On harmonic functions on trees

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1. Introduction.

In this note we study the asymptotic boundary behaviour of harmonic and p-harmonic functions ($1 < p < \infty$) on trees.

1.1. Vector calculus and trees.

By a *tree* T we mean a connected graph such that every subgraph obtained from T by removing any of its edges is not connected. In what follows we will only consider trees in which we distinguish a vertex v_0 as an origin. As usual we denote by V and E the set of vertices and the set of edges (respectively) of the tree. If v and w are the boundary vertices of an edge, we say that they are neighbours and we write $v \sim w$; we denote by $[v, w]$ the edge that joins the vertices v and w. We assume (except for sections 2 and 5) that the set of edges E is *symmetric*, *i.e.* $[v, w] \in E$ if and only if $[w, v] \in E$.

By a *function on* T we mean a function with real values defined on the set V of vertices of T and by a *vector field* we mean a function with real values defined on the set E of edges of T.

If u is a function on T, its *gradient* ∇u is the vector field defined by the formula

$$\nabla u(v, w) \,=\, \nabla u([v, w]) \,=\, u(w) - u(v)\,.$$

If U is a vector field on T, its *divergence*, denoted by $div\, U$, is the function on T defined by the formula

$$div\, U(v) \,=\, \sum_{w \sim v} U([v, w])\,.$$

The Laplacian and the p-Laplacian of a function u on T ($1 < p < \infty$) are the functions on T defined respectively as

$$\Delta u = \,-div\,(\,\nabla u)\,, \qquad \Delta_p u = \,-div\,(\,\nabla u\,|\nabla u|^{p-2})\,.$$

Notice that $\Delta_2 = \Delta$.

We say that a function u defined on T is *p-subharmonic, p-superharmonic or p-harmonic* if $\Delta_p u \le 0$, $\Delta_p u \ge 0$ or $\Delta_p u = 0$, respectively. When $p = 2$, these functions will be called simply subharmonic, superharmonic and harmonic functions.

A good reference for the study of p-harmonic functions in $\mathbf{R}^N$ is the book [HKM].

NOTATION. In the following, for simplicity, we will use the expression

$$t^{\alpha} = t|t|^{\alpha-1}\,, \qquad \text{for } \alpha > 0\,,\ t \in \mathbf{R}\,. \tag{1.1}$$

We mean that $0^{\alpha} = 0$. Observe that $(t^{\alpha})^{\beta} = t^{\alpha\beta}$, for any $\alpha, \beta > 0$ and $t \in \mathbf{R}$. In particular, $t^2 = t|t|$ is negative if t is negative, and so it is different from the usual notion. Everywhere in the paper we shall use t^{α} only with the meaning (1.1) and no other.

With this notation the *p-Laplacian* of a function u at a vertex v is

$$\Delta_p u(v) = -\sum_{w \sim v} (u(w) - u(v))^{p-1}\,. \tag{1.2}$$

1.2. Fatou's and Bourgain's theorems.

The classical Fatou's Theorem asserts that any bounded holomorphic function in the unit disk $\mathbf{D}$ of the complex plane has radial limits except at most for a set of directions with zero length. The analogue of this result for bounded harmonic functions in a tree is a well known result.

The radial variation of a function f holomorphic in $\mathbf{D}$ at a point $e^{i\theta} \in \partial\mathbf{D}$ is defined as

$$V_f(e^{i\theta}) = \int_0^1 |f'(re^{i\theta})|\, dr\,.$$

Thus $V_f(e^{i\theta})$ is simply the euclidean length of the image under f of the radius ending at $e^{i\theta}$. If at $e^{i\theta}$ we have $V_f(e^{i\theta}) < \infty$, then f has a finite radial limit at $e^{i\theta}$.

Rudin initiated in [R] the study of the set $\{\theta \in [0, 2\pi) : V_f(e^{i\theta}) < \infty\}$ for functions f bounded and holomorphic in $\mathbf{D}$. He proved that there exist bounded holomorphic functions in $\mathbf{D}$ such that

$$|\{\theta \in [0, 2\pi) : V_f(e^{i\theta}) < \infty\}| = 0\,.$$

He raised the question whether there are f's as above with

$$\{\theta \in [0, 2\pi) : V_f(e^{i\theta}) < \infty\} = \emptyset\,.$$

Recently Bourgain [B1], see also [M], proved a counterpart of Fatou's theorem, namely:

Theorem A. *Let $f : \mathbf{D} \to \mathbf{C}$ be a bounded holomorphic function. Then*

$$Dim \left\{\theta \in [0, 2\pi) : \int_0^1 |f'(re^{i\theta})|\, dr < \infty\right\} = 1\,,$$

where Dim denotes Hausdorff dimension.

It should be observed that there are functions f holomorphic in $\mathbf{D}$ belonging to the Hardy space H^2, even to BMOA, such that

$$\{\theta \in [0, 2\pi) : V_f(e^{i\theta}) < \infty\} = \emptyset\,,$$

for instance,

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n}\, z^{2^n}\,.$$

In fact, as Bourgain remarks in [B1], the same argument proves Theorem A for a bounded harmonic function u in the unit disk, if we replace the derivative of f by the gradient of u. Theorem A is also true for positive harmonic functions in the unit disk as Bourgain has recently proved [B2].

It is an open question if the analogue of Theorem A is true for bounded or positive harmonic functions in the unit ball of $\mathbf{R}^N$ for $N \geq 3$.

The aim of this paper is to extend Bourgain's Theorem to trees (under certain restrictions). Our extension works also for p-harmonic functions. Now, on the one hand, very regular trees are discrete models of the unit ball of $\mathbf{R}^N$ (endowed with hyperbolic geometry) and, on the other hand, graphs have important connections with Potential Theory on Riemannian manifolds (see for example [K1], [K2], [K3], [HS], [S1], [S2]) and this perhaps could allow us to expect to prove Bourgain's theorem for functions defined in the unit ball of $\mathbf{R}^N$ via graphs, and to give sharp estimates on the size of the Fatou set of p-harmonic functions in the unit ball of $\mathbf{R}^N$, an interesting open problem.

Also, we have obtained a similar result to Rudin's example: for each $1 < p < \infty$, there exists a regular directed tree T and a p-harmonic function u on T such that the Lebesgue measure of the set $BV(u)$ is zero. This proves that our results are sharp, in some sense.

1.3. The main result.

In order to formulate our extension of Bourgain's Theorem we will need some definitions:

By a *path* we mean a sequence of vertices $\{v_1, \dots, v_n, \dots\}$ (finite or infinite) such that $[v_i, v_{i+1}] \in E$ for all $i \geq 1$. We can define in V a natural distance given by

$$d(v, w) = \inf\{\text{length}\,\gamma : \ \gamma \text{ is a path from } v \text{ to } w\}\,,$$

where we are assigning to all edges a length equal to one.

The *degree* of a vertex is the number of its neighbours, *i.e.* the number of vertices at distance 1 from it. A graph has *bounded degree* if there is an upper bound for the degree of its vertices.

We will denote by S_n the n-sphere (with center v_0) of V, *i.e.*

$$S_n = \{v \in V : \ d(v, v_0) = n\}\,.$$

Given a vertex $v \in S_n$ the *children* of v are the neighbours of v which are in S_{n+1}. The set of children of v will be denoted by H_v.

A tree T is *regular* if all vertices (except at most v_0) have the same degree. Following the notations of Lyons [L] we will say that a tree is *spherically symmetric* if, for each n, all the vertices in S_n have the same degree. In particular, every regular tree is spherically symmetric.

Given a tree T, we define the boundary of T, denoted by ∂T, as the set of all the paths

$$\{v_0,\, v_1, \dots, v_n, \dots\}$$

satisfying $v_{j+1} \in H_{v_j}$ for all $j \geq 0$.

If u is a function on T we define the *variation* of u along the path $\gamma = \{v_0,\, v_1, \dots, v_n, \dots\}$ as

$$V(u, \gamma) := \sum_{n=0}^{\infty} |\nabla u(v_n, v_{n+1})| = \sum_{n=0}^{\infty} |u(v_{n+1}) - u(v_n)|\,.$$

We say that a function u on T has *bounded variation* along the path $\gamma = \{v_0,\, v_1, \dots, v_n, \dots\}$, if

$$V(u, \gamma) \, < \, \infty\,.$$

We will denote by $BV(u)$ the set of paths in ∂T along of which u has bounded variation. Let us observe that if we denote by $u(\gamma)$ the limit of u along γ,

$$u(\gamma) := \lim_{n\to\infty} u(v_n)\,,$$

we have that

$$\sum_{n=0}^{\infty}(u(v_{n+1}) - u(v_n)) = u(\gamma) - u(v_0)\,.$$

Therefore, if u has bounded variation along γ, we have that there exists the limit of u along γ.

Next we need to define the notion of Hausdorff dimension of a subset of ∂T. If T is a spherically symmetric tree we can identify ∂T with the interval $[0,1]$ via the following association:

If $H_{v_0} = \{v_1^1, \dots, v_1^N\}$ we can identify each v_1^j with the subinterval $[j/N, (j+1)/N]$ $(j = 0, \dots, N-1)$. By induction, if the subinterval $[a,b]$ has been associated to $v_n \in S_n$ and $H_{v_n} = \{v_{n+1}^1, \dots, v_{n+1}^M\}$, then we associate to each v_{n+1}^j the subinterval $[a + (b-a)j/M, a + (b-a)(j+1)/M]$ $(j = 0, \dots, M-1)$. Now, we associate to a given path $\{v_0, \dots, v_n, \dots\}$ in ∂T the unique point in $[0,1]$ which belongs to all the subintervals identified with the succesive v_n (for all $n \geq 0$).

Now we can pull back the notion of Hausdorff dimension (initially defined for subsets of $[0,1]$) for subsets of ∂T via this identification. Therefore we have the normalization $Dim\,(\partial T) = 1$.

This definition of Hausdorff dimension coincides with the usual one in the context of stochastic processes, see for example [B]. Observe that the definition of Hausdorff dimension in [L] although it is essentially the same, uses a different normalization.

Our main result is an extension of Bourgain's Theorem to bounded harmonic functions on trees.

Theorem 1. *Let T be a regular tree. Let u be a positive superharmonic function on T. Then,*

$$Dim\,(BV(u)) = Dim\,(\partial T) = 1\,.$$

In fact, we can prove a more general result.

Theorem 2. *Let T be a spherically symmetric tree with bounded degree. For $1 < p < \infty$, there exists a constant $\phi(p) > 0$, satisfying $\phi(2) = 1$, such that for any bounded above p-subharmonic function u (or bounded below p-superharmonic function), we have that*

$$Dim\,(BV(u)) \geq \phi(p)\,.$$

Recall that if u has bounded variation along a path γ, then u has also limit along γ. Therefore we have

Corollary. *Let T be a spherically symmetric tree with bounded degree. For $1 < p < \infty$, there exists a constant $\phi(p) > 0$, satisfying $\phi(2) = 1$, such that for any bounded above p-subharmonic function u (or* *bounded below p-superharmonic function), we have that the Hausdorff dimension of the set $F(u)$ of paths along which u has limit is greater or equal than $\phi(p)$.*

Let us recall that the precise dimension of $F(u)$ for a p-harmonic function u in the unit ball of $\mathbf{R}^n$ ($p \neq 2$) is a very interesting problem. See [FGMS] for some bounds.

We want to remark that the hypothesis of bounded degree appearing in Theorem 2 is usual in the context of Potential Theory on graphs (see for example, [K1], [K2], [K3], [HS], [S1], [S2]).

The outline of the paper is as follows: In Section 2, we consider a simpler version of Theorem 2; it will serve the purpose, we hope, of exhibiting the main ideas. In Section 3 we collect some technical results used in Section 4 where we will prove Theorem 2. Finally, in Section 5 we construct an anologue to Rudin's example in this setting, proving that there exist p-harmonic functions u so that the Lebesgue measure of $BV(u)$ is zero.

2. A simple case.

Let $T = T_D$ be a directed regular tree (*i.e.* its vertices have the same number of children). The term *directed* means that we have chosen a direction in each edge. Therefore, if $[v, w] \in E$, we have that $[w, v] \notin E$. We choose the direction in the following way: $[v, w] \in E$ if and only if $w \in H_v$. This fact has the consequence that the p-Laplacian of a function u on T_D (in a vertex v) is equal to

$$\Delta_p u(v) = -\sum_{w \in H_v} (u(w) - u(v))^{p-1}\,.$$

Recall that we are using here the notation in (1.1). Observe that in this definition we do not take into account the edge that ends at v.

It is worth to mention that solving the Laplace equation for a directed tree is equivalent to solving the heat equation on $\mathbf{Z}$. Namely, let u be a harmonic function in a directed tree T, (we will assume T to be a 2-regular tree for simplicity), for $v_1 \in S_{n-1}$ and $v_0, v_2 \in S_n$ so that, $v_0, v_1 \in H_{v_1}$ we have

$$\triangle u = 0 \Leftrightarrow \frac{u(v_0) + u(v_2)}{2} = u(v_1).$$

Now if we consider j as the space variable and n as the time variable, the above equation becomes,

$$\frac{u(0,n) + u(2,n)}{2} = u(1, n-1).$$

Or equivalently,

$$\frac{u(0,n) + u(2,n) - 2u(1,n)}{2} = u(1, n+1) - u(1,n).$$

That is, we are solving the discrete version of the heat equation

$$-\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Notice the sign.

Now we can prove the following discrete extension of Bourgain's theorem.

Theorem 3. *Let T_D be a directed regular tree. For $1 < p < \infty$, there exists a constant $\psi(p) > 0$, satisfying $\psi(2) = 1$, such that for any upper bounded p-harmonic function u on T_D we have that*

$$Dim\,(BV(u)) \geq \psi(p)\,.$$

In what follows, in order to work with Hausdorff dimension, we need to talk about measures in the boundary of a tree T. Let us consider a function $m : V \longrightarrow [0, \infty)$ with the property that for each vertex v in V the following holds:

$$\sum_{w \in H_v} m(w) = 1\,.$$

To each such m we may associate a consistent sequence of measures μ_n in S_n, for all $n \geq 0$, in the following way: If $P_n = \{v_0, v_1, \ldots, v_n\}$ is the path beginning at v_0 and ending at S_n we define

$$\mu_0(v_0) = 1\,, \qquad \mu_n(v_n) = m(v_1)\cdots m(v_n)\,.$$

It is clear that

$$\mu_{n-1}(v) = \mu_n(v^1) + \cdots + \mu_n(v^k)\,, \qquad \text{for all } \; v \in S_{n-1}\,, \; \text{ with } \; H_v = \{v^1, \ldots, v^k\}\,.$$

Therefore, if we identify the set of all paths in ∂T containing a vertex $v_n \in S_n$ with the vertex v_n, we can define a measure μ in ∂T by the formula

$$\mu(v_n) = \mu_n(v_n)\,.$$

The set of measures defined in this way will be denoted by $\mathcal{M}_T$. In what follows we will use these identifications between paths and vertices, and between μ and μ_n.

