_j^k = (-1)^{[k/2]}, \quad \text{for} \quad k = 0, ..., n+1 \tag{2}$$

where $[x]$ denotes the integer part of x. Notice that (2) defines a linear system of $n+2$ equations and $n+2$ unknowns, with the VanderMonde matrix as the coefficient matrix of the system.

As in Theorem 1, define the extension of h given by

$$f(x+iy) = \frac{1}{2}\left(A(x+iy) + A(x-iy)\right) + \frac{i}{2}\left(A(x+iy) - A(x-iy)\right).$$

In this case, condition (2) forces some of the p_j's to be negative and therefore we cannot conclude that f is a homeomorphism.

First, we shall show that for each $x \in \mathbb{R}$, there exists a neighbourhood of x, $V_x \subset \mathbb{R} \cup \mathbb{H}$ such that $|f_z(z)| > 0$ for $z \in V_x$.

Since $h \in C^1(\mathbb{R})$, the expression of f can be differentiated and evaluating at $z = x + iy$,

$$\begin{aligned} 4f_{\overline{z}}(x+iy) = &\ \textstyle\sum_{j=0}^{n+1} w_j h'(x + a_j y) + \overline{w}_j h'(x - a_j y), \\ 4f_z(x+iy) = &\ i \textstyle\sum_{j=0}^{n+1} \overline{w}_j h'(x + a_j y) - w_j h'(x - a_j y) \end{aligned}$$

where $w_j = p_j((1-a_j) + i\,((1+a_j))$.

For $y = 0$, condition (2) on a_j's and p_j's implies,

$$4f_z(x) = -i\sum_j (w_j - \overline{w}_j)h'(x) = 2\sum_j p_j(1+a_j)h'(x) = 4h'(x) \neq 0,$$

since $\log h'$ is locally bounded in $\mathbb{R}$.

On the other hand, condition (2) yields,

$$4f_{\overline{z}}(x) = \sum_j (w_j + \overline{w}_j)h'(x) = 2\sum_j p_j(1-a_j)h'(x) = 2(1-1)h'(x) = 0.$$

Therefore,

$$J_f(x) = |f_z(x)|^2 - |f_{\overline{z}}(x)|^2 = (h'(x))^2 > 0$$

where J_f denotes the Jacobian of f. Since J_f is continuous (h is a $C^1(\mathbb{R})$ homeomorphism) there exists a neighbourhood of x, V_x, so that $J_f(z) > 0$ in V_x. In particular, $|f_z(z)| > 0$ for every $z \in V_x$.

Next we show that there exists a neighbourhood of x, U_x, such that $\operatorname{ess\,sup}_{\xi \in U_x \cap \mathbb{R},\ t \leq y} |\mu_f(\xi + it)| = \mathcal{O}(y^\beta)$ for y such that $\xi + iy \in U_x$.

Since $h' \in \mathcal{P}^\beta$ there exists a polynomial of degree at most n, $P_n^x(t)$, that approximates $h'(x+t)$ for small t. Write $P_n^x(t) = \sum_{m=0}^n c_m(x) t^m$ where $c_0(x) = h'(x)$. Take $U_x \subset V_x$, $x \in U_x$, such that for any $\xi + it \in U_x$ the polynomial approximation of h' exists at $\xi + a_j t$, $j = 0, \ldots, n+1$.

Let $\xi + it = x + s + it \in U_x$, so if we replace h' by its approximation,

$$
\begin{aligned}
4 f_{\overline{z}}(\xi + it) &= \sum_{j=0}^{n+1} w_j h'(\xi + a_j t) + \overline{w}_j h'(\xi - a_j t) \\
&= \sum_{j=0}^{n+1} w_j \left(\sum_{m=0}^{n} c_m(x)(s + a_j t)^m \right) \\
&+ \sum_{j=0}^{n+1} \overline{w}_j \left(\sum_{m=0}^{n} c_m(x)(s - a_j t)^m \right) + C(x)\mathcal{O}(t^\beta) \\
&= \sum_{m=0}^{n} \sum_{k=0}^{m} \left(\sum_{j=0}^{n+1} (w_j + (-1)^k \overline{w}_j) a_j^k \right) \binom{m}{k} c_m(x) t^k s^{m-k} \\
&+ C(x)\mathcal{O}(t^\beta).
\end{aligned}
$$

To calculate $\sum_{j=0}^{n+1} (w_j + (-1)^k \overline{w}_j) a_j^k$, $k = 0, \ldots, n$, we consider separately the cases of even and odd k.

- For even k, observe that, $[k/2] = [(k+1)/2]$. By condition (2),

$$
\sum_{j=0}^{n+1} (w_j + \overline{w}_j) a_j^k = 2 \sum_j p_j (1 - a_j) a_j^k = 2((-1)^{[k/2]} - (-1)^{[(k+1)/2]}) = 0.
$$

- For odd k, observe that, $[k/2] + 1 = [(k+1)/2]$. By condition (2),

$$
\sum_{j=0}^{n+1} (w_j - \overline{w}_j) a_j^k = 2i \sum_j p_j (1 + a_j) a_j^k = 2i((-1)^{[k/2]} + (-1)^{[(k+1)/2]}) = 0.
$$

Therefore for every $\xi + it \in U_x$,

$$
4 f_{\overline{z}}(\xi + it) = C(x)\mathcal{O}(t^\beta).
$$

Analogously, for f_z,

$$\begin{aligned}
4f_z(\xi+it) &= i\sum_{j=0}^{n+1}\overline{w}_j h'(\xi+a_jt) - w_jh'(\xi-a_jt)\\
&= i\sum_{m=0}^{n}\sum_{k=0}^{m}\left(\sum_{j=0}^{n+1}(\overline{w}_j+(-1)^{k+1}w_j)a_j^k\right)\binom{m}{k}t^ks^{m-k}c_m(x)\\
&+C(x)\mathcal{O}(t^\beta)\\
&= \sum_{k=0}^{n}B_k(x,s)t^k + C(x)\mathcal{O}(t^\beta).
\end{aligned}$$

where, $B_k(x,s) = 4b_k\sum_{m=k}^n\binom{m}{k}c_m(x)s^{m-k}$ and $b_k = \begin{cases} (-1)^{[k/2]}, & \text{if } k \text{ even}\\ i(-1)^{[k/2]}, & \text{if } k \text{ is odd.}\end{cases}$

Observe that, for every k, $B_k(x,s)$ is continuous in s and $B_0(x,0) = c_0(x) = h'(x) \neq 0$.

So the dilatation of f at $\xi+it \in U_x$,

$$\mu_f(\xi+it) = \frac{C(x)\mathcal{O}(t^\beta)}{\sum_{k=0}^n B_k(x,s)t^k + C(x)\mathcal{O}(t^\beta)},$$

and thus,

$$\text{ess sup}_{\xi\in U_x\cap\mathbb{R},\ t\le y}|\mu_f(\xi+it)| = \mathcal{O}(y^\beta).$$

□

Remarks

1. In this case, in order to obtain (2), we need some p_j to be negative. In Theorem 1 it was necessary to require each p_j to be positive in order to obtain a homeomorphic extension.
2. This technique of cancelling the intermediate derivatives of a function by adding up the values of the function evaluated at different points is known as Richardson's extrapolation method. See [3], p.372 and [12], p.194.
3. Notice that if $h \in \Lambda^\beta$ and $\log h'$ is continuous then $h' \in \mathcal{P}^\beta$, and therefore there exists a local extension of h in $\mathcal{Q}_\beta$.

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Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195-4350, USA

Current address: Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain

E-mail address: `acanton@mat.uab.es`