Now, given a function u on T, let $d : V\backslash\{v_0\} \longrightarrow \mathbf{R}$ be the function defined by

$$d(w) = \nabla u(v, w) = u(w) - u(v)\,, \qquad \text{if } \; w \in H_v\,.$$

Also, we will denote by u_n and d_n the functions given by

$$u_n = \begin{cases} u\,, & \text{on } \; S_n\,, \\ 0\,, & \text{elsewhere}\,, \end{cases} \qquad d_n = \begin{cases} d\,, & \text{on } \; S_n\,, \\ 0\,, & \text{elsewhere}\,. \end{cases}$$

Proof of Theorem 3. Let v be a vertex of T_D and $H_v = \{v^1, \ldots, v^k\}$ be the set of its children. Let us observe that the number k is the same for any vertex of T_D. The p-harmonicity of u in v means that

$$d(v^1)^{p-1} + \cdots + d(v^k)^{p-1} = 0\,.$$

Consider the closed sets

$$\Omega := \left\{ \mathbf{x} \in \mathbf{R}^k : \ \|\mathbf{x}\|_1 = 1 \,, \ \sum_{j=1}^{k} x_j^{p-1} = 0 \,, \ x_1 \geq x_2 \geq \cdots \geq x_k \right\},$$

$$\Omega_0 := \left\{ \mathbf{x} \in \mathbf{R}^k : \ \sum_{j=1}^{k} x_j^{p-1} = 0 \,, \ x_1 \geq x_2 \geq \cdots \geq x_k \right\},$$

where $\|\mathbf{x}\|_1$ is the usual ℓ^1-norm in $\mathbf{R}^k$, $\|\mathbf{x}\|_1 = \sum_{j=1}^k |x_j|$.

Let us observe that if $\mathbf{x} \in \Omega$ we have that $x_1 > 0$. In other case, the conditions

$$x_1^{p-1} + x_2^{p-1} + \cdots + x_k^{p-1} = 0 \,, \qquad x_1 \geq x_2 \geq \cdots \geq x_k \,,$$

would imply that $x_1 = x_2 = \cdots = x_k = 0$ in Ω, which contradicts $\|\mathbf{x}\|_1 = 1$.

Let us consider a positive number q such that

$$q > \Lambda_p := \max_{x \in \Omega} \frac{x_2 + \cdots + x_k}{(x_2^{p-1} + \cdots + x_k^{p-1})^{1/(p-1)}} = \max_{x \in \Omega} \frac{-x_2 - \cdots - x_k}{x_1} \,.$$

It is clear that we always have $\Lambda_p \geq 1$ (to see this, it is enough to take $\mathbf{x} = (1/2, 0, \dots, 0, -1/2) \in \Omega$).

Therefore we have in Ω

$$q\, x_1 + x_2 + \cdots + x_k > 0 \,,$$

and this implies that there exists a positive number δ such that

$$\text{(2.1)} \qquad \frac{q}{q+k-1}\, x_1 + \frac{1}{q+k-1}\, x_2 + \cdots + \frac{1}{q+k-1}\, x_k \geq \delta \,, \qquad \text{for } \mathbf{x} \in \Omega,$$

since Ω is a compact set. We have that

$$\text{(2.2)} \qquad \frac{q}{q+k-1}\, x_1 + \frac{1}{q+k-1}\, x_2 + \cdots + \frac{1}{q+k-1}\, x_k \geq \delta\, \|\mathbf{x}\|_1 \,, \qquad \text{for } \mathbf{x} \in \Omega_0.$$

The statement (2.2) is trivial for $\mathbf{x} = 0$ and for $\mathbf{x} \neq 0$ it is a consequence of (2.1).

We now construct a measure $\mu \in \mathcal{M}_{T_D}$ and the corresponding function m, in the following inductive way: Let v be any vertex of T_D and $H_v = \{v^1, \dots, v^k\}$. Fix a child v^i verifying

$$d(v^i) = \max\{d(v^1), \dots, d(v^k)\} \,.$$

We define $m|_{H_v}$ as the function

$$m(v^j) := \begin{cases} \dfrac{q}{q+k-1} \,, & \text{for } j = i, \\[2mm] \dfrac{1}{q+k-1} \,, & \text{for } j \neq i. \end{cases}$$

Let us recall that we always have $q > \Lambda_p \geq 1$. This fact implies that the measure μ gives more mass to the vertex maximizing (in H_v) the function d.

Let us observe that there is a rearrangement of the vector $(d(v^1), \dots, d(v^k))$ which belongs to Ω_0. Therefore, if $v \in S_{n-1}$, (2.2) implies that

$$\int_{H_v} d_n \, d\mu \geq \delta\, \|d_n|_{H_v}\|_1 \, \mu(v) \geq \delta \int_{H_v} |d_n| \, d\mu \,,$$

and consequently

$$\int d_n\, d\mu \geq \delta \int |d_n|\, d\mu\,. \tag{2.3}$$

Let us observe that the constant $\delta > 0$ depends on q and k, but neither on u nor n.

Lemma 4.1 (see Section 4 below) and (2.3) give that

$$\int u_m\, d\mu = u(v_0) + \sum_{n=1}^{m} \int d_n\, d\mu \geq u(v_0) + \delta \sum_{n=1}^{m} \int |d_n|\, d\mu\,.$$

If M is an upper bound of the function u, this inequality implies that

$$\int \sum_{n=1}^{m} |d_n|\, d\mu \leq \delta^{-1}(M - u(v_0))\,,$$

and then

$$\int \sum_{n=1}^{\infty} |d_n|\, d\mu \leq \delta^{-1}(M - u(v_0))\,.$$

Therefore

$$\sum_{n=1}^{\infty} |d_n| < \infty$$

almost everywhere with respect to μ and consequently $\mu(BV(u)) = 1$.

On the other hand, since $q > 1$, if $v \in S_n$,

$$\mu(v) \leq \Big(\frac{q}{q+k-1}\Big)^n = \Big(\frac{1}{k}\Big)^{n(\log(q+k-1)-\log q)/\log k} = |v|^{(\log(q+k-1)-\log q)/\log k}\,,$$

where $|v| = k^{-n}$ is the Lebesgue measure of v in T_D (which is generated by the function $m_0 \equiv 1/k$). This fact, $\mu(BV(u)) = 1$ and Lemma 4.2 (see Section 4 below) give that

$$Dim(BV(u)) \geq \frac{\log(q+k-1) - \log q}{\log k}\,,$$

for any $q > \Lambda_p$. Consequently we deduce that

$$Dim(BV(u)) \geq \frac{\log(\Lambda_p + k - 1) - \log \Lambda_p}{\log k} =: \psi(p)\,.$$

Finally, let us observe that $\psi(2) = 1$, since $\Lambda_2 = 1$.

3. Technical results.

In what follows we will consider, for $\eta > 0$, the function

$$\eta(t) = \begin{cases} 1\,, & \text{if } t \geq 0\,, \\ \eta\,, & \text{if } t < 0\,. \end{cases} \tag{3.1}$$

Observe that, with the definition of the power $t^\alpha = t|t|^{\alpha-1}$ given in the Introduction, we have that:

$$(t^\alpha)' = \alpha\, |t|^{\alpha-1}\,.$$

Lemma 3.1. *Let α, a, b, c, d positive constants. Consider the function*

$$F(t) = a\,t + \frac{b}{\eta(c - dt^\alpha)}\,(c - dt^\alpha)^{1/\alpha}\,, \qquad t \geq t_0 := \left(\frac{c}{1+d}\right)^{1/\alpha},$$

Denote by t_2 the number

$$t_2 := \left(\frac{c}{d - \left(\frac{bd}{a\eta}\right)^{\alpha/(\alpha-1)}}\right)^{1/\alpha}.$$

We have the following assertions:

(A) *If $0 < \alpha < 1$, $a\eta > bd^{1/\alpha}$, $a > bd$, then F is an increasing function in the interval $[t_0, \infty)$.*

(B) *If $\alpha > 1$, $a\eta < bd^{1/\alpha}$, $a < bd$, then F is a decreasing function in the interval $[t_0, \infty)$.*

(C) *If $0 < \alpha < 1$, $a\eta < bd^{1/\alpha}$, $a < bd$, then F attains its maximum in the interval $[t_0, \infty)$ either at the point t_0 or at the point t_2.*

(D) *If $\alpha > 1$, $a\eta > bd^{1/\alpha}$, $a > bd$, then F attains its minimum in the interval $[t_0, \infty)$ either at the point t_0 or at the point t_2.*

Besides,

$$F(t_2) = \frac{ac^{1/\alpha}}{d}\left(d - \left(\frac{bd}{a\eta}\right)^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}.$$

Proof. First of all observe that

$$\lim_{t\to\infty} F(t) = \begin{cases} \infty\,, & \text{if } \ a\eta > bd^{1/\alpha}\,, \\ -\infty\,, & \text{if } \ a\eta < bd^{1/\alpha}\,, \end{cases}$$

and

$$F'(t) = a - \frac{bdt^{\alpha-1}}{\eta(c - dt^\alpha)}\,|c - dt^\alpha|^{(1-\alpha)/\alpha}\,, \qquad \text{if } \ t \neq t_1 := (c/d)^{1/\alpha}\,.$$

Therefore, $F'(t_0) = a - bd$ and

$$F'(t_1) = \begin{cases} -\infty\,, & \text{if } \ \alpha > 1\,, \\ a\,, & \text{if } \ 0 < \alpha < 1\,. \end{cases}$$

On the other hand it is easy to see that

- $F'(t)$ vanishes exactly once in the interval (t_0, t_1) if $(a/(bd))^{\alpha/(1-\alpha)} < 1$ and $F'(t) \neq 0$ for all $t \in (t_0, t_1)$ in other case.

- $F'(t)$ annihilates exactly once in the interval (t_1, ∞) if $d > (bd/(a\eta))^{\alpha/(\alpha-1)}$, and this critical point is t_2, and $F'(t) \neq 0$ for all $t \in (t_1, \infty)$ in other case.

Observe now that the condition $d > (bd/(a\eta))^{\alpha/(\alpha-1)}$ is equivalent to the two following ones:

$$\begin{aligned} a\eta > bd^{1/\alpha}\,, &\qquad \text{if } \ \alpha > 1\,, \\ a\eta < bd^{1/\alpha}\,, &\qquad \text{if } \ 0 < \alpha < 1\,. \end{aligned}$$

Collecting now all this information it is easy to see in each case that:

(A) $F' > 0$ in the interval $[t_0, \infty)$.

(B) $F' < 0$ in $[t_0, t_1) \cup (t_1, \infty)$ and $F'(t_1) = -\infty$.

(C) $F'(t_0) < 0$, $F'(t_1) > 0$, F' annihilates exactly once in (t_0, t_1), and exactly once (at the point t_2) in (t_1, ∞), and $\lim_{t\to\infty} F(t) = -\infty$.

(D) $F'(t_0) > 0$, $F'(t_1) = -\infty$, F' annihilates exactly at two points, one of them in the interval (t_0, t_1) and the other at the point $t_2 \in (t_1, \infty)$, and $\lim_{t\to\infty} F(t) = \infty$.

Finally, the expression for $F(t_2)$ follows from a straightforward computation.

Proposition 3.1. *Let $\mathbf{k} = (k_1, \ldots, k_{n+1})$ be a vector with strictly positive integer entries, $N = \sum_i k_i$, $0 < \eta < 1$ and $\alpha > 0$. Let $\eta(t)$ be the function whose values are 1 if $t \geq 0$, and the constant η elsewhere. Given $0 < \varepsilon < 1$, consider the numbers*

$$\varepsilon_1 = \frac{N - k_1}{k_1}\,\varepsilon\,, \qquad \varepsilon_2 = \cdots = \varepsilon_{n+1} = -\varepsilon\,.$$

Then, the function defined by

$$f(\mathbf{x}) = \sum_{i=1}^{n} k_i\,(1+\varepsilon_i)\,\frac{x_i}{\eta(x_i)} + k_{n+1}\,(1+\varepsilon_{n+1})\,\frac{\big((1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha)/k_{n+1}\big)^{1/\alpha}}{\eta(1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha)}$$

satisfies

$$\min_{\mathbf{x}\in D} f(\mathbf{x}) = f(N^{-1/\alpha}, \ldots, N^{-1/\alpha}) = N^{(\alpha-1)/\alpha}\,, \qquad \textit{for} \;\; 1 > \varepsilon > \varepsilon(\alpha, \eta, \mathbf{k}) > 0\,, \tag{3.2}$$

where $\varepsilon(\alpha, \eta, \mathbf{k})$ decreases when η grows and

$$D := \Big\{\mathbf{x} \in \mathbf{R}^n :\; x_1 \geq \cdots \geq x_n \geq \Big(\frac{1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha}{k_{n+1}}\Big)^{1/\alpha}\Big\}\,.$$

Observe that if we define x_{n+1} as

$$x_{n+1} = \Big(\frac{1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha}{k_{n+1}}\Big)^{1/\alpha}\,, \tag{3.3}$$

we have $x_1 \geq \cdots \geq x_n \geq x_{n+1}$ and $k_1 x_1^\alpha + \cdots + k_n x_n^\alpha + k_{n+1} x_{n+1}^\alpha = 1$, and therefore $x_1 > 0$.

Remark. Although we will use this proposition in Section 4 only for the case $k_1 = \cdots = k_{n+1} = 1$, to prove this particular case we will need the general one.

Proof. We will use induction in n.

If $n = 1$ we have

$$f(x) = k_1(1+\varepsilon_1)\,x + k_2(1-\varepsilon)\frac{\big((1 - k_1 x^\alpha)/k_2\big)^{1/\alpha}}{\eta(1 - k_1 x^\alpha)}\,, \qquad k_1 + k_2 = N\,,\; \varepsilon_1 = \frac{k_2}{k_1}\,\varepsilon\,,$$

and

$$x \in D \Longleftrightarrow x \geq \Big(\frac{1-k_1x^\alpha}{k_2}\Big)^{1/\alpha} \Longleftrightarrow x \geq N^{-1/\alpha}\,.$$

Therefore, in this case $D = [N^{-1/\alpha}, \infty)$.

This function (and its domain) coincides with the one in Lemma 3.1, if we take

$$a = k_1(1+\varepsilon_1)\,, \quad b = k_2(1-\varepsilon)\,, \quad c = \frac{1}{k_2}\,, \quad d = \frac{k_1}{k_2}\,,$$

and therefore $a > bd$ always.

Observe now that $\lim_{x\to\infty} f(x) = \infty$ if

$$k_1(1+\varepsilon_1) > \frac{k_2(1-\varepsilon)}{\eta}\Big(\frac{k_1}{k_2}\Big)^{1/\alpha}\,. \tag{3.4}$$

This inequality is trivially true for $\varepsilon = 1$, and then a continuity argument shows that (3.4) is true for $1 > \varepsilon > \varepsilon_1(\alpha, \eta, \mathbf{k})$ (this condition is the same that $a\eta > bd^{1/\alpha}$ in the notation of Lemma 3.1), where $\varepsilon_1(\alpha, \eta, \mathbf{k})$ decreases when η grows. Besides an easy computation gives

$$\varepsilon_1(1, \eta, \mathbf{k}) = \frac{1-\eta}{1+\eta\, k_2/k_1}\,.$$

We will consider now three cases:

- If $0 < \alpha < 1$ we are in the case (A) of Lemma 3.1, and therefore

$$f(x) \geq f(N^{-1/\alpha}) = N^{(\alpha-1)/\alpha}\,, \qquad x \geq N^{-1/\alpha}\,. \tag{3.5}$$

- If $\alpha = 1$, (3.5) is also true since f is an increasing function by (3.4).

- If $\alpha > 1$, we are in the case (D) of Lemma 3.1. A continuity argument gives $f(t_2) > N^{(\alpha-1)/\alpha} = f(N^{-1/\alpha}) = f(t_0)$ if $1 > \varepsilon > \varepsilon_2(\alpha, \eta, \mathbf{k})$, where $\varepsilon_2(\alpha, \eta, \mathbf{k})$ decreases when η grows. Hence, (3.5) is also true in this case.

This ends the proof of the case $n = 1$.

Suppose now that the proposition is true for the case $n-1$. We will prove it for the case n.

First, we will see that f attains a minimum in the domain D. We have

$$x_1 \geq \cdots \geq x_n \geq \Big(\frac{1-k_1x_1^\alpha-\cdots-k_nx_n^\alpha}{k_{n+1}}\Big)^{1/\alpha} \geq -\Big(\frac{k_1x_1^\alpha+\cdots+k_nx_n^\alpha}{k_{n+1}}\Big)^{1/\alpha}\,.$$

Then, for $i = 1, \dots, n$, we can write $x_i = m_i x_1$, where $\mathbf{m} = (m_1, \dots, m_n)$ is in the compact set

$$M := \Big\{\mathbf{m} \in \mathbf{R}^n : \; m_1 = 1 \geq m_2 \geq \cdots \geq m_n \geq -\Big(\frac{k_1m_1^\alpha+\cdots+k_nm_n^\alpha}{k_{n+1}}\Big)^{1/\alpha}\Big\}\,.$$

Observe now that

$$f(\mathbf{x}) = \sum_{i=1}^n k_i\,(1+\varepsilon_i)\,\frac{m_ix_1}{\eta(m_i)} + k_{n+1}\,(1+\varepsilon_{n+1})\,\frac{\big((1-k_1m_1^\alpha x_1^\alpha-\cdots-k_nm_n^\alpha x_1^\alpha)/k_{n+1}\big)^{1/\alpha}}{\eta(1-k_1m_1^\alpha x_1^\alpha-\cdots-k_nm_n^\alpha x_1^\alpha)} \geq u_{\mathbf{m}}(\varepsilon)\,x_1\,,$$

where

$$u_{\mathbf{m}}(\varepsilon) = \sum_{i=1}^{n} k_i\,(1+\varepsilon_i)\,\frac{m_i}{\eta(m_i)} - k_{n+1}\,(1+\varepsilon_{n+1})\,\frac{\big((k_1 m_1^\alpha + \cdots + k_n m_n^\alpha)/k_{n+1}\big)^{1/\alpha}}{\eta(-k_1 m_1^\alpha - \cdots - k_n m_n^\alpha)}\,.$$

As M is compact the function $u(\varepsilon) := \min_{\mathbf{m}\in M} u_{\mathbf{m}}(\varepsilon)$ is continuous in ε. Since $u_{\mathbf{m}}(1) = N$, we have $u(1) = N$, and a continuity argument gives

$$u(\varepsilon) > 0 \quad \text{if} \;\; 1 > \varepsilon > \varepsilon_3(\alpha, \eta, \mathbf{k})\,,$$

where $\varepsilon_3(\alpha, \eta, \mathbf{k})$ decreases when η grows. Hence

$$u_{\mathbf{m}}(\varepsilon) \geq u(\varepsilon) > 0\,, \qquad \text{for all } \mathbf{m} \in M\,.$$

It follows that $f(\mathbf{x}) \geq u(\varepsilon)\,x_1$ and therefore $f(\mathbf{x}) \to \infty$ "uniformly" as $\mathbf{x} \to \infty$.

This implies that there exists the minimum of f in the domain D and that this minimum is attained either on the boundary of D, either on the critical points of f, or on the points in which f is not differentiable. We will study each of this cases separately:

1) *f can not attain its minimum in the interior points of D in which f is not differentiable.* To prove this observe that, for $i = 1, \ldots, n$,

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = k_i\,(1+\varepsilon_i)\,\frac{1}{\eta(x_i)} - k_i\,(1+\varepsilon_{n+1})\,\frac{\big|(1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha)/k_{n+1}\big|^{(1-\alpha)/\alpha}}{\eta(1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha)}\,|x_i|^{\alpha-1}\,.$$

We need to distinguish several cases:

- If $x_i = 0$ for some $i \in \{2, \ldots, n\}$, we have (see (3.3)) $x_{n+1} < 0$ and then

$$\frac{\partial f}{\partial x_i}(\mathbf{x})\Big|_{x_i=0} = -\infty\,, \qquad \text{if} \;\; 0 < \alpha < 1\,,$$

$$\left\{\begin{array}{l} \dfrac{\partial f}{\partial x_i}(\mathbf{x})\Big|_{x_i=0^+} = k_i(1+\varepsilon_i) > 0\,, \\ \dfrac{\partial f}{\partial x_i}(\mathbf{x})\Big|_{x_i=0^-} = k_i(1+\varepsilon_i)/\eta > 0\,, \end{array}\right. \qquad \text{if} \;\; \alpha > 1\,.$$

If $\alpha \neq 1$, this implies that, in this case, f can not attain its minimum in the interior points of D where f is not differentiable.

- If $x_{n+1} = 0$, then $x_i > 0$ for all $i \leq n$ and

$$\frac{\partial f}{\partial x_i}(\mathbf{x})\Big|_{x_{n+1}=0} = -\infty\,, \qquad \text{if} \;\; \alpha > 1\,,$$

$$\frac{\partial f}{\partial x_i}(\mathbf{x})\Big|_{x_{n+1}=0} = k_i(1+\varepsilon_i) > 0\,, \qquad \text{if} \;\; 0 < \alpha < 1\,,$$

for all $i \leq n$. If $\alpha \neq 1$, this implies again that, also in this case, f can not attain its minimum in the interior points of D where f is not differentiable.

- If $\alpha = 1$ and $x_i = 0$ for some $1 < i \leq n$, then $x_{n+1} < 0$ and we have, for all these $\mathbf{x}$ in the interior of D, that

$$\frac{\partial f}{\partial x_1}(\mathbf{x}) = k_1(1+\varepsilon_1) - k_1(1-\varepsilon)/\eta > 0$$

if $1 > \varepsilon > \varepsilon_4(1, \eta, \mathbf{k})$, where

$$\varepsilon_4(1, \eta, \mathbf{k}) = \frac{1-\eta}{1 + (N - k_1)\,\eta/k_1} \, .$$

- If $\alpha = 1$ and $x_{n+1} = 0$, then $x_i > 0$ for all $i \le n$, and

$$\frac{\partial f}{\partial x_1}(\mathbf{x})\Big|_{x_{n+1}=0^+} = k_1(1+\varepsilon_1) - k_1(1-\varepsilon) > 0 \, ,$$
$$\frac{\partial f}{\partial x_1}(\mathbf{x})\Big|_{x_{n+1}=0^-} = k_1(1+\varepsilon_1) - k_1(1-\varepsilon)/\eta > 0 \, ,$$

if $1 > \varepsilon > \varepsilon_4(1, \eta, \mathbf{k})$.

This implies that, also in the two last cases, f can not attain its minimum in the interior points of D where f is not differentiable.

2) *f can not attain its minimum in the critical points belonging to the interior of D.* It is easy to see that a interior point $\mathbf{x}$ is a critical point of f if and only if $x_i \neq 0$ for all $i \in \{1, \dots, n+1\}$ and

(3.6) $$(1+\varepsilon_1)\, x_1^{1-\alpha} = (1-\varepsilon)\, \frac{|x_2|^{1-\alpha}}{\eta(x_2)} = \cdots = (1-\varepsilon)\, \frac{|x_n|^{1-\alpha}}{\eta(x_n)} = (1-\varepsilon)\, \frac{|x_{n+1}|^{1-\alpha}}{\eta(x_{n+1})} \, .$$

We need again to distinguish several cases:

- If $\alpha = 1$, then (3.6) implies that $x_2, \dots, x_{n+1} < 0$, since $1 + \varepsilon_1 > 1 - \varepsilon$, and therefore a fortiori we must have $1 + \varepsilon_1 = (1-\varepsilon)/\eta$, but this is a contradiction with $1 > \varepsilon > \varepsilon_4(1, \eta, \mathbf{k})$. Therefore, in this case, f can not attain its minimum on the critical points.

- If $\alpha \neq 1$ and $n \ge 3$, there are not critical points in the interior of D since we have

$$\frac{|x_2|^{1-\alpha}}{\eta(x_2)} = \frac{|x_3|^{1-\alpha}}{\eta(x_3)} = \frac{|x_4|^{1-\alpha}}{\eta(x_4)} \, .$$

But this implies that $x_i = x_{i+1}$ for some i, *i.e.* that $\mathbf{x} \in \partial D$.

- If $\alpha \neq 1$ and $n = 2$, a critical point must verify $x_1 > x_2 > 0 > x_3$ since if $\operatorname{sgn} x_2 = \operatorname{sgn} x_3$, arguing as in the last case it is easy to see that then $x_2 = x_3$. On the other hand, if $\mathbf{x}$ is a critical point of f, then

$$x_2 = \Big(\frac{1-\varepsilon}{1+\varepsilon_1}\Big)^{1/(\alpha-1)} x_1 \, .$$

Therefore, $x_1 > x_2$ if and only if $\alpha > 1$ and so there are not critical points when $0 < \alpha < 1$.

If $x_1 > x_2$ and $\alpha > 1$, we have

$$x_2 = \Big(\frac{1-\varepsilon}{1+\varepsilon_1}\Big)^{1/(\alpha-1)} x_1 \, , \qquad x_3 = -\Big(\frac{1}{\eta}\, \frac{1-\varepsilon}{1+\varepsilon_1}\Big)^{1/(\alpha-1)} x_1 \, ,$$

and then

$$x_1 = \left(\frac{1}{k_1 + k_2\Big(\frac{1-\varepsilon}{1+\varepsilon_1}\Big)^{\alpha/(\alpha-1)} - k_3\Big(\frac{1}{\eta}\, \frac{1-\varepsilon}{1+\varepsilon_1}\Big)^{\alpha/(\alpha-1)}} \right)^{1/\alpha} ,$$

and, if $\mathbf{x}_0$ is this critical point,

$$f(\mathbf{x}_0) = \Big(1 + \frac{k_2 + k_3}{k_1}\varepsilon\Big)\Big(k_1 + k_2\Big(\frac{1-\varepsilon}{1+\varepsilon_1}\Big)^{\alpha/(\alpha-1)} - k_3\Big(\frac{1}{\eta}\, \frac{1-\varepsilon}{1+\varepsilon_1}\Big)^{\alpha/(\alpha-1)}\Big)^{(\alpha-1)/\alpha} \, .$$

We can assure that $f(\mathbf{x}_0) > N^{(\alpha-1)/\alpha}$ if $1 > \varepsilon > \varepsilon_5(\alpha, \eta, \mathbf{k})$ by a continuity argument, where $\varepsilon_5(\alpha, \eta, \mathbf{k})$ decreases when η grows.

This implies that the mimimum of f in D can not be attained at $\mathbf{x}_0$.

3) Therefore *the minimum of f is attained in ∂D.* The point where this minimum is attained must verify $x_j = x_{j+1}$ for some $j \in \{1, \dots, n\}$. If we substitute this relation in the function f and in the domain D, we obtain a function and a domain of the same type, but now with $n-1$ variables, and with a different $\mathbf{k}'$ also satisfying $\sum_i k'_i = N$.

Hence, the induction hypothesis gives that

$$\min_{\mathbf{x} \in \partial D} f(\mathbf{x}) \; = \; N^{(\alpha-1)/\alpha} \qquad \text{for} \quad 1 > \varepsilon > \varepsilon_6(\alpha, \eta, \mathbf{k}') \, ,$$

where $\varepsilon_6(\alpha, \eta, \mathbf{k}')$ decreases when η grows by the induction hypothesis. As we have used along the induction process only a finite number of functions $\varepsilon_i(\alpha, \eta, \cdot)$, the proof is finished.

Proposition 3.2. *Let $\mathbf{k} = (k_1, \dots, k_{n+1})$ be a vector with strictly positive integer entries, $N = \sum_i k_i$, $0 < \eta < 1$ and $\alpha > 0$. Let $\eta(t)$ be the function whose values are 1 if $t \geq 0$, and the constant η elsewhere. Given $0 < \varepsilon < 1$, consider the number*

$$\varepsilon_{n+1} = \frac{N - k_{n+1}}{k_{n+1}} \, \varepsilon \, ,$$

Then, the function defined by

$$g(\mathbf{x}) = \sum_{i=1}^{n} k_i \, (1-\varepsilon) \, x_i \, \eta(x_i) + k_{n+1} \, (1 + \varepsilon_{n+1}) \left(\frac{1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha}{k_{n+1}} \right)^{1/\alpha} \eta(1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha)$$

satisfies

$$\max_{\mathbf{x} \in D} g(\mathbf{x}) = g(N^{-1/\alpha}, \dots, N^{-1/\alpha}) = N^{(\alpha-1)/\alpha} \, , \qquad \textit{for} \;\; 1 > \varepsilon > \varepsilon'(\alpha, \eta, \mathbf{k}) > 0 \, , \tag{3.7}$$

where $\varepsilon'(\alpha, \eta, \mathbf{k})$ decreases when η grows and

$$D := \left\{ \mathbf{x} \in \mathbf{R}^n : \; x_1 \geq \cdots \geq x_n \geq \left(\frac{1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha}{k_{n+1}} \right)^{1/\alpha} \right\} .$$

Recall that if we define x_{n+1} as

$$x_{n+1} = \left(\frac{1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha}{k_{n+1}} \right)^{1/\alpha} ,$$

we have $x_1 \geq \cdots \geq x_n \geq x_{n+1}$ and $k_1 x_1^\alpha + \cdots + k_n x_n^\alpha + k_{n+1} x_{n+1}^\alpha = 1$, and therefore $x_1 > 0$.

Proof. We will use induction in n.

If $n = 1$ we have

$$g(x) = k_1 (1 - \varepsilon) \, x + k_2 (1 + \varepsilon_2) \left(\frac{1 - k_1 x^\alpha}{k_2} \right)^{1/\alpha} \eta(1 - k_1 x^\alpha) \, , \qquad k_1 + k_2 = N \, , \; \varepsilon_2 = \frac{k_1}{k_2} \, \varepsilon \, ,$$

and

$$x \in D \Longleftrightarrow x \geq \Big(\frac{1-k_1 x^\alpha}{k_2}\Big)^{1/\alpha} \Longleftrightarrow x \geq N^{-1/\alpha} .$$

Therefore, in this case $D = [N^{-1/\alpha}, \infty)$.

This function (and its domain) coincides with the one in Lemma 3.1, if we take

$$a = k_1(1-\varepsilon) , \quad b = k_2(1+\varepsilon_2) , \quad c = \frac{1}{k_2} , \quad d = \frac{k_1}{k_2} ,$$

and we consider η^{-1} instead of η.

Observe that we have $a < bd$ always.

On the other hand we have that $\lim_{x\to\infty} g(x) = -\infty$ if

$$k_1(1-\varepsilon) < k_2(1+\varepsilon_2) \Big(\frac{k_1}{k_2}\Big)^{1/\alpha} \eta \tag{3.8}$$

and we can assure this for $1 > \varepsilon > \varepsilon_1'(\alpha, \eta, \mathbf{k})$ by a continuity argument (this condition is the same that $a < bd^{1/\alpha}\eta$ in the notation of Lemma 3.1), where $\varepsilon_1'(\alpha, \eta, \mathbf{k})$ decreases when η grows. Besides an easy computation gives

$$\varepsilon_1'(1, \eta, \mathbf{k}) = \frac{1-\eta}{1+\eta\, k_1/k_2} .$$

We will consider now three cases:

- If $\alpha > 1$ we are in the case (B) of Lemma 3.1, and therefore

$$g(x) \leq g(N^{-1/\alpha}) = N^{(\alpha-1)/\alpha} , \qquad x \geq N^{-1/\alpha} . \tag{3.9}$$

- If $\alpha = 1$, (3.9) is also true since g is a decreasing function by (3.8).

- If $0 < \alpha < 1$, we are in the case (C) of Lemma 3.1. A continuity argument gives $g(t_2) < N^{(\alpha-1)/\alpha} = g(N^{-1/\alpha}) = g(t_0)$ if $1 > \varepsilon > \varepsilon_2'(\alpha, \eta, \mathbf{k})$, where $\varepsilon_2'(\alpha, \eta, \mathbf{k})$ decreases when η grows. Hence, (3.9) is also true in this case.

This ends the proof of the case $n = 1$.

Suppose now that the proposition is true for the case $n-1$. We will prove it for the case n.

First, we will see that g attains a maximum in the domain D. We have, as in Proposition 3.1,

$$x_1 \geq \cdots \geq x_n \geq \Big(\frac{1 - k_1 x_1^\alpha - \cdots - k_n x_n^\alpha}{k_{n+1}}\Big)^{1/\alpha} \geq -\Big(\frac{k_1 x_1^\alpha + \cdots + k_n x_n^\alpha}{k_{n+1}}\Big)^{1/\alpha} .$$

Then, for $i = 1, \ldots, n$, we can write $x_i = m_i x_1$, where $\mathbf{m} = (m_1, \ldots, m_n)$ is in the compact set

$$M := \Big\{\mathbf{m} \in \mathbf{R}^n : \; m_1 = 1 \geq m_2 \geq \cdots \geq m_n \geq -\Big(\frac{k_1 m_1^\alpha + \cdots + k_n m_n^\alpha}{k_{n+1}}\Big)^{1/\alpha}\Big\} .$$

We will consider the auxiliary function

$$\begin{aligned}\tilde{g}(x_1) &= \Big(\sum_{i=1}^{n} k_i\,(1-\varepsilon)\, m_i\, \eta(m_i) - \eta\, k_{n+1}\,(1+\varepsilon_{n+1})\,\Big(\frac{k_1 m_1^\alpha + \cdots + k_n m_n^\alpha}{k_{n+1}}\Big)^{1/\alpha} \eta(-k_1 m_1^\alpha - \cdots - k_n m_n^\alpha)\Big)\, x_1 \\ &:= v_{\mathbf{m}}(\varepsilon)\, x_1 .\end{aligned}$$

Now observe that

$$\eta \le \frac{(Ax_1^\alpha - 1)^{1/\alpha}}{A^{1/\alpha}x_1}\,, \qquad \text{if } \ A^{1/\alpha}x_1 \ge c_1(\alpha,\eta) > 0\,,$$

where $c_1(\alpha,\eta)$ is an increasing function in η. This implies that

$$(1 - Ax_1^\alpha)^{1/\alpha}\, \eta(1 - Ax_1^\alpha) \le -\, A^{1/\alpha}x_1\eta^2\,, \qquad \text{if } \ A^{1/\alpha}x_1 \ge c_1(\alpha,\eta) > 0\,.$$

Taking $A = k_1m_1^\alpha + \cdots + k_nm_n^\alpha$, we obtain that

$$g(\mathbf{x}) \le \tilde{g}(x_1)\,, \qquad \text{if } \ (k_1m_1^\alpha + \cdots + k_nm_n^\alpha)^{1/\alpha}x_1 \ge c_1(\alpha,\eta) > 0\,.$$

Now let be $m_{n+1} := -(A/k_{n+1})^{1/\alpha}$. Then

$$1 = m_1 \ge m_2 \ge \cdots \ge m_n \ge m_{n+1}\ ,$$

$$k_1m_1^\alpha + \cdots + k_nm_n^\alpha + k_{n+1}m_{n+1}^\alpha = 0\,.$$

These conditions imply that $m_{n+1} < 0$ in the compact set M. Therefore $m_{n+1} \le -c_2(\alpha,\mathbf{k}) < 0$ in M. This means that

$$A^{1/\alpha} \ge c_3(\alpha,\mathbf{k}) := c_2(\alpha,\mathbf{k})\, k_{n+1}^{1/\alpha}\,.$$

Hence,

$$g(\mathbf{x}) \le \tilde{g}(x_1)\,, \qquad \text{if } \ x_1 \ge \frac{c_1(\alpha,\eta)}{c_3(\alpha,\mathbf{k})}\,.$$

As M is compact the function $v(\varepsilon) := \max_{\mathbf{m}\in M} v_{\mathbf{m}}(\varepsilon)$ is continuous in ε. Since $v_{\mathbf{m}}(1) = \eta^2Nm_{n+1} \le -\eta^2Nc_2(\alpha,\mathbf{k}) < 0$ we have $v(1) \le -\eta^2Nc_2(\alpha,\mathbf{k}) < 0$, and a continuity argument gives

$$v(\varepsilon) < 0\,, \qquad \text{if } \ 1 > \varepsilon > \varepsilon_3'(\alpha,\eta,\mathbf{k})\,,$$

where $\varepsilon_3'(\alpha,\eta,\mathbf{k})$ decreases when η grows. Hence

$$v_{\mathbf{m}}(\varepsilon) \le v(\varepsilon) < 0\,, \qquad \text{for all } \mathbf{m} \in M\,.$$

It follows that $g(\mathbf{x}) \le \tilde{g}(x_1) = v_m(\varepsilon)\,x_1 \le v(\varepsilon)\,x_1$, if $x_1 \ge c_1(\alpha,\eta)/c_3(\alpha,\mathbf{k})$ and therefore $g(\mathbf{x}) \to -\infty$ "uniformly" as $\mathbf{x} \to \infty$.

This implies that there exists the maximum of g in the domain D and that this maximum is attained either on the boundary of D, either on the critical points of g, or on the points in which g is not differentiable. We will study each of this cases separately:

1) *g can not attain its maximum in the interior points of D in which g is not differentiable.* To prove this observe that

$$\frac{\partial g}{\partial x_i}(\mathbf{x}) = k_i\,(1-\varepsilon)\,\eta(x_i) - k_i\,(1+\varepsilon_{n+1})\left|\frac{1 - k_1x_1^\alpha - \cdots - k_nx_n^\alpha}{k_{n+1}}\right|^{(1-\alpha)/\alpha} \eta(1 - k_1x_1^\alpha - \cdots - k_nx_n^\alpha)\;|x_i|^{\alpha-1}\,.$$

We need to distinguish several cases:

- If $x_i = 0$ for some $i \in \{2,\ldots,n\}$, we have (see (3.3)) $x_{n+1} < 0$ and then

$$\frac{\partial g}{\partial x_i}(\mathbf{x})\Big|_{x_i=0} = -\infty\,, \qquad \text{if } \ 0 < \alpha < 1\,,$$

$$\begin{cases} \dfrac{\partial g}{\partial x_i}(\mathbf{x})\Big|_{x_i=0^+} = k_i(1-\varepsilon) > 0\,, \\ \dfrac{\partial g}{\partial x_i}(\mathbf{x})\Big|_{x_i=0^-} = k_i(1-\varepsilon)\eta > 0\,, \end{cases} \qquad \text{if } \ \alpha > 1\,.$$

If $\alpha \neq 1$, this implies that, in this case, g can not attain its maximum in the interior points of D where g is not differentiable.

- If $x_{n+1} = 0$, then $x_i > 0$ for all $i \leq n$ and

$$\frac{\partial g}{\partial x_i}(\mathbf{x})\Big|_{x_{n+1}=0} = -\infty\,, \qquad \text{if} \ \ \alpha > 1\,,$$

$$\frac{\partial g}{\partial x_i}(\mathbf{x})\Big|_{x_{n+1}=0} = k_i(1-\varepsilon) > 0\,, \qquad \text{if} \ \ 0 < \alpha < 1\,,$$

for all $i \leq n$. If $\alpha \neq 1$, this implies again that, also in this case, g can not attain its maximum in the interior points of D where g is not differentiable.

- If $\alpha = 1$ and $x_i = 0$ for some $1 < i \leq n$, then $x_{n+1} < 0$ and we have, for all these $\mathbf{x}$ in the interior of D, that

$$\frac{\partial g}{\partial x_1}(\mathbf{x}) = k_1(1-\varepsilon) - k_1(1+\varepsilon_{n+1})\eta < 0$$

if $1 > \varepsilon > \varepsilon_4'(1, \eta, \mathbf{k})$, where

$$\varepsilon_4'(1, \eta, \mathbf{k}) = \frac{1-\eta}{1 + (N - k_{n+1})\,\eta/k_{n+1}}\ .$$

- If $\alpha = 1$ and $x_{n+1} = 0$, then $x_i > 0$ for all $i \leq n$, and

$$\frac{\partial g}{\partial x_1}(\mathbf{x})\Big|_{x_{n+1}=0^+} = k_1(1-\varepsilon) - k_1(1+\varepsilon_{n+1}) < 0\,,$$

$$\frac{\partial g}{\partial x_1}(\mathbf{x})\Big|_{x_{n+1}=0^-} = k_1(1-\varepsilon) - k_1(1+\varepsilon_{n+1})\,\eta < 0\,,$$

if $1 > \varepsilon > \varepsilon_4'(1, \eta, \mathbf{k})$.

This implies again that, also in the two last cases, g can not attain its maximum in the interior points of D where g is not differentiable.

2) *g have not critical points in the interior of D.* It is easy to see that $\mathbf{x}$ is a critical point of g if and only if $x_i \neq 0$ for all $i \in \{1, \ldots, n+1\}$ and

$$(3.10) \quad (1-\varepsilon)\,x_1^{1-\alpha} = (1-\varepsilon)\,|x_2|^{1-\alpha}\eta(x_2) = \cdots = (1-\varepsilon)\,|x_n|^{1-\alpha}\eta(x_n) = (1+\varepsilon_{n+1})\,|x_{n+1}|^{1-\alpha}\eta(x_{n+1})\,.$$

We need again to distinguish several cases:

- If $\alpha = 1$, then (3.10) implies that $x_1, x_2, \ldots, x_n > 0$ and $x_{n+1} < 0$, since $1 + \varepsilon_{n+1} > 1 - \varepsilon$, and therefore a fortiori we must have $(1+\varepsilon_{n+1})\,\eta = (1-\varepsilon)$, but this is a contradiction with $1 > \varepsilon > \varepsilon_4'(1, \eta, \mathbf{k})$. Therefore, in this case, g can not have critical points.

- If $\alpha \neq 1$ and $n \geq 3$, there are not critical points in the interior of D since we have

$$x_1^{1-\alpha} = |x_2|^{1-\alpha}\eta(x_2) = |x_3|^{1-\alpha}\eta(x_3)\ .$$

But this implies that $x_i = x_{i+1}$ for some i, *i.e.* that $\mathbf{x} \in \partial D$.

• If $\alpha \neq 1$ and $n = 2$, we must have $x_1 > 0 > x_2 > x_3$ in order to $\mathbf{x}$ be in the interior of D, since if $x_2 > 0$, arguing as in the last case it is easy to see that $x_1 = x_2$. On the other hand, if $\mathbf{x}$ is a critical point of g, then

$$x_2 = \Big(\frac{1+\varepsilon_3}{1-\varepsilon}\Big)^{1/(1-\alpha)} x_3 \ .$$

Therefore, $0 > x_2 > x_3$ if and only if $\alpha > 1$ and so there are not critical points when $0 < \alpha < 1$.

If $0 > x_2 > x_3$ and $\alpha > 1$, we have $L(\varepsilon)\, x_1 = 1$, where

$$L(\varepsilon) = k_1 - k_2\, \eta^{1/(\alpha-1)} - k_3 \Big(\eta\, \frac{1+\varepsilon_3}{1-\varepsilon}\Big)^{1/(\alpha-1)} \ .$$

We can assure that $L(\varepsilon) < 0$ if $1 > \varepsilon > \varepsilon_5'(\alpha, \eta, \mathbf{k})$ by a continuity argument, where $\varepsilon_5'(\alpha, \eta, \mathbf{k})$ decreases when η grows.

This and the fact that $x_1 > 0$ imply that, also in this last case, there are not critical points of g in the interior of D.

3) Therefore *the maximum of g is attained in ∂D*. The point where the maximum is attained must verify $x_j = x_{j+1}$ for some $j \in \{1, \dots, n\}$. If we substitute this relation in the function g and in the domain D, we obtain a function and a domain of the same type, but now with $n-1$ variables, and with a different $\mathbf{k}'$ also satisfying $\sum_i k_i' = N$.

The induction hypothesis gives that

$$\max_{\mathbf{x} \in \partial D} g(\mathbf{x}) \ = \ N^{(\alpha-1)/\alpha} \qquad\qquad \text{for} \quad 1 > \varepsilon > \varepsilon_6'(\alpha, \eta, \mathbf{k}')\,,$$

where $\varepsilon_6'(\alpha, \eta, \mathbf{k}')$ decreases when η grows. As we have used along the induction hypothesis only a finite number of functions $\varepsilon_i'(\alpha, \eta, \cdot)$, the proof is finished.

Proposition 3.3. *Let $\mathbf{k} = (k_1, \dots, k_{n+1})$ be a vector with strictly positive integer entries, $N = \sum_i k_i$, $0 < \eta < 1$ and $\alpha > 0$. Let $\eta(t)$ be the function whose values are 1 if $t \geq 0$, and the constant η elsewhere. Given $0 < \varepsilon < 1$, consider the numbers*

$$\varepsilon_1 = \frac{N - k_1}{k_1}\, \varepsilon\,, \qquad \varepsilon_2 = \cdots = \varepsilon_{n+1} = -\varepsilon\,.$$

Then, the function defined by

$$h(\mathbf{x}) = \sum_{i=1}^{n} k_i\, (1+\varepsilon_i)\, \frac{x_i}{\eta(x_i)} - k_{n+1}\, (1+\varepsilon_{n+1})\, \frac{((k_1 x_1^\alpha + \cdots + k_n x_n^\alpha)/k_{n+1})^{1/\alpha}}{\eta(-k_1 x_1^\alpha - \cdots - k_n x_n^\alpha)}$$

satisfies

$$\min_{\mathbf{x} \in D_0} h(\mathbf{x}) = h(\mathbf{0}) = 0\,, \qquad \textit{for} \ \ 1 > \varepsilon > \varepsilon''(\alpha, \eta, \mathbf{k}) > 0\,, \tag{3.11}$$

where $\varepsilon''(\alpha, \eta, \mathbf{k})$ decreases when η grows and

$$D_0 := \Big\{ \mathbf{x} \in \mathbf{R}^n : \ x_1 \geq \cdots \geq x_n \geq -\Big(\frac{k_1 x_1^\alpha + \cdots + k_n x_n^\alpha}{k_{n+1}}\Big)^{1/\alpha} \Big\}\,.$$

Remark. Observe that in fact $\varepsilon''(\alpha,\eta,\mathbf{k}) = \varepsilon_3(\alpha,\eta,\mathbf{k})$ (see the proof of Proposition 3.1). This implies that (3.11) is true for $1 > \varepsilon > \varepsilon(\alpha,\eta,\mathbf{k})$ where this last function is the one appearing in Proposition 3.1.

Proof. If we define now

$$x_{n+1} := -\Big(\frac{k_1 x_1^\alpha + \cdots + k_n x_n^\alpha}{k_{n+1}}\Big)^{1/\alpha},$$

we have

$$x_1 \ge \cdots \ge x_n \ge x_{n+1} \qquad \text{and} \qquad k_1 x_1^\alpha + \cdots + k_n x_n^\alpha + k_{n+1} x_{n+1}^\alpha = 0\,.$$

This implies that $x_1 \ge 0$.

Then, for $i = 1, \ldots, n$, we can write $x_i = m_i x_1$, where $\mathbf{m} = (m_1, \ldots, m_n)$ is in the compact set

$$M := \Big\{\mathbf{m} \in \mathbf{R}^n : \ m_1 = 1 \ge m_2 \ge \cdots \ge m_n \ge -\Big(\frac{k_1 m_1^\alpha + \cdots + k_n m_n^\alpha}{k_{n+1}}\Big)^{1/\alpha}\Big\}\,.$$

Observe now that

$$h(\mathbf{x}) = \Bigg(\sum_{i=1}^n k_i\,(1+\varepsilon_i)\,\frac{m_i}{\eta(m_i)} - k_{n+1}\,(1+\varepsilon_{n+1})\,\frac{((k_1 m_1^\alpha + \cdots + k_n m_n^\alpha)/k_{n+1})^{1/\alpha}}{\eta(-k_1 m_1^\alpha - \cdots - k_n m_n^\alpha)}\Bigg)\,x_1 := u_{\mathbf{m}}(\varepsilon)\,x_1\ .$$

Observe that this function $u_{\mathbf{m}}$ is the same function that appears in the proof of Proposition 3.1.

As M is compact the function $u(\varepsilon) := \min_{\mathbf{m}\in M} u_{\mathbf{m}}(\varepsilon)$ is continuous in ε. Since $u_{\mathbf{m}}(1) = N$ we have $u(1) = N > 0$, and a continuity argument gives

$$u(\varepsilon) > 0 \quad \text{if} \ \ 1 > \varepsilon > \varepsilon''(\alpha,\eta,\mathbf{k})\,,$$

where $\varepsilon''(\alpha,\eta,\mathbf{k}) = \varepsilon_3(\alpha,\eta,\mathbf{k})$ decreases when η grows. Hence

$$u_{\mathbf{m}}(\varepsilon) \ge u(\varepsilon) > 0\,, \qquad \text{for all } \mathbf{m} \in M\,.$$

Therefore,

$$h(\mathbf{x}) \ge u(\varepsilon)\,x_1 \ge 0\,, \qquad \text{if} \ \ 1 > \varepsilon > \varepsilon''(\alpha,\eta,\mathbf{k})\,.$$

4. Proof of Theorem 2.

Recall that in Section 2, we defined the set of measures $\mathcal{M}_T$ and the functions d, u_n, d_n. We will use these definitions in what follows.

Lemma 4.1. *If T is a tree, $\mu \in \mathcal{M}_T$ and u is a function on T, then*

$$\int u_n\,d\mu = u(v_0) + \sum_{j=1}^n \int d_j\,d\mu\,.$$

PROOF. We will use induction in n. If $n = 1$ and $H_{v_0} = \{v^1, \ldots, v^k\}$ the lemma follows from

$$u(v_0) + \int d_1\, d\mu = u(v_0) + (u(v^1) - u(v_0))\, \mu(v^1) + \cdots + (u(v^k) - u(v_0))\, \mu(v^k) = \int u_1\, d\mu\,.$$

Finally, if we assume that the lemma is true for n then, if $v_n \in S_n$ and $H_{v_n} = \{v^1, \ldots, v^r\}$, we have

$$u(v^1)\, \mu(v^1) + \cdots + u(v^r)\, \mu(v^r) - u(v_n)\, \mu(v_n) = (u(v^1) - u(v_n))\, \mu(v^1) + \cdots + (u(v^r) - u(v_n))\, \mu(v^r)\,.$$

Summing this equalities for all vertices in S_n we obtain

$$\int u_{n+1}\, d\mu - \int u_n\, d\mu = \int d_{n+1}\, d\mu\,.$$

This formula and the induction hypothesis give the case $n + 1$.

Proposition 4.1. *Let T be a spherically symmetric tree with bounded degree. Given a p-subharmonic function u on T $(1 < p < \infty)$ and a constant $0 < \eta < 1$, there exist a function $\varepsilon(p, \eta)$ independent of u, which decreases when η grows, and a measure $\mu_\varepsilon \in \mathcal{M}_T$ for each $1 > \varepsilon > \varepsilon(p, \eta)$, such that*

$$\int d_n\, d\mu_\varepsilon \ge \frac{1-\eta}{1+\eta} \int |d_n|\, d\mu_\varepsilon \tag{4.1}$$

for all n.

PROOF. Given $1 > \varepsilon > 0$ we will define the measure μ_ε in the following way: $\mu_\varepsilon(v_0) = 1$ and, if $v \in V$ and $H_v = \{v^1, \ldots, v^N\}$ is indexed such that

$$d(v^1) \ge \cdots \ge d(v^N)\,,$$

we define

$$\mu_\varepsilon(v^1) = \mu_\varepsilon(v)\, \frac{1 + (N-1)\varepsilon}{N}\,, \qquad \mu_\varepsilon(v^2) = \cdots = \mu_\varepsilon(v^N) = \mu_\varepsilon(v)\, \frac{1-\varepsilon}{N}\,.$$

With this definition of μ_ε, we have as a consequence of Proposition 3.3 with $k_i = 1$ for all i (see Section 3; see also (4.6) below) that for $n = 1$

$$\eta \int (d_1)_+\, d\mu_\varepsilon - \int (d_1)_-\, d\mu_\varepsilon \ge 0\,, \tag{4.2}$$

where h_+ and h_- are the usual positive and negative parts of the function h. This inequality is true since u is a p-subharmonic function at the point v_0, *i.e.*

$$d(v^1)^{p-1} + \cdots + d(v^k)^{p-1} \ge 0\,,$$

where $H_{v_0} = \{v^1, \ldots, v^k\}$.

We will prove by induction that the condition (4.2) is true for all n and $1 > \varepsilon > \varepsilon(p, \eta)$, *i.e.* that

$$\eta \int (d_n)_+\, d\mu_\varepsilon - \int (d_n)_-\, d\mu_\varepsilon \ge 0\,. \tag{4.3}$$

If we suppose that (4.3) is true, we have that

$$\int (d_n)_+ \, d\mu_\varepsilon + \int (d_n)_- \, d\mu_\varepsilon \; - \eta \int (d_n)_+ \, d\mu_\varepsilon - \eta \int (d_n)_- \, d\mu_\varepsilon$$
$$\leq \int (d_n)_+ \, d\mu_\varepsilon - \int (d_n)_- \, d\mu_\varepsilon \; + \eta \int (d_n)_+ \, d\mu_\varepsilon - \eta \int (d_n)_- \, d\mu_\varepsilon \,,$$

i.e. that

$$(1-\eta) \int |d_n| \, d\mu_\varepsilon \, \leq \, (1+\eta) \int d_n \, d\mu_\varepsilon \,,$$

and then the proposition will be proved.

Therefore to finish the proof we can suppose that (4.3) is true for n. If $S_n = \{v_1, \dots, v_m\}$, we consider $H_{v_j} = \{v_j^1, \dots, v_j^N\}$ (where $N = N(n)$ is independent of j since T is spherically symmetric), ordered such that $d_{n+1}(v_j^1) \geq \cdots \geq d_{n+1}(v_j^N)$. If we define $y_k := d_{n+1}(v_j^k)$, then

$$\Delta_p u(v_j) \; = \; -y_1^{p-1} - \cdots - y_N^{p-1} + d_n(v_j)^{p-1} \leq 0 \,.$$

- If $d_n(v_j) > 0$, the numbers $z_k := y_k / d_n(v_j)$ satisfy

$$z_1^{p-1} + \cdots + z_N^{p-1} \; \geq \; 1 \qquad \text{and} \qquad z_1 \geq \cdots \geq z_N \,.$$

Defining now $x_k := z_k$ for $k < N$, and $x_N = (1 - z_1^{p-1} - \cdots - z_{N-1}^{p-1})^{1/(p-1)} \leq z_N$, we have that

$$x_1^{p-1} + \cdots + x_N^{p-1} \; = \; 1 \qquad \text{and} \qquad x_1 \geq \cdots \geq x_N \,.$$

We will be interested in the expression

$$\begin{aligned} A &:= N\eta \Big(d_{n+1}(v_j^1) \, \mu_\varepsilon(v_j^1) + \frac{d_{n+1}(v_j^2)}{\eta(d_{n+1}(v_j^2))} \, \mu_\varepsilon(v_j^2) \cdots + \frac{d_{n+1}(v_j^N)}{\eta(d_{n+1}(v_j^N))} \, \mu_\varepsilon(v_j^N) \Big) \\ &= \eta \, d_n(v_j) \, \mu_\varepsilon(v_j) \Big((1+\varepsilon_1) \, z_1 + (1-\varepsilon) \, \frac{z_2}{\eta(z_2)} + \cdots + (1-\varepsilon) \frac{z_N}{\eta(z_N)} \Big) \\ &\geq \eta \, d_n(v_j) \, \mu_\varepsilon(v_j) \Big((1+\varepsilon_1) \, x_1 + (1-\varepsilon) \, \frac{x_2}{\eta(x_2)} + \cdots + (1-\varepsilon) \frac{x_N}{\eta(x_N)} \Big) \\ &= \eta \, d_n(v_j) \, \mu_\varepsilon(v_j) \, f(\mathbf{x}) \,, \end{aligned}$$

where $\mathbf{x} = (x_1, \dots, x_{N-1}) \in D$ (see Proposition 3.1 for the definition of D; we are using here the case $k_i = 1$ for all i and $\alpha = p-1$).

Proposition 3.1 gives

$$A \geq \eta \, d_n(v_j) \, \mu_\varepsilon(v_j) \, N^{(p-2)/(p-1)} \,, \tag{4.4}$$

if $1 > \varepsilon > \varepsilon_1(p, \eta)$.

- If $d_n(v_j) < 0$, the numbers $z_k := y_k / d_n(v_j)$ satisfy

$$z_1^{p-1} + \cdots + z_N^{p-1} \; \leq \; 1 \qquad \text{and} \qquad z_1 \leq \cdots \leq z_N \,.$$

Defining now $x_k := z_{N-k+1}$ for $k \geq 2$, and $x_1 = (1 - z_1^{p-1} - \cdots - z_{N-1}^{p-1})^{1/(p-1)} \geq z_1$, we have that

$$x_1^{p-1} + \cdots + x_N^{p-1} \; = \; 1 \qquad \text{and} \qquad x_1 \geq \cdots \geq x_N \,.$$

We will be interested in the expression

$$\begin{aligned} B &:= N\Big(d_{n+1}(v_j^1)\,\mu_\varepsilon(v_j^1)\,\eta(-d_{n+1}(v_j^1)) + \cdots + d_{n+1}(v_j^N)\,\mu_\varepsilon(v_j^N)\,\eta(-d_{n+1}(v_j^N))\Big) \\ &= d_n(v_j)\,\mu_\varepsilon(v_j)\Big((1+\varepsilon_1)\,z_1\,\eta(z_1) + (1-\varepsilon)\,z_2\,\eta(z_2) + \cdots + (1-\varepsilon)\,z_N\,\eta(z_N)\Big) \\ &\geq d_n(v_j)\,\mu_\varepsilon(v_j)\,g(\mathbf{x})\,, \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_{N-1}) \in D$ (see Proposition 3.2 for the definition of D; we are using here the case $k_i = 1$ for all i and $\alpha = p-1$).

Proposition 3.2 gives

$$B \geq d_n(v_j)\,\mu_\varepsilon(v_j)\,N^{(p-2)/(p-1)}\,, \tag{4.5}$$

if $1 > \varepsilon > \varepsilon_2(p, \eta)$.

- If $d_n(v_j) = 0$, the numbers y_k satisfy

$$y_1^{p-1} + \cdots + y_N^{p-1} \;\geq\; 0 \qquad \text{and} \qquad y_1 \geq \cdots \geq y_N\,.$$

Defining now $x_k := y_k$ for $k < N$, and $x_N = -(y_1^{p-1} + \cdots + y_{N-1}^{p-1})^{1/(p-1)} \leq y_N$, we have that

$$x_1^{p-1} + \cdots + x_N^{p-1} \;=\; 0 \qquad \text{and} \qquad x_1 \geq \cdots \geq x_N\,.$$

We are interested now in the expression

$$\begin{aligned} C &:= (1+\varepsilon_1)\,\frac{y_1}{\eta(y_1)} + (1-\varepsilon)\,\frac{y_2}{\eta(y_2)} + \cdots + (1-\varepsilon)\,\frac{y_N}{\eta(y_N)} \\ &\geq (1+\varepsilon_1)\,\frac{x_1}{\eta(x_1)} + (1-\varepsilon)\,\frac{x_2}{\eta(x_2)} + \cdots + (1-\varepsilon)\,\frac{x_N}{\eta(x_N)} = h(\mathbf{x})\,, \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_{N-1}) \in D_0$ (see Proposition 3.3 for the definition of D_0; we are using here the case $k_i = 1$ for all i and $\alpha = p-1$).

Proposition 3.3 gives

$$C \geq 0\,, \tag{4.6}$$

if $1 > \varepsilon > \varepsilon_1(p, \eta)$.

Recall that the vertices $v_1, \ldots, v_m$ have the same number of children since T is spherically symmetric. Therefore, summing the expressions A, B and C, for all vertices $v_1, \ldots, v_m$ in S_n and using (4.4), (4.5) and (4.6), we obtain that

$$N\Big(\eta\int (d_{n+1})_+\,d\mu_\varepsilon \;-\; \int (d_{n+1})_-\,d\mu_\varepsilon\Big) \geq N^{(p-2)/(p-1)}\Big(\eta\int (d_n)_+\,d\mu_\varepsilon \;-\; \int (d_n)_-\,d\mu_\varepsilon\Big)\,,$$

and

$$\eta\int (d_{n+1})_+\,d\mu_\varepsilon \;-\; \int (d_{n+1})_-\,d\mu_\varepsilon \geq N^{-1/(p-1)}\Big(\eta\int (d_n)_+\,d\mu_\varepsilon \;-\; \int (d_n)_-\,d\mu_\varepsilon\Big) \;\geq\; 0\,,$$

by the induction hypothesis. This finishes the proof of Proposition 4.1.

Proof of Theorem 2. Without loss of generality we can assume that u is an bounded above p-subharmonic function.

Let $0 < \eta < 1$ and μ_ε the measure of Proposition 4.1 with $1 > \varepsilon > \varepsilon(p,\eta)$. Proposition 4.1 and Lemma 4.1 give

$$\int u_m \, d\mu_\varepsilon \;=\; u(v_0) \,+\, \sum_{n=1}^{m} \int d_n \, d\mu_\varepsilon \;\geq\; u(v_0) \,+\, \frac{1-\eta}{1+\eta} \sum_{n=1}^{m} \int |d_n| \, d\mu_\varepsilon \, .$$

If M is an upper bound of the function u, this inequality implies

$$\sum_{n=1}^{m} \int |d_n| \, d\mu_\varepsilon \;\leq\; \frac{1+\eta}{1-\eta} \, (M - u(v_0)) \, ,$$

and then

$$\int \Big(\sum_{n=1}^{\infty} |d_n| \Big) \, d\mu_\varepsilon \;\leq\; \frac{1+\eta}{1-\eta} \, (M - u(v_0)) \, .$$

Therefore

$$\sum_{n=1}^{\infty} |d_n| \;<\; \infty \, , \tag{4.7}$$

almost everywhere with respect to μ_ε and consequently $\mu_\varepsilon(BV(u)) = 1$.

It is well known the following fact

Lemma 4.2. *If μ is a Borel measure over $\mathbf{R}$ and there are positive constants c, d such that for all interval I, $\mu(I) \leq c|I|^d$, and H is a set with $\mu(H) > 0$, we have $Dim\,(H) \geq d$.*

Let $\{v_0, v_1, \dots\} \in \partial T$ be an infinite path. If we prove

$$\mu_\varepsilon(v_n) \;\leq\; c\,|v_n|^d \, ,$$

with constants c, d independent of the vertices, then Lemma 4.2 gives $Dim\,(BV(u)) \geq d$. Here we are using the identification between ∂T and $[0,1)$. More concretely, if we denote by N_{k+1} the cardinal of the set H_{v_k} and by $|\cdot|$ the “Lebesgue measure” in ∂T, then

$$|v_1| = \frac{1}{N_1} \, , \qquad \frac{|v_k|}{|v_{k-1}|} = \frac{1}{N_k} \, ,$$

and

$$|v_n| = \frac{1}{N_1} \cdots \frac{1}{N_n} \, .$$

On the other hand, for the measure μ_ε we have

$$\frac{\mu_\varepsilon(v_k)}{\mu_\varepsilon(v_{k-1})} = \begin{cases} \dfrac{1 + (N_k - 1)\,\varepsilon}{N_k} \\ \text{or} \\ \dfrac{1-\varepsilon}{N_k} \end{cases} \qquad \text{for} \;\; k \geq 1 \, ,$$

and

$$\mu_\varepsilon(v_n) = \frac{a(1,\varepsilon)}{N_1} \, \frac{a(2,\varepsilon)}{N_2} \cdots \frac{a(n,\varepsilon)}{N_n} \, ,$$

where

$$a(k,\varepsilon) = \begin{cases} 1+(N_k-1)\,\varepsilon \\ \text{or} \\ 1-\varepsilon \end{cases} \qquad \text{for} \quad k \geq 1\,.$$

Therefore,

$$\mu_\varepsilon(v_n) \leq \frac{1+(N_1-1)\,\varepsilon}{N_1} \cdots \frac{1+(N_n-1)\,\varepsilon}{N_n}\,.$$

Let now N be the number defined by

$$N = \limsup_{k\to\infty} N_k\ .$$

Then, $N_k \leq N$ if $k \geq k_0$ and therefore, if $n \geq k_0$, we have that

$$\mu_\varepsilon(v_n) \leq \frac{1+(N_{k_0}-1)\,\varepsilon}{N_{k_0}} \cdots \frac{1+(N_n-1)\,\varepsilon}{N_n}\,.$$

On the other hand if we take

$$d := 1 - \frac{\log\left(1+(N-1)\,\varepsilon\right)}{\log N}\,,$$

it is not difficult to see that

$$d \leq 1 - \frac{\log\left(1+(N_k-1)\,\varepsilon\right)}{\log N_k}\,, \qquad \text{for} \quad k \geq k_0\,, \tag{4.8}$$

by using the fact that, for each integer $m \geq 2$, the function

$$A(\varepsilon) := \log m\, \log\left(1+m\,\varepsilon\right) - \log\left(m+1\right)\, \log\left(1+(m-1)\,\varepsilon\right)$$

satisfies $A(\varepsilon) \geq 0$ for all $\varepsilon \in [0,1]$.

Then, (4.8) implies

$$\frac{1+(N_k-1)\,\varepsilon}{N_k} \leq \Big(\frac{1}{N_k}\Big)^d, \qquad \text{for} \quad k \geq k_0\ .$$

This gives

$$\mu_\varepsilon(v_n) \leq \Big(N_1\,N_2 \cdots N_{k_0-1}\Big)^d \Big(\frac{1}{N_1}\,\frac{1}{N_2}\,\cdots\,\frac{1}{N_n}\Big)^d = C\,|v_n|^d\,,$$

for all $n \geq 1$. Hence

$$Dim\left(BV(u)\right) \geq 1 - \frac{\log\left(1+(N-1)\,\varepsilon\right)}{\log N}\,, \qquad \text{for} \quad 1 > \varepsilon > \varepsilon(p,\eta)\,,$$

and consequently

$$Dim\left(BV(u)\right) \geq 1 - \frac{\log\left(1+(N-1)\,\varepsilon(p,\eta)\right)}{\log N}\,.$$

If we choose $\phi(p)$ as the function defined by

$$\phi(p) := \lim_{\eta\to 1}\Big(1 - \frac{\log\left(1+(N-1)\,\varepsilon(p,\eta)\right)}{\log N}\Big)\,, \tag{4.9}$$

the Theorem is proved unless we need yet to show that $\phi(2)=1$.

Observe that the function $\varepsilon(2,\eta)$ appearing in Proposition 4.1 (and (4.9)) tends to zero as $\eta \to 1$ since the functions $\varepsilon_1(1,\eta,\mathbf{k})$, $\varepsilon_3(1,\eta,\mathbf{k})$, $\varepsilon_4(1,\eta,\mathbf{k})$ (appearing in the proof of Proposition 3.1), and $\varepsilon_1'(1,\eta,\mathbf{k})$,

$\varepsilon_3'(1,\eta,\mathbf{k})$, $\varepsilon_4'(1,\eta,\mathbf{k})$ (appearing in the proof of Proposition 3.2) tend also to zero as $\eta \to 1$. In fact, remember that

$$\varepsilon_1(1,\eta,\mathbf{k}) = \frac{1-\eta}{1+\eta\, k_2/k_1}\,, \qquad \varepsilon_4(1,\eta,\mathbf{k}) = \frac{1-\eta}{1+(N-k_1)\,\eta/k_1}\,,$$

$$\varepsilon_1'(1,\eta,\mathbf{k}) = \frac{1-\eta}{1+\eta\, k_1/k_2}\,, \qquad \varepsilon_4'(1,\eta,\mathbf{k}) = \frac{1-\eta}{1+(N-k_{n+1})\,\eta/k_{n+1}}\,.$$

Therefore it only remains to find good upper bounds of $\varepsilon_3(1,\eta,\mathbf{k})$ and $\varepsilon_3'(1,\eta,\mathbf{k})$.

We have seen that

$$u_{\mathbf{m}}(\varepsilon) = \sum_{i=1}^{n} k_i\,(1+\varepsilon_i)\,\frac{m_i}{\eta(m_i)} - k_{n+1}\,(1+\varepsilon_{n+1})\,\frac{\big((k_1 m_1^\alpha + \cdots + k_n m_n^\alpha)/k_{n+1}\big)^{1/\alpha}}{\eta(-k_1 m_1^\alpha - \cdots - k_n m_n^\alpha)}\,,$$

and that $\varepsilon_3(\alpha,\eta,\mathbf{k})$ is defined by

$$u_{\mathbf{m}}(\varepsilon) > 0\,, \qquad \text{for all } \mathbf{m}\in M\,,$$

if $1 > \varepsilon > \varepsilon_3(\alpha,\eta,\mathbf{k})$.

Observe that, if $\alpha = 1$ and $m_r \geq 0 > m_{r+1}$, we have

$$u_{\mathbf{m}}(\varepsilon) \;=\; k_1\,(1+\varepsilon_1) - k_1\frac{1-\varepsilon}{\eta} + \sum_{i=2}^{r} k_i\,(1-\varepsilon)\,m_i\Big(1-\frac{1}{\eta}\Big)\,,$$

and then

$$u_{\mathbf{m}}(\varepsilon) \;\geq\; k_1\,(1+\varepsilon_1) - k_1\frac{1-\varepsilon}{\eta} - (N-k_1)\,(1-\varepsilon)\,\frac{1-\eta}{\eta}\,.$$

An easy computation gives that the last right hand is positive if and only if $\varepsilon > 1-\eta$, and therefore

$$\varepsilon_3(1,\eta,\mathbf{k}) \leq 1-\eta\,.$$

Also, we have seen that

$$v_{\mathbf{m}}(\varepsilon) = \sum_{i=1}^{n} k_i\,(1-\varepsilon)\,m_i\,\eta(m_i) - \eta\,k_{n+1}\,(1+\varepsilon_{n+1})\,\Big(\frac{k_1 m_1^\alpha + \cdots + k_n m_n^\alpha}{k_{n+1}}\Big)^{1/\alpha}\,\eta(-k_1 m_1^\alpha - \cdots - k_n m_n^\alpha)\,,$$

and that $\varepsilon_3'(\alpha,\eta,\mathbf{k})$ is defined by

$$v_{\mathbf{m}}(\varepsilon) < 0\,, \qquad \text{for all } \mathbf{m}\in M\,,$$

if $1 > \varepsilon > \varepsilon_3'(\alpha,\eta,\mathbf{k})$.

Observe that, if $\alpha = 1$ and $m_r \geq 0 > m_{r+1}$, we have

$$v_{\mathbf{m}}(\varepsilon) \;=\; (1-\varepsilon-\eta^2(1+\varepsilon_{n+1}))\sum_{i=1}^{n} k_i\,m_i - (1-\varepsilon)(1-\eta)\sum_{i=r+1}^{n} k_i\,m_i\,.$$

In order to obtain the inequality $1-\varepsilon-\eta^2(1+\varepsilon_{n+1}) < 0$, we impose the condition

$$\varepsilon > \frac{1-\eta^2}{1+(N-k_{n+1})\eta^2/k_{n+1}}\,, \tag{4.10}$$

and then

$$v_{\mathbf{m}}(\varepsilon) \;\leq\; (1-\varepsilon-\eta^2(1+\varepsilon_{n+1}))\,A + (1-\varepsilon)(1-\eta)\,B\,,$$

where $A > 0$ and $B \geq 0$ are constants which are independent of $\mathbf{m}$ and whose existence we can assure since M is a compact set. An easy computation gives that the last right hand is negative if and only if

$$\varepsilon > \frac{B(1-\eta) + A(1-\eta^2)}{B(1-\eta) + A(1 + (N - k_{n+1})\eta^2/k_{n+1})} .$$

This condition implies, in particular, (4.10), and therefore

$$\varepsilon_3(1, \eta, \mathbf{k}) \leq \frac{B(1-\eta) + A(1-\eta^2)}{B(1-\eta) + A(1 + (N - k_{n+1})\eta^2/k_{n+1})} .$$

Remark. If $p \neq 2$, we have $\phi(p) < 1$. This can be deduced by considering $\varepsilon_2(p-1, \eta, \mathbf{k})$ (if $p > 2$) and $\varepsilon_2'(p-1, \eta, \mathbf{k})$ (if $1 < p < 2$).

5. Some examples.

In this section we are going to prove an analogue to Rudin's result for a tree. Precisely, we construct a bounded harmonic function on a directed tree with infinite variation along almost every path in ∂T. Rudin's example is based on lacunary series while our construction will be based on a probabilistic approach.

We shall be able to find also, examples of bounded p-harmonic functions of infinite variation along almost every path.

In order to make the exposition clearer we are going to introduce some concepts that we will need trough out this section.

5.1. A general one-dimensional random walk.

In this section we are going to deal with a general notion of one-dimensional random walk. We are going to consider the path described by a particle that starting from a position $\alpha_0 k$, has probability p_j to move at time n from x to $x+\alpha_n j$ for each integer j. The number $|\alpha_n|$ will be the step of the random walk at time n.

In other words, the position of the particle following the n-th trial is the point,

$$\mathcal{S}_n = \alpha_0 k + \alpha_1 X_1 + ... + \alpha_n X_n \, ,$$

where $\{X_k\}$ are mutually independent random variables identically distributed such that $P(X_k = j) = p_j$, $j \in \mathbf{Z}$. Here $P(A)$ denotes the probability of the event A.

Note that if $\alpha_0 = \alpha_1 = \dots$, then $\mathcal{S}_n$ is the traditionally called generalized random walk (see [F, p.363]). If moreover $p_j = 0$ for all $j \neq -1, 1$, that is, if the particle can only jump one unit up or one unit down, then $\mathcal{S}_n$ is the usual random walk, which is termed symmetric whenever $p_{-1} = p_1 = 1/2$.

Troughout this section we are always going to refer to this notion of random walk where only a finite number of probabilities p_j are different from zero.

5.2. Sequence of temporary absorbing barriers.

Let us consider the random walk as described in Section 5.1. Suppose that at certain time n_0 the position of the random walk is between two levels M and N, that is, $M < S_{n_0} < N$. Then we can stop when the particle reaches the positions M or N for the first time after n_0, that is, we stop the process whenever,

$$\begin{aligned} \mathcal{S}_n &= \alpha_0 k + \alpha_1 X_1 + \cdots + \alpha_n X_n \leq M \, , \quad \text{or,} \\ \mathcal{S}_n &= \alpha_0 k + \alpha_1 X_1 + \cdots + \alpha_n X_n \geq N \, , \end{aligned}$$

for $n \geq n_0$. In such a case we say that the particle performs a random walk with absorbing barriers at M and N (usually n_0 is zero).

Now we can also let the barriers act for a while, that is, we can stop the process only for n with $n_0 \le n \le n_1$, and let it start again after time n_1. In this case we say that the particle performs a random walk with temporary absorbing barriers. This allows us to confine the random walk in a band during a while, and change the band afterwards if it is necessary. In this way we could also construct a sequence of temporary absorbing barriers.

The period of time in which the barriers are active either could be given a priori, that is, we could fix a sequence of times n_k's, or could be chosen by a stopping time argument, that is, given n_{k-1}, and barriers M_{k-1}, N_{k-1} we let the process to evolve and wait until something has happened. We take the time n_k to be the one in which it has occured what we have been waiting for. To give an example, n_k could be the time at which the probability that the particle has reached a barrier is big enough.

For convenience we will consider that $X_n \equiv 0$ if the process stops at time n. Note that in spite $\{X_n\}$ will not be identically distributed any more it make sense to refer to the position of the particle at any time $n \in \mathbf{N}$ by $\mathcal{S}_n = \alpha_0 k + \alpha_1 X_1 + \cdots + \alpha_n X_n$.

5.3. Existence theorem and general ideas of the proof.

We can now state the main theorem of this section.

Theorem 4. *For $1 < p < \infty$, there exists a process $\mathcal{S}_n$, such that*

i) $\sum_{j \in \mathbf{Z}} j^{p-1} p_j = 0$, *where only a finite number of p_j's are different from zero.*
ii) *It is bounded.*
iii) *For $\triangle \mathcal{S}_n = \mathcal{S}_n - \mathcal{S}_{n-1} = \alpha_n X_n$, we have that,*

$$P(\{\sum_{n=1}^{\infty} |\triangle \mathcal{S}_n| = \infty\}) = 1\,.$$

The behaviour of the random walk depends very strongly on the expectation of X_n, $E(X_n)$. If $E(X_n) = 0$, the random walk has no "preference" for any direction, the particle will move "equally" up or down. Nevertheless if $E(X_n) > 0$ (or $E(X_n) < 0$) the random walk will have a drift towards the top barrier (or the bottom barrier).

When $p = 2$, i) implies that $E(X_n) = 0$. A random walk without barriers will oscilate infinitely often around its initial position. We are going to take the advantage of this fact, but since we require it to be bounded we need to put some barriers. Nevertheless once the process reaches a barrier is "trapped"and then $|\triangle S_n| = 0$. So the idea is to put the barriers further and further away by making the step smaller and smaller with respect to them.

In the case when $p \neq 2$, we will take the advantage of the drift to force the random walk to oscillate infinitely often. Note that since it oscillates it will not escape to infinity (will be bounded) and will be running all the time and so, $\sum_{n=1}^{\infty} |\triangle \mathcal{S}_n|$ will have more chances to diverge.

5.4. Proof for $p = 2$.

To prove Theorem 4 in this special case we need the following well-known lemma due to Kolmogorov, see [W, p.138].

Lemma 5.1. *Let $\{X_n\}$ be a sequence of zero mean random variables in L^2. Define $\sigma_k^2 = Var(X_k)$. Write $\mathcal{S}_n := a + X_1 + X_2 + \cdots + X_n$. Then for $C > 0$,*

$$P\left(\left\{\sup_{k\le n} |\mathcal{S}_k| \geq C\right\}\right) \leq \frac{\sum_{k=1}^{n} \sigma_k^2}{(C-a)^2} .$$

Now we are going to construct a process with decreasing step $\{\alpha_n\}$. The particle starts at position 0 and has probability 1/2 to jump one unit up or down, and has absorbing barriers at M_1 and $-M_1$. At time n_1 we change the barriers to M_2 and $-M_2$ with $M_2 > M_1$. n_1 is chosen so that the barriers $M_2, -M_2$ appear further away with step α_{n_2} than the barriers $M_1, -M_1$ with step α_1. Again we let the particle to move freely until time n_2 when we change the barriers once more. We continue the process in this way indefinitely.

In other words, we are going to construct a symmetric random walk that starts at position 0 with steps α_n and temporary absorbing barriers $M_k, -M_k$ for $n_{k-1} < n \leq n_k$. So the position of the particle at time n is given by,

$$\mathcal{S}_n = \alpha_1 X_1 + \cdots + \alpha_n X_n ,$$

where X_j is either a Bernoulli trial so $P(X_j = 1) = 1/2 = P(X_j = -1)$, or $X_j \equiv 0$ and then $P(X_j = 0) = 1$.

In any case, $E(X_j) = 0$, and therefore the process is a martingale and property i) holds for $p = 2$.

Now we are going to choose the steps, the barriers and the time intervals in which they are active.

First the steps are a decreasing sequence of positive numbers $\{\alpha_n\}$ such that,

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\alpha_1 < 1$ and,

(2) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

For technical reasons we will also require that $\alpha_n/\alpha_{n+1} \in \mathbf{N}$.

Next, we are going to choose the barriers. Take $\{\nu_j\}$ a sequence of natural numbers so that $\sum_{j=1}^{\infty} \alpha_{\nu_j} \leq 1$ and $\nu_1 = 0$. Let M much bigger than $\frac{1}{\alpha_1}$ and define

$$m_j = \alpha_{\nu_j} M \qquad \text{and} \qquad M_k = \sum_{j=1}^{k} m_j \,.$$

And finally, to define the interval of time in which the barriers will be active, take a subsequence of ν_j, $n_k = \nu_{j_k}$ such that for a fixed $\varepsilon \in (0,1)$,

(3) $\displaystyle\sum_{n=1+n_k}^{n_{k+1}} \alpha_n > 1$, for all $k \in \mathbf{N}$ and,

(4) $\displaystyle\sum_{n=n_k}^{\infty} \alpha_n^2 \leq \varepsilon (m_{k+1})^2$ for all $k \in \mathbf{N}$.

Note that we can always choose such a sequence by properties (1) and (2).

Clearly with this choice of the barriers the process is bounded since given $n \in \mathbf{N}$ take k so that $n \leq n_k$ and then $|\mathcal{S}_n| \leq M_k < M$.

So we are left to show that

$$P\left(\left\{\sum_{n=1}^{\infty} |\triangle \mathcal{S}_n| < \infty\right\}\right) = 0\,.$$

For simplicity and only throghout this proof, we say that a process reaches a barrier M_j if it reaches either M_j or $-M_j$, that is, $\mathcal{S}_n = M_j$ or $\mathcal{S}_n = -M_j$ for $n_{j-1} < n \leq n_j$.

The path described by the particle in its evolution can either avoid infinitely many barriers or only avoid a finite number of them (and cannot do anything else).

Suppose that it avoids infinitely many barriers, say $\{M_{k_j}\}$, then,

$$|\mathcal{S}_n| < M_{k_j}, \text{ for } n_{k_j - 1} < n \leq n_{k_j}\,,$$

and then

$$\triangle \mathcal{S}_n = \alpha_n X_n, \qquad X_n \neq 0, \qquad \text{for } \; n_{k_j - 1} < n \leq n_{k_j}\,.$$

So,

$$\sum_{n=1}^{\infty} |\triangle \mathcal{S}_n| \geq \sum_{j=1}^{\infty} \sum_{n=n_{k_j-1}}^{n_{k_j}} \alpha_n \geq \sum_{j=1}^{\infty} 1 = \infty\,,$$

where the last inequality follows from the condition (3).

Therefore, a necessary condition for $\sum_{n=1}^{\infty} |\triangle \mathcal{S}_n| < \infty$ to hold is that the particle avoids a finite number of barriers. So it is enough to show that

$$P(\text{a finite number of barriers is avoided}) = 0\,.$$

Notice that,

$$\begin{aligned} & P(\text{a finite number of barriers is avoided}) \\ \leq & \sum_{k=1}^{\infty} P(\text{the particle reaches all the barriers after } M_k). \end{aligned}$$

Denote the event "*the particle reaches all barriers after* M_k" by B_k, and the event "*the particle reaches the barrier* M_k" by C_k. Then,

$$B_k = \bigcap_{j=k+1}^{\infty} C_j \,.$$

Notice that the events C_j are independent since the process is markovian and $n_k > n_{k-1} + 1$ (due to $\alpha_1 < 1$ and by property (3)). Therefore,

$$P(B_k) = \prod_{j=k+1}^{\infty} P(C_j | C_{j-1}) \,.$$

Now, observe that,

$$\begin{aligned} C_j &= \{\text{particle reaches the barrier } M_j\} \\ &= \{|\mathcal{S}_n| = M_j \text{ for some } n,\ n_{j-1} < n \leq n_j\} \\ &= \{\sup_{n_{j-1} < n \leq n_j} |\mathcal{S}_n| \geq M_j\}, \end{aligned}$$

where the second equality holds because the random walk cannot "trespass" the barrier M_j for time $n \leq n_j$ since $\frac{\alpha_n}{\alpha_{n+1}} \in \mathbf{N}$. On the other hand, observe that for $n > n_{j-1}$,

$$\begin{aligned} \mathcal{S}_n &= \mathcal{S}_{n_{j-1}} + \alpha_{n_{(j-1)}+1} X_{n_{(j-1)}+1} + \cdots + \alpha_n X_n \\ \mathcal{S}_n &\leq M_{j-1} + \alpha_{n_{(j-1)}+1} X_{n_{(j-1)}+1} + \cdots + \alpha_n X_n \\ \mathcal{S}_n &\geq -M_{j-1} + \alpha_{n_{(j-1)}+1} X_{n_{(j-1)}+1} + \cdots + \alpha_n X_n, \end{aligned}$$

thus, by Lemma 5.1,

$$\begin{aligned} P(C_j | C_{j-1}) &= P(\{\sup_{n_{j-1} < n \leq n_j} |\mathcal{S}_n| \geq M_j\}) \\ &\leq \frac{\sum_{i=n_{j-1}}^{n_j} \alpha_i^2}{(M_j - M_{j-1})^2} < \frac{\sum_{i=n_{j-1}}^{\infty} \alpha_i^2}{(m_j)^2}. \end{aligned}$$

Therefore by property (4),

$$P(C_j | C_{j-1}) < \varepsilon.$$

So we have,

$$P(B_k) = \prod_{j=k+1}^{\infty} P(C_j | C_{j-1}) \leq \prod_{j=k+1}^{\infty} \varepsilon = 0 \,,$$

and then Theorem 4 is proved for $p = 2$.

5.5. Proof for $p \neq 2$.

As we have mentioned above, in this case the expectation of X_n could be positive or negative and it determinates a drift towards the top barrier or the bottom barrier. To make more precise this fact we need the following lemma, (see [F, p. 366]).

Lemma 5.2. *Let $\mathcal{S}_n$ be a random walk of constant step $\alpha > 0$, that starts at position k, that is, $\mathcal{S}_n = k + \alpha(X_1 + \cdots + X_n)$. Suppose further that $p_{-a} = p$ and $p_b = 1 - p$ for a, b integers such that $a, b > 0$ and $a < k - M, b < N - k$. Let u_k be the probability that the particle reaches a position $\leq M$ before a position $\geq N$. Then,*

$$u_k \geq \frac{\sigma^{\frac{N-M}{\alpha}} - \sigma^{\frac{k-M}{\alpha}}}{\sigma^{\frac{N-M}{\alpha}} - \sigma^{-a+1}},$$

where $\sigma \neq 1$ is the positive root of the equation

$$p_{-a}x^{-a} + p_b x^b = 1 .$$

Corollary 5.1. *Let M, N, k, α and σ be as in Lemma 5.2. Let v_k^n denote the probability that a particle that starts at position k reaches a position $\leq M$ before reaching a position $\geq N$, before time n. If $\sigma > 1$ there exists $\tilde{n} \in \mathbf{N}$ such that*

$$v_k^{\tilde{n}} \geq \frac{\sigma^{\frac{N-M}{\alpha}} - \sigma^{\frac{k-M}{\alpha}}}{\sigma^{\frac{N-M}{\alpha}}} .$$

To prove the case $p \neq 2$, the idea is to construct a process with steps α_n and temporarily absorbing barriers M_k and $-M_k$ for $n_{k-1} < n \leq n_k$. The step will be constant for $n_{k-1} < n \leq n_k$, let us denote the step by α_k. We will choose the signs of α_k so that $E(\alpha_{2k}X_n) > 0$ for $n_{2k-1} < n \leq n_{2k}$ and $E(\alpha_{2k+1}X_n) < 0$ for $n_{2k} < n \leq n_{2k+1}$. Therefore the process will have a drift towards the top barrier and the bottom barrier alternately, that will make it oscillate and therefore will keep it bounded and $\sum |\triangle \mathcal{S}_n|$ will have more chances to diverge.

Suppose first that $p > 2$.

The position of the particle at time n, for $n_{k-1} < n \leq n_k$ is given by,

$$\mathcal{S}_n = \alpha_1 X_1 + \cdots + \alpha_k X_n ,$$

where, as we have mentioned above, the step α_k is constant during the time the barriers $-M_k$ and M_k are active, that is, for $n_{k-1} < n \leq n_k$. Moreover we choose α_k so that $\operatorname{sgn}(\alpha_k) = (-1)^{k-1}, k \geq 1$ and $|\alpha_k| \searrow 0$ as $k \to \infty$, and also we choose the sequence $\{M_k\}$ to be increasing and $0 < M_k < M$ for every k and some number M.

For $n_{k-1} < n \leq n_k$, we take X_n be a random variable so that,

- if $|\mathcal{S}_{n-1}| < M_k$ then,

$$P(X_n = -1) = \frac{2^{p-1}}{2^{p-1}+1}$$
$$P(X_n = 2) = \frac{1}{2^{p-1}+1},$$

- and if $|\mathcal{S}_{n-1}| = M_k$ then, $X_n \equiv 0$.

Notice that with this choice of $\{\alpha_n\}$ and $\{X_n\}$ it is easy to see that i) and ii) hold.

First, $\sum_j j^{p-1}p_j = 0$, since

$$\sum_j j^{p-1}p_j = 2^{p-1}\frac{1}{2^{p-1}+1} - \frac{2^{p-1}}{2^{p-1}+1} = 0\,.$$

Also, $\mathcal{S}_n$ is bounded. Take $n \in \mathbf{N}$ and let k be so that $n_{k-1} < n \leq n_k$, then

$$|\mathcal{S}_n| \leq M_k < M\,.$$

We are left to show that,

$$P\left(\left\{\sum |\triangle \mathcal{S}_n| < \infty\right\}\right) = 0.$$

Here it will be crucial the alternance of the signs of the α_k's. They have been chosen so that $E(\alpha_k X_{n_k}) > 0$ if k is even and $E(\alpha_k X_{n_k}) < 0$ if k is odd. This fact will allow us to make the process fluctuate.

Now we are going to choose the steps $\{\alpha_k\}$ and the barriers $\{M_k\}$. Take $\{\alpha_k\}$ such that

(1) $\displaystyle\sum_{k=1}^{\infty} |\alpha_k| \leq 1.$

As above, we will require that $|\alpha_k/\alpha_{k+1}| \in \mathbf{N}$ for technical reasons.

Let $\sigma > 1$ be the root of

$$\frac{2^{p-1}}{2^{p-1}+1}x^{-1} + \frac{1}{2^{p-1}+1}x^2 = 1\,,$$

notice that $\sigma > 1$ since $p > 2$. Take $M \in \mathbf{N}$, $M > 1/\alpha_1$ so that,

(2) $\displaystyle\frac{\sigma^M - 1}{\sigma^M} > 1/2.$

We choose,

(3) $\displaystyle M_k = M\sum_{j=1}^{k} |\alpha_k|.$

The period of time in which the barriers are active is a sequence of natural numbers $\{n_k\}$ so that,

(4) $|\alpha_k|(n_k - n_{k-1}) \geq 1$, and

(5) $n_k \geq \tilde{n}_k + n_{k-1}$, where $\tilde{n}_k \in \mathbf{N}$ is given by Corollary 5.1, and the process we are considering starts at M_{k-1} and reaches the barrier $-M_k$ before the barrier M_k. That is, we are chosing the n_k's by a stopping

time argument, we wait until the probability that the process is in the bottom barrier ($-M_k$) is high enough even though the process has started very close to the top barrier. We can get such a "high" probability because $E(X_n) < 0$ (since $p > 2$) and therefore there is a drift towards the bottom barrier.

To evaluate $P(\sum |\triangle \mathcal{S}_n| < \infty)$ observe that the particle in its evolution can:

(a) avoid infinitely many top and bottom barriers at the same time, that is there exist infinitely many j's so that neither M_j nor $-M_j$ are reached for $n_{j-1} < n \leq n_j$,

(b) reach infinitely many top barriers and reach also infinitely many bottom barriers,

(c) avoid at most a finite number of top barriers,

(d) avoid at most a finite number of bottom barriers,

and there are not other posibilities.

Suppose first, that we are in the case (a), that is, the particle avoids infinitely many top and bottom barriers at the same time. Then there exists a sequence $\{k_j\}$ so that the particle does not reach the barrier M_{k_j} nor the barrier $-M_{k_j}$, that is,

$$|\mathcal{S}_n| < M_{k_j}, \qquad \text{for} \ \ n_{k_j-1} < n \leq n_{k_j},$$

and then,

$$|\triangle \mathcal{S}_n| = |\alpha_{k_j} X_n| \geq |\alpha_{k_j}|, \qquad \text{for} \ \ n_{k_j-1} < n \leq n_{k_j}.$$

So we have,

$$\begin{aligned}\sum_{n=1}^{\infty} |\triangle \mathcal{S}_n| &\geq \sum_{j=1}^{\infty} \sum_{n=1+n_{k_j-1}}^{n_{k_j}} |\alpha_{k_j}| \\ &= \sum_{j=1}^{\infty} |\alpha_{k_j}|(n_{k_j} - n_{k_j-1}) \\ &\geq \sum_{j=1}^{\infty} 1 = \infty,\end{aligned}$$

where the last inequality follows from property (4).

Let us assume now that the particle reaches infinitely many top barriers and infinitely many bottom barriers, that is we are in the case (b). We define,

$$\begin{aligned}&N_0 := \inf\{n : \mathcal{S}_n \geq 1\}, \\ &N_k := \inf\{n > N_{k-1} : |\mathcal{S}_n| \geq 1 \ \ \text{and} \quad \operatorname{sgn}(\mathcal{S}_{N_k}) \neq \operatorname{sgn}(\mathcal{S}_{N_{k-1}})\}\,.\end{aligned}$$

Observe that,

$$\sum_{n=N_{k-1}}^{N_k} |\triangle \mathcal{S}_n| \geq |\mathcal{S}_{N_k} - \mathcal{S}_{N_{k-1}}| \geq 2\,.$$

We have that $M_k > 1$ since $M > \frac{1}{\alpha_1}$.

Since $\mathcal{S}_n$ reaches infinitely many top barriers and infinitely many bottom barriers, $\mathcal{S}_n \geq 1$ and $\mathcal{S}_n \leq -1$ infinitely often and therefore $\{N_k\}$ is an infinite sequence so we have,

$$\sum_{n=1}^{\infty} |\triangle \mathcal{S}_n| \geq \sum_{k=1}^{\infty} \sum_{n=N_{k-1}}^{N_k} |\triangle \mathcal{S}_n| \geq \sum_{k=1}^{\infty} 2 = \infty \, .$$

Thus the only chance for the particle to perform a process so that $\sum |\triangle \mathcal{S}_n| < \infty$ is either to avoid a finite number of top barriers or to avoid a finite number of bottom barriers. That is to be either in the case (c) or the case (d), then,

$$\begin{aligned} P\left(\left\{\sum |\triangle \mathcal{S}_n| < \infty\right\}\right) \leq\ & P(\text{ avoiding a finite number of top barriers }) \\ & + P(\text{ avoiding a finite number of bottom barriers }). \end{aligned}$$

And by symmetry, it is enough to prove,

$$P(\text{ avoiding a finite number of top barriers }) = 0.$$

Notice that,

$$\begin{aligned} & P(\text{ avoiding a finite number of top barriers }) \\ & \leq \sum_{k=1}^{\infty} P(\text{ reaching all top barriers after } M_k \text{ }). \end{aligned}$$

Let us denote the event "*reaching all top barriers after* M_k" by A_k, the event "*reaching the top barrier* M_k" by B_k and the event "*reaching the bottom barrier* $-M_k$" by C_k. Then,

$$A_k = \bigcap_{j=k+1}^{\infty} B_j \quad \text{and} \quad B_j^c \supset C_j \, ,$$

where A^c denotes the complementary set of A.

Using the fact that the events B_j are independent (the process is markovian and $n_k > 1 + n_{k-1}$) and by elementary properties of the probability, we obtain,

$$P(A_k) = \prod_{j \geq k+1} P(B_j | B_{j-1}) \quad \text{and} \quad P(B_j | B_{j-1}) \leq 1 - P(C_j | B_{j-1}).$$

Notice also that,

$$\begin{aligned} C_j &= \{ \text{ reaching the bottom barrier } - M_j \} \\ &= \{ \text{ reaching the bottom barrier } - M_j \text{ before the barrier } M_j \text{ for } n \leq n_j \} \end{aligned}$$

Observe that the position of the particle at time n is bounded by,

$$\mathcal{S}_n \leq M_{j-1} + \alpha_j X_{n_{j-1}} + ... + \alpha_j X_n \qquad \text{for} \qquad n_{j-1} < n \leq n_j \, .$$

Then by property (5) of n_k's and applying Corollary 5.1 with $M = -M_{2j+1}$, $N = M_{2j+1}$, $k = M_{2j}$ and $\alpha = \alpha_{2j+1}$ (recall that $\alpha_{2j+1} > 0$), we obtain,

$$P(C_{2j+1}|B_{2j}) \geq \frac{\sigma^{\frac{2M_{(2j+1)}}{\alpha_{(2j+1)}}} - \sigma^{\frac{M_{(2j+1)}+M_{(2j)}}{\alpha_{(2j+1)}}}}{\sigma^{\frac{2M_{(2j+1)}}{\alpha_{(2j+1)}}}} = \frac{\sigma^M - 1}{\sigma^M} > 1/2,$$

where the equality and the last inequality follow from properties (3) and (2) respectively. That is, there is a probability greater than 1/2 of reaching the bottom barrier $-M_{2j+1}$ before time n_{2j+1}, and therefore there is probabilty less than 1/2 of reaching the top barrier M_{2j+1} before time n_{2j+1}, that is,

$$P(B_{2j+1}) \leq 1 - P(C_{2j+1}) < 1/2\,.$$

Then,

$$P(A_k) = \prod_{j \geq k+1} P(B_j|B_{j-1}) \leq \prod_{j \geq k+1} P(B_{2j+1}|B_{2j}) \leq \prod_{j \geq k+1} 1/2 = 0\,.$$

So we obtain,

$$P(\text{ avoiding a finite number of top barriers }) \leq \sum_{k=1}^{\infty} P(A_k) = 0.$$

And then Theorem 4 is proved for $p > 2$.

Suppose now that $p < 2$.

Take $\{\alpha_n\}$ and $\{X_n\}$ those chosen in the case $p > 2$.

Let $\gamma > 1$ be the root of,

$$\frac{2^{p-1}}{2^{p-1}+1}x + \frac{1}{2^{p-1}+1}x^{-2} = 1,$$

notice that $\gamma > 1$ since $p < 2$.

Take $M' \in \mathbf{N}$, $M' > 1/\alpha_1$ so that,

(2') $\dfrac{\gamma^{M'} - 1}{\gamma^{M'}} > 1/2.$

Define $M_k = M' \sum_{j \leq k} |\alpha_j|$ and $m_j = |\alpha_j| M'$.

And now take n_k's so that (4) and (5) hold for the corresponding steps and for the corresponding barriers.

The rest of the proof follows in the same way that the case $p > 2$ but with the roles of the top barriers and the bottom barriers interchanged (notice that $E(X_n) > 0$ unlike the case $p > 2$ where $E(X_n) < 0$).

5.6. Theorem 4 stated for a tree.

In this section we are going to state Theorem 4 for a tree, namely,

Theorem 5. *For $p \in (1, \infty)$ there exist a directed regular tree T, and $u : T \to \mathbf{R}$ a function on the tree, such that,*

(1) *u is p-harmonic,*
(2) *u is bounded,*
(3) *$|BV(u)| = 0$.*

Proof. We are going to divide the proof into two cases. Both are essentially the same but in the second one there are minor technical dificulties that do not appear in the first case.

Case 1: $2^{p-1} \in \mathbf{N}$.

Let N be the degree of T and take $N := 2^{p-1} + 1$. From now on, given $n \in \mathbf{N}$, $k(n)$ will denote the integer so that,

$$n_{k(n)-1} < n \leq n_{k(n)} \,,$$

where n_k's are given in Section 5.4 for $p = 2$ and in Section 5.5 for $p \neq 2$.

We are going to define the function u recursively:

-For $v = v_0$, we define $u(v_0) = 0$.

-Assume now that u is defined for all vertices in S_{n-1}. Take $v \in S_{n-1}$ and let $v_j \in H_v$ for $j = 1, \ldots, N$. Then,

- If $|u(v)| < M_{k(n)}$, we define,

$$u(v_j) = u(v) + \alpha_{k(n)} Z_n(v_j), \qquad j = 1, \ldots, N,$$

where α_k and M_k are given in Section 5.4 for $p = 2$ and in Section 5.5 for $p \neq 2$; and $Z_n(v_j)$ is so that,

$$Z_n(v_j) = \begin{cases} -\;\;1, & \text{for } j = 1, \ldots, N-1 \\ \;\;\;\;2, & \text{for } j = N\,. \end{cases}$$

- If $|u(v)| = M_{k(n)}$ we define,

$$u(v_j) = u(v), \qquad \text{for } j = 1, \ldots, N,$$

that is, $Z_n(v_j) \equiv 0$.

Notice that $Z_n : S_n \to \mathbf{R}$ is only defined for vertices in S_n and that $|u(v)| \leq M_{k(n)}$ for all $v \in S_n$.

With this definition the function u has the following properties.

(1) u is p-harmonic. Given $v \in S_n$,

$$\triangle_p u(v) = -\sum_{j=1}^{N} (u(v_j) - u(v))^{p-1} \,.$$

(i) If $|u(v)| < M_{k(n)}$ then,

$$u(v_j) - u(v) = \alpha_{k(n)} \cdot \begin{cases} -\quad 1, & \text{if} \quad j = 1, \dots, N-1, \\ \quad 2, & \text{if} \quad j = N, \end{cases}$$

and so,

$$\triangle_p u(v) = \alpha_{k(n)}^{p-1}(-(N-1) + 2^{p-1}) = 0 \,.$$

Recall that $N = 2^{p-1} + 1$.

(ii) If $|u(v)| = M_{k(n)}$, then $u(v) = u(v_j)$ and trivially, $\triangle_p u(v) = 0$.

(2) u is bounded. Given $v \in T$ let n be so that $v \in S_n$. Clearly,

$$|u(v)| \leq M_{k(n)} < M \,.$$

(3) $|BV(u)| = 0$.

We define in ∂T the random variables $\{X_n\}$ such that,

$$P(X_n = a) = |\{\gamma \in \partial T : Z_n(v_n) = a, v_n \in S_n \text{ and } v_n \in \gamma\}| \,,$$

for $a = -1, 2, 0$. Then,

$$u(\gamma) = \sum_{n=1}^{\infty} \alpha_{k(n)} Z_n(v_n) = \sum_{n=1}^{\infty} \alpha_{k(n)} X_n = \lim_{n\to\infty} \mathcal{S}_n \,,$$

where $\mathcal{S}_n$ is the process described in Section 5.4 for $p = 2$ and in Section 5.5 for $p \neq 2$. Therefore,

$$\begin{aligned} V(u,\gamma) &= \sum_{n=0}^{\infty} |\nabla u(v_n, v_{n+1})| = \sum_{n=0}^{\infty} |u(v_{n+1}) - u(v_n)| \\ &= \sum_{n=1}^{\infty} |\alpha_{k(n)} Z_n(v_n)| = \sum_{n=1}^{\infty} |\alpha_{k(n)} X_n| = \sum_{n=1}^{\infty} |\triangle \mathcal{S}_n| \,. \end{aligned}$$

And so,

$$|BV(u)| = P\left(\left\{\sum_{n=1}^{\infty} |\triangle \mathcal{S}_n| < \infty\right\}\right) = 0 \,,$$

where we have used the results obtained in previous sections.

Case 2: $2^{p-1} \notin \mathbf{N}$.

In this case $2^{p-1} + 1$ is not a natural number, and so we cannot use the stochastic proof of the theorem in an inmediate way, as we have done for $2^{p-1} \in \mathbf{N}$. Nevertheless the changes required are only technical.

Choose $N \in \mathbf{N}$, $N > 2$ and so that $p < N$. Let us denote,

$$s(p) := \operatorname{sgn}(p-2) = \begin{cases} -\quad 1, & p < 2 \\ \quad 1, & p > 2 \,. \end{cases}$$

We take $\sigma > 1$ the root of

$$\frac{N-1}{N}x^{-s(p)} + \frac{1}{N}x^{s(p)(N-1)^{\beta}} = 1,$$

where $\beta = \dfrac{1}{p-1}$.

Let $M > 0$ be so that,

$$\frac{\sigma^M - 1}{\sigma^M} > 1/2.$$

And take $\{\alpha_k\}, \{M_k\}$ and $\{n_k\}$ with properties (1), (3), (4) and (5) described in Section 5.5. Note that we do not require now that $|\frac{\alpha_k}{\alpha_{k+1}}| \in \mathbf{N}$.

As above we will define u recursively,

For v_0, define $u(v_0) = 0$.

Suppose that u is defined for all vertices in S_{n-1}. Take $v \in S_{n-1}$ and let $v_j \in H_v$ for $j = 1, \ldots, N$. Then,

- If $|u(v)| < M_{k(n)}$ we define,

$$u(v_j) = u(v) + \alpha_{k(n)} Z_n(v_j), \qquad j = 1, \ldots, N$$

where,

$$Z_n(v_j) = \begin{cases} -s(p) & j = 1, \ldots, N-1 \\ s(p)(N-1)^{\beta}, & j = N. \end{cases}$$

- If $|u(v)| \geq M_{k(n)}$, we define,

$$u(v_j) = u(v), \qquad j = 1, \ldots, N,$$

that is, $Z_n(v_j) \equiv 0$.

Notice that $|u(v)| \leq M_{k(n)} + (N-1)^{\beta}\alpha_{k(n)}$ for all $v \in S_n$.

Following the proof above it is clear that **(1)**, **(2)** and **(3)** hold for u defined in this way.

